HW Chiral ring. Let $W(z_1, \dots z_N)$ be a quasi-homog. poly which defines an isolated hypersurf. sing.

Say $W(\lambda^{q_1} z_1, \dots, \lambda^{q_N} z_N) = \lambda W(z)$. $R = \mathbb{C}[z_1 \dots z_N] / \langle \partial_1 W, \dots, \partial_N W \rangle$

$\Rightarrow P(t) = \sum_w \dim R_w \cdot t^w = \prod_{i=1}^N \left(\frac{1-t^{1-q_i}}{1-t^{q_i}} \right)$

pf. Let $R_i = \mathbb{C}[z_1, \dots z_N] / \langle \partial_1 W, \dots \partial_i W \rangle$.

$(W=0)$ isolated sing. $\Rightarrow$ $(\underline{\partial_i W})$ is a regular sequence $\Rightarrow$ exact seq. $0 \to R_{i-1} \xrightarrow{\cdot \partial_i W} R_{i-1} \to R_i \to 0$

So we get $P(R_{i-1}) = t^{1-q_i} P(R_{i-1}) + P(R_i) \Rightarrow P(R_i) = (1 - t^{1-q_i}) P(R_{i-1})$

$\partial_i W$ has weight $wt(W) - wt(z_i) = 1 - q_i$

By induction, $\underset{P(t)}{\overset{\|}{P(R)}} = \prod_{i=1}^N (1-t^{1-q_i}) \cdot P(\mathbb{C}[z_1, \dots z_N]) = \prod \left(\frac{1-t^{1-q_i}}{1-t^{q_i}} \right)$

$P(\mathbb{C}[z_1, \dots z_N]) = \prod \frac{1}{1-t^{q_i}}$

Let $t \to 1^-$, we get $\dim_{\mathbb{C}} R = \prod \frac{1-q_i}{q_i}$.

Let $D = \sum_i (1-2q_i)$. Then $P(\frac{1}{t}) = \prod \left(\frac{1-t^{q_i-1}}{1-t^{-q_i}} \right) = t^{\sum(2q_i-1)} \cdot \prod \left(\frac{1-t^{1-q_i}}{1-t^{q_i}} \right) = t^{-D} P(t)$

$\Rightarrow \dim R_w = \dim R_{D-w}$ (Poincaré duality).

$$L = \underbrace{\frac{1}{2} g_{i\bar{j}} \dot{x}^i \dot{x}^{\bar{j}}}_{\overset{\shortparallel}{\mathrm{I}}} + \underbrace{\sqrt{-1}\, g_{i\bar{j}} \bar{\psi}^i \nabla_t \psi^{\bar{j}}}_{\overset{\shortparallel}{\mathrm{II}}} \underbrace{- \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^{i\bar{j}k\bar{l}}}_{\overset{\shortparallel}{\mathrm{III}}}$$

$$\delta \mathrm{I} = \frac{1}{2} g_{i\bar{j},k} (\underbrace{\varepsilon \bar{\psi}^k}_{(1)} - \underbrace{\bar{\varepsilon} \psi^k}_{(2)}) \dot{x}^i \dot{x}^{\bar{j}} + g_{i\bar{j}} (\underbrace{\varepsilon \dot{\psi}^i}_{(1)} - \underbrace{\bar{\varepsilon} \dot{\psi}^i}_{(3)}) \dot{x}^{\bar{j}}$$

$$\delta \mathrm{II} = \sqrt{-1}\, g_{i\bar{j},k} (\underbrace{\varepsilon \psi^{\bar{k} i} \dot{\psi}^{\bar{j}}}_{(4)} + \underbrace{\varepsilon \Gamma^{\bar{j}}_{pq} \dot{x}^p \psi^{\bar{k} i q}}_{(9)} - \underbrace{\bar{\varepsilon} \psi^{k i} \dot{\psi}^{\bar{j}}}_{(5)} - \underbrace{\bar{\varepsilon} \Gamma^{\bar{j}}_{pq} \dot{x}^p \psi^{k i q}}_{(10)})$$

$$+ \sqrt{-1}\, g_{i\bar{j}} \Big(\bar{\varepsilon} (- \underbrace{\sqrt{-1}\, \dot{x}^i \dot{\psi}^{\bar{j}}}_{(3)} - \underbrace{\sqrt{-1}\, \Gamma^{\bar{j}}_{kl} \dot{x}^i \dot{x}^l \psi^k}_{(2)} - \underbrace{\Gamma^i_{pq} \psi^{\bar{p} q} \dot{\psi}^{\bar{j}}}_{(5)} - \underbrace{\Gamma^i_{pq} \Gamma^{\bar{j}}_{kl} \dot{x}^l \psi^{\bar{p} q k}}_{(10)} \Big)$$

$$+ \bar{\psi}^i \Big(\varepsilon (\underbrace{\sqrt{-1}\, \ddot{x}^{\bar{j}}}_{(1)} - \underbrace{\Gamma^{\bar{j}}_{pq} \dot{\bar{\psi}}^p \psi^q}_{(8)} - \underbrace{\Gamma^{\bar{j}}_{pq} \bar{\psi}^p \dot{\psi}^q}_{(4)} - \underbrace{\Gamma^{\bar{j}}_{pq,m} \dot{x}^m \psi^{\bar{p} q}}_{(9)})$$

$$+ \Gamma^{\bar{j}}_{kl,m} (\underbrace{\varepsilon \bar{\psi}^m}_{(9)} - \underbrace{\bar{\varepsilon} \psi^m}_{(10)}) \dot{x}^k \psi^l + \Gamma^{\bar{j}}_{kl} (\underbrace{\varepsilon \dot{\bar{\psi}}^k}_{(8)} - \underbrace{\bar{\varepsilon} \dot{\psi}^k}_{(5)}) \psi^l$$

$$+ \Gamma^{\bar{j}}_{kl} \dot{x}^k \varepsilon (\underbrace{\sqrt{-1}\, \dot{x}^l}_{(1)} - \underbrace{\Gamma^l_{pq} \psi^{\bar{p} q}}_{(9)}) \Big)$$

$$\delta \mathrm{III} = -\frac{1}{2} \left(R_{i\bar{j}k\bar{l};m} - \Gamma^s_{im} R_{s\bar{j}k\bar{l}} - \Gamma^s_{jm} R_{isk\bar{l}} - \Gamma^s_{km} R_{i\bar{j}s\bar{l}} - \Gamma^s_{lm} R_{i\bar{j}ks} \right) (\underbrace{\varepsilon \bar{\psi}^m}_{(6)} - \underbrace{\bar{\varepsilon} \psi^m}_{(7)}) \psi^{i\bar{j}k\bar{l}}$$

$$- \frac{1}{2} R_{i\bar{j}k\bar{l}}\, \varepsilon (\underbrace{\sqrt{-1}\, \dot{x}^i}_{(9)} - \underbrace{\Gamma^i_{pq} \psi^{\bar{p} q}}_{(6)}) \psi^{\bar{j}k\bar{l}} - \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^i \bar{\varepsilon} (- \sqrt{-1}\, \dot{x}^{\bar{j}} - \underbrace{\Gamma^i_{pq} \psi^{\bar{p} q}}_{(7)}) \psi^{k\bar{l}}$$

$$- \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^{i\bar{j}}\, \varepsilon (\underbrace{\sqrt{-1}\, \dot{x}^k}_{(9)} - \underbrace{\Gamma^k_{pq} \psi^{\bar{p} q}}_{(6)}) \psi^{\bar{l}} - \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^{i\bar{j}k}\, \bar{\varepsilon} (- \sqrt{-1}\, \dot{x}^l - \underbrace{\Gamma^l_{pq} \psi^{\bar{p} q}}_{(7)})$$

$$(1) = \varepsilon\Big(\tfrac{1}{2} g_{ij,k}\bar\psi^k \dot x^i \dot x^j + g_{ij}\dot{\bar\psi}^i \dot x^j + \underbrace{g_{ij}\bar\psi^i \ddot x^j}_{\text{IBP}} + \underbrace{g_{ij}\Gamma^j_{kl}}\bar\psi^i \dot x^k \dot x^l\Big) = 0$$

$$\left(-g_{ij,k}\bar\psi^i \dot x^j \dot x^k - g_{ij}\dot{\bar\psi}^i \dot x^j\right) \qquad \tfrac{1}{2}\big(g_{il,k} + g_{ki,l} - g_{kl,i}\big)$$

$$\| \ (k \leftrightarrow l)$$

$$(2) = \bar\varepsilon\Big(-\tfrac{1}{2} g_{ij,k}\dot x^i \dot x^j \psi^k + \underbrace{g_{ij}\Gamma^j_{kl}}\dot x^i \dot x^l \psi^k\Big)\Big) = 0 \qquad \big(g_{il,k} - \tfrac{1}{2} g_{kl,i}\big)$$

$$(3) = 0 \text{ is trivial.} \qquad \tfrac{1}{2}\big(g_{il,k} + \underbrace{g_{ki,l} - g_{kl,i}}_{0\ (i\leftrightarrow l)}\big)$$

$$(4) = \varepsilon\Big(\sqrt{-1}\, g_{ij,k}\psi^{k\bar i}\psi^j + \sqrt{-1}\,\underbrace{g_{ij}\Gamma^j_{pq}}\psi^{\bar i p}\psi^q\Big) = 0$$

$$\tfrac{1}{2}\big(g_{iq,p} + \underbrace{g_{pi,q}}_{0\ (i\leftrightarrow p)} - \underbrace{g_{pq,i}}_{g_{iq,p}}\big)$$

$$(5) = \bar\varepsilon\Big(-\sqrt{-1}\,\underbrace{g_{ij,k}}_{g_{im}\Gamma^m_{kj} + g_{mj}\Gamma^m_{ik}}\psi^{k\bar i}\bar\psi^j - \sqrt{-1}\, g_{ij}\Gamma^i_{pq}\psi^{\bar p q}\bar\psi^j - \sqrt{-1}\, g_{ij}\Gamma^j_{kl}\psi^{\bar i l}\bar\psi^k\Big) = 0$$

$$(6) = \varepsilon\Big(-\tfrac{1}{2}\big(\underbrace{R_{ijkl;m}} - \underbrace{\Gamma^s_{im}R_{sjkl}}_{(a)} - \underbrace{\Gamma^s_{jm}R_{iskl}}_{0\ (j\leftrightarrow m)} - \underbrace{\Gamma^s_{km}R_{ijsl}}_{(b)} - \underbrace{\Gamma^s_{lm}R_{ijks}}_{0\ (l\leftrightarrow m)}\big)\psi^{\bar m i\bar j k\bar l}$$

$$R_{ijkl;m} = \underbrace{R_{ijkm;l}}_{-R_{ijkl;m}\ (m\leftrightarrow l)} + \underbrace{R_{ijml;k}}_{0} \quad (R_{i[jml]} = 0) \quad \Rightarrow (=0)$$

$$+\tfrac{1}{2}\underbrace{R_{ijkl}\Gamma^i_{pq}\psi^{\bar p q\bar j k\bar l}}_{(a)} + \tfrac{1}{2}\underbrace{R_{ijkl}\Gamma^k_{pq}\psi^{i\bar j\bar p q\bar l}}_{(b)} = 0$$

$$(a) = 0 \text{ by } \begin{pmatrix} i\to q \\ m\to p\end{pmatrix} \qquad (b) = 0 \text{ by } \begin{pmatrix} k\to q \\ m\to p\end{pmatrix}$$

$$(7) = 0 \text{ similarly.}$$

$$(8) = 0 \text{ is trivial.} \qquad \big(g_{im}\Gamma^m_{kj} + \underbrace{g_{mj}\Gamma^m_{ik}}_{0\ (i\leftrightarrow k)}\big)$$

$$(9) = \varepsilon\Big(\sqrt{-1}\, g_{ij,k}\Gamma^j_{pq}\dot x^p\psi^{k\bar i q} + \sqrt{-1}\, g_{ij}\Gamma^j_{pq,m}\dot x^m\psi^{\bar i p q} - \sqrt{-1}\, g_{ij}\Gamma^j_{kl,m}\dot x^k\psi^{\bar i m l}$$

$$+\sqrt{-1}\, g_{ij}\Gamma^j_{kl}\Gamma^l_{pq}\dot x^k\psi^{\bar i p q} - \tfrac{\sqrt{-1}}{2}R_{ijkl}\dot x^i\psi^{\bar j k\bar l} - \tfrac{\sqrt{-1}}{2}R_{ijkl}\dot x^k\psi^{i\bar j\bar l}\Big)$$

$$(i\leftrightarrow k,\ j\leftrightarrow l)$$

$$= \sqrt{-1}\,\varepsilon\Big(g_{js}\Gamma^s_{ml}\Gamma^m_{ik} + g_{ls}\Gamma^s_{jk,i} - g_{ls}\Gamma^s_{ik,j} + g_{ls}\Gamma^s_{im}\Gamma^m_{jk} - R_{ijkl}\Big)\dot x^i\psi^{\bar j k\bar l} = 0$$

$$\updownarrow$$

$$-g_{ls}\Gamma^s_{mj}\Gamma^m_{ik}$$

by def of R.

$$(10) = 0 \text{ similarly.}$$

HW. Assume $\psi_-(t,s) = e^{-2\pi i a}\psi_-(t,s+2\pi) = e^{-2\pi i b}\psi_-(t+2\pi\tau_2, s+2\pi\tau_1)$ $\quad(*_-)$

$\psi_+(t,s) = e^{2\pi i\tilde{a}}\psi_+(t,s+2\pi) = e^{2\pi i\tilde{b}}\psi_+(t+2\pi\tau_2, s+2\pi\tau_1)$ $\quad(*_+)$

Check: $A^{0,1} = 2\pi i\,\dfrac{b-\tau a}{2\tau_2}\,d\bar{z}$. $\tilde{A}^{1,0} = 2\pi i\,\dfrac{\tilde{b}-\bar{\tau}\tilde{a}}{2\tau_2}\,dz$.

and solve the general solution of the form.

Consider $\psi'_\pm(t,s) = e^{w_\pm(t,s)}\psi_\pm(t,s)$ where $w(t,s) = u_\pm t + v_\pm s$.

We want to find $u_\pm, v_\pm$ s.t. $\psi'_\pm(t,s) = \psi'_\pm(t,s+2\pi) = \psi'_\pm(t+2\pi\tau_2, s+2\pi\tau_1)$.

A direct computation shows that

$$\begin{cases} 2\pi v_- = -2\pi i a \\ 2\pi\tau_2 u_- + 2\pi\tau_1 v_- = -2\pi i b \\ 2\pi v_+ = 2\pi i\tilde{a} \\ 2\pi\tau_2 u_+ + 2\pi\tau_1 v_+ = 2\pi i\tilde{b} \end{cases}$$

which gives $w_-(t,s) = i\left(\dfrac{a\tau_1-b}{\tau_2}t - as\right)$, $w_+(t,s) = i\left(\dfrac{\tilde{b}-\tilde{a}\tau_1}{\tau_2}t + \tilde{a}s\right)$

$= \pi\left(\dfrac{a\bar{\tau}-b}{\tau_2}z - \dfrac{a\tau-b}{\tau_2}\bar{z}\right)$ $\qquad = \pi\left(\dfrac{\tilde{b}-\tilde{a}\bar{\tau}}{\tau_2}z - \dfrac{\tilde{b}-\tilde{a}\tau}{\tau_2}\bar{z}\right)$

Any $\psi_\pm$ satisfies $(*_\pm)$ can be regarded as sections of the complex line bundles

$L_\pm = e^{-w_\pm(z)}$ of the torus $\mathbb{C}/\langle 1,\tau\rangle$.

Then $\nabla(e^{-w_\pm}) = -dw_\pm e^{-w_\pm}$ $\Rightarrow$ $A = -dw_- = \pi\left(\dfrac{b-a\bar{\tau}}{\tau_2}dz - \dfrac{b-a\tau}{\tau_2}d\bar{z}\right)$

$\tilde{A} = -dw_+ = \pi\left(\dfrac{\tilde{a}\bar{\tau}-\tilde{b}}{\tau_2}dz - \dfrac{\tilde{a}\tau-\tilde{b}}{\tau_2}d\bar{z}\right)$.

For the general solution, $\psi'_\pm(t,s) = e^{w_\pm}\psi_\pm(t,s)$ and the periodicity of $\psi'_\pm$ gives

$$\psi_-(t,s) = \left(\sum_n \psi_n(t)e^{ins}\right)e^{-w_-} = \sum_{r\in\mathbb{Z}+a}\psi_r(t)\,e^{i\left(rs+\frac{b-a\tau_1}{\tau_2}t\right)}$$

$$\bar{\psi}_-(t,s) = \left(\sum_n \bar{\psi}_n(t)e^{ins}\right)e^{w_-} = \sum_{r\in\mathbb{Z}-a}\bar{\psi}_r(t)\,e^{i\left(rs-\frac{b-a\tau_1}{\tau_2}t\right)}$$

$$\psi_+(t,s) = \left(\sum_n \tilde{\psi}_n(t)e^{-ins}\right)e^{-w_+} = \sum_{\tilde{r}\in\mathbb{Z}+\tilde{a}}\tilde{\psi}_n(t)\,e^{-i\left(\tilde{r}s+\frac{\tilde{b}-\tilde{a}\tau_1}{\tau_2}t\right)}$$

$$\bar{\psi}_+(t,s) = \left(\sum_n \bar{\tilde{\psi}}_n(t)e^{-ins}\right)e^{w_+} = \sum_{\tilde{r}\in\mathbb{Z}-\tilde{a}}\hat{\psi}_n(t)\,e^{-i\left(\hat{r}s-\frac{\tilde{b}-\tilde{a}\tau_1}{\tau_2}t\right)}$$

HW. Check $\langle e^{ik_1 x(t_1,s_1)} e^{ik_2 x(t_2,s_2)} \rangle = 2\pi \delta(k_1+k_2) \left[(z_1-z_2)(\tilde z_1 - \tilde z_2)\right]^{\frac{k_1k_2}{2}}$

$$\langle e^{ik_1 x(t_1,s_1)} e^{ik_2 x(t_2,s_2)} \rangle := \langle 0 | T[:e^{ik_1x(t_1,s_1)}::e^{ik_2x(t_2,s_2)}:]|0\rangle$$

$$x(t,s) = x_0 + tp_0 + \frac{i}{\sqrt2}\sum_{n\neq0}\frac1n\left(\alpha_n e^{-in(t-s)} + \tilde\alpha_n e^{-in(t+s)}\right)$$

$$\Rightarrow :e^{ikx(t,s)}: = e^{\frac{k}{\sqrt2}\sum_{n=1}^\infty \frac1n(\alpha_{-n}z^n + \tilde\alpha_{-n}\tilde z^n)} e^{ikx_0} e^{iktp_0} e^{-\frac{k}{\sqrt2}\sum_{n=1}^\infty\frac1n(\alpha_n z^{-n}+\tilde\alpha_n\tilde z^{-n})}$$

where $z = e^{i(t-s)}$, $\tilde z = e^{i(t+s)}$.

So, when $t_1>t_2$,

$T[:e^{ik_1x(t_1,s_1)}::e^{ik_2x(t_2,s_2)}:]|0\rangle$

$$= e^{\frac{k_1}{\sqrt2}\sum_{n=1}^\infty \frac1n(\alpha_{-n}z_1^n + \tilde\alpha_{-n}\tilde z_1^n)} e^{ik_1x_0} e^{ik_1t_1p_0} e^{-\frac{k_1}{\sqrt2}\sum_{n=1}^\infty\frac1n(\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n})}$$

$$\cdot e^{\frac{k_2}{\sqrt2}\sum_{n=1}^\infty \frac1n(\alpha_{-n}z_2^n + \tilde\alpha_{-n}\tilde z_2^n)} e^{ik_2x_0} e^{ik_2t_2p_0} \underbrace{e^{-\frac{k_2}{\sqrt2}\sum_{n=1}^\infty\frac1n(\alpha_n z_2^{-n}+\tilde\alpha_n\tilde z_2^{-n})}|0\rangle}_{=|0\rangle}$$

$|0\rangle$ (∵ $\alpha_n, \tilde\alpha_n$ kills $|0\rangle$)

$$= e^{\frac{k_1}{\sqrt2}\sum_{n=1}^\infty \frac1n(\alpha_{-n}z_1^n + \tilde\alpha_{-n}\tilde z_1^n)} \underbrace{e^{-\frac{k_1}{\sqrt2}\sum_{n=1}^\infty\frac1n(\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n})} e^{\frac{k_2}{\sqrt2}\sum_{n=1}^\infty \frac1n(\alpha_{-n}z_2^n + \tilde\alpha_{-n}\tilde z_2^n)}}_{=\prod_{n=1}^\infty e^{-\frac{k_1}{\sqrt2}\frac1n\alpha_n z_1^{-n}} e^{\frac{k_2}{\sqrt2}\frac1n\alpha_n z_2^n}\cdot\prod_{n=1}^\infty e^{-\frac{k_1}{\sqrt2}\frac1n\tilde\alpha_n\tilde z_1^{-n}} e^{\frac{k_2}{\sqrt2}\frac1n\tilde\alpha_{-n}\tilde z_2^n}}$$

$$e^{ik_1x_0} e^{ik_1t_1p_0} e^{ik_2x_0} e^{ik_2t_2p_0}|0\rangle \quad (*)$$

$e^{ik_2t_2p_0}|0\rangle = |0\rangle$

$e^{ik_2x_0}|0\rangle = |k_2\rangle$

$e^{ik_1t_1p_0}|k_2\rangle = e^{ik_1k_2t_1}|k_2\rangle$

$e^{ik_1x_0}e^{ik_1k_2t_1}|k_2\rangle = e^{ik_1k_2t_1}|k_1+k_2\rangle$

Note that
$$e^{-\frac{k_1}{\sqrt2}\frac1n\alpha_n z_1^{-n}} e^{\frac{k_2}{\sqrt2}\frac1n\alpha_{-n}z_2^n}|0\rangle_n = \sum_l \frac1{l!}\left(-\frac{k_1}{\sqrt2}\frac1n\alpha_n z_1^{-n}\right)^l \sum_m \frac1{m!}\left(\frac{k_2}{\sqrt2}\frac1n\alpha_{-n}z_2^n\right)^m|0\rangle_n$$

$$=\sum_{l,m}\frac{1}{l!m!}\left(-\frac{k_1}{\sqrt2}\frac{z_1^{-n}}{n}\right)^l\left(\frac{k_2}{\sqrt2}\frac{z_2^n}{n}\right)^m$$

$$\alpha_n^l\alpha_{-n}^m|0\rangle_n$$

$$= mn\,\alpha_n^{l-1}\alpha_{-n}^{m-1}|0\rangle_n$$

$$= \begin{cases}0 & \text{if } l>m\\ \frac{m!}{(m-l)!}n^l\alpha_{-n}^{m-l}|0\rangle_n & \text{if } l\le m\end{cases}$$

$$=\sum_{l\le m}\frac{n^l}{l!(m-l)!}\left(-\frac{k_1}{\sqrt2}\frac{z_1^{-n}}{n}\right)^l\left(\frac{k_2z_2^n}{\sqrt2 n}\right)^m\alpha_{-n}^{m-l}|0\rangle_n$$

$m=l+j$

$$=\sum_{l,j}\frac{n^l}{l!j!}\left(-\frac{k_1k_2}{2n^2}\left(\frac{z_2}{z_1}\right)^n\right)^l\left(\frac{k_2z_2^n}{\sqrt2 n}\right)^j\alpha_{-n}^j|0\rangle_n$$

$$= e^{-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n} e^{\frac{k_2z_2^n}{\sqrt2 n}\alpha_{-n}}|0\rangle_n$$

So

$$(*) = e^{\frac{k_1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n} z_1^n + \tilde{\alpha}_{-n}\tilde{z}_1^n)}\left(\prod_{n=1}^{\infty} e^{-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n - \frac{k_1k_2}{2n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n + \frac{k_2 z_2^n}{2\sqrt{n}}\alpha_{-n} + \frac{k_2\tilde{z}_2^n}{2\sqrt{n}}\tilde{\alpha}_{-n}}\right) e^{ik_1k_2t_1} |k_1+k_2\rangle$$

$$\langle 0| T[:e^{ik_1 x(t_1,s_1)}::e^{ik_2 x(t_2,s_2)}:]|0\rangle$$

$$= 2\pi\delta(k_1+k_2)\cdot \underbrace{\prod_{n=1}^{\infty} e^{-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n - \frac{k_1k_2}{2n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n}\cdot e^{ik_1k_2t_1}} = 2\pi\delta(k_1+k_2)\,[(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)]^{\frac{k_1k_2}{2}}.$$

$$\parallel$$

$$e^{k_1k_2\left(\underbrace{it_1}_{\pi} - \underbrace{\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z_2}{z_1}\right)^n}_{\parallel} - \underbrace{\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n}_{\parallel}\right)} = [(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)]^{\frac{k_1k_2}{2}}$$

$$\log(z_1\tilde{z}_1) + \frac{1}{2}\log\left(1-\frac{z_2}{z_1}\right) + \frac{1}{2}\log\left(1-\frac{\tilde{z}_2}{\tilde{z}_1}\right)$$

$$L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \sqrt{-1}\,g_{ij}\bar{\psi}^i\underbrace{\nabla_t\psi^j}_{\parallel} - \frac{1}{2}R_{ijkl}\psi^{i\bar{j}k\bar{l}}$$

$$\dot{\psi}^j + \Gamma^j_{pq}\dot{x}^p\psi^q$$

$$\Rightarrow \delta\int L\,dt = \int g_{ij}(\dot{\varepsilon}\bar{\psi}^i - \dot{\bar{\varepsilon}}\psi^i)\dot{x}^j + \sqrt{-1}\,g_{ij}\bar{\psi}^i\left(\dot{\varepsilon}(\sqrt{-1}\dot{x}^j - \Gamma^j_{pq}\psi^{\bar{p}q}) + \Gamma^j_{pq}(\varepsilon\bar{\psi}^p - \bar{\varepsilon}\psi^p)\psi^q\right)$$

$$\underbrace{-\dot{\varepsilon}g_{ij}\bar{\psi}^i\dot{x}^j}\ \text{IBP last time}\ dt$$

$$= \int \dot{\varepsilon}\left(g_{ij}\bar{\psi}^i\dot{x}^j - \sqrt{-1}g_{ij}\bar{\psi}^i(\cancel{\sqrt{-1}\dot{x}^j} - \cancel{\Gamma^j_{pq}\psi^{\bar{p}q}} + \cancel{\Gamma^j_{pq}\psi^{\bar{p}q}}) - \cancel{g_{ij}\bar{\psi}^i\dot{x}^j}\right)$$

$$+\dot{\bar{\varepsilon}}\left(-g_{ij}\psi^i\dot{x}^j + \sqrt{-1}g_{ij}\underbrace{\Gamma^j_{pq}\bar{\psi}^i\psi^p\psi^q}_{\substack{\parallel \\ 0\ (p\leftrightarrow q)}}\right)dt = \int -i\dot{\varepsilon}Q - i\dot{\bar{\varepsilon}}\bar{Q}$$

where $Q = ig_{ij}\bar{\psi}^i\dot{x}^j$, $\bar{Q} = -ig_{ij}\psi^i\dot{x}^j$.

B06201034 李雙言

HW1: Let $\Phi = \varphi + \theta^{=}\varphi_{=} + \theta^{\mp}\varphi_{\mp} + \theta^{=\mp}\varphi_{=\mp}$ $\Rightarrow \begin{cases} \varphi_{=} + \theta^{\mp}\varphi_{=\mp} = i\theta^{-}\partial_{-}\Phi \\ \varphi_{\mp} - \theta^{=}\varphi_{=\mp} = i\theta^{+}\partial_{+}\Phi \end{cases}$

$$\Rightarrow \begin{cases} \varphi_{=} = i\theta^{-}\partial_{-}\varphi \\ \varphi_{\mp} = i\theta^{+}\partial_{+}\varphi \\ \varphi_{=\mp} = i\theta^{-}\partial_{-}\varphi_{\mp} \\ \varphi_{=\mp} = -i\theta^{+}\partial_{+}\varphi_{=} \end{cases}$$

So $\varphi_{=} = \theta^{-}\underbrace{\eta_{-=}}_{i\partial_{-}\eta} + \theta^{+-}\underbrace{\eta_{+-=}}_{-i\partial_{-}\eta_{+}}$, $\varphi_{\mp} = \theta^{+}\underbrace{\eta_{+\mp}}_{i\partial_{+}\eta} + \theta^{+-}\underbrace{\eta_{+-\mp}}_{i\partial_{+}\eta_{-}}$, $\varphi_{=\mp} = \theta^{+-}\underbrace{\eta_{+-=\mp}}_{\partial_{-}\partial_{+}\eta}$

$$\Rightarrow \Phi = \eta + \theta^{+}\eta_{+} + \theta^{-}\eta_{-} + \theta^{+-}\eta_{+-} + \theta^{=}(\theta^{-}i\partial_{-}\eta - \theta^{+-}i\partial_{-}\eta_{+})$$
$$+\theta^{\mp}(\theta^{+}i\partial_{+}\eta + \theta^{+-}i\partial_{+}\eta_{-}) + \theta^{4}\partial_{-}\partial_{+}\eta.$$

Since $\eta_\alpha(y) = \eta_\alpha - \partial_{+}\eta_\alpha \cdot i\theta^{+\mp} - \partial_{-}\eta_\alpha \cdot i\theta^{-=} - \partial_{-}\partial_{+}\eta_\alpha \cdot \theta^{4}$,

$$\Phi = \underbrace{\eta(y)}_{\phi} + \theta^{+}\underbrace{\eta_{+}(y)}_{\psi_{+}} + \theta^{-}\underbrace{\eta_{-}(y)}_{\psi_{-}} + \theta^{+-}\underbrace{\eta_{+-}(y)}_{F}.$$

HW2. Calculate $Q_{\pm}$, $\bar{Q}_{\pm}$, F_A, F_V ...

Since $Q_{\pm}\eta = \frac{\partial}{\partial\theta^{\pm}}\eta + i\bar\theta^{\pm}\partial_{\pm}\eta = \partial_{\pm}\eta\cdot(-i\bar\theta^{\pm}) + i\bar\theta^{\pm}\partial_{\pm}\eta = 0$

$\bar{Q}_{\pm}\eta = -\frac{\partial}{\partial\bar\theta^{\pm}}\eta - i\theta^{\pm}\partial_{\pm}\eta = -\partial_{\pm}\eta\cdot i\theta^{\pm} - i\theta^{\pm}\partial_{\pm}\eta = -2i\theta^{\pm}\partial_{\pm}\eta$ for any $\eta = \eta(y)$

$$\delta\Phi = (\pm\varepsilon_{\pm}Q_{\mp} \pm \bar\varepsilon_{\mp}\bar{Q}_{\pm})(\phi + \theta^{\pm}\psi_{\pm} + \theta^{2}F)$$
$$= \pm\theta^{\pm}\cdot 2i\bar\varepsilon_{\mp}\partial_{\pm}\phi + \underbrace{(\delta\theta^{\pm})}_{\mp\varepsilon_{\mp}}\psi_{\pm} \mp \underbrace{\theta^{\pm}\cdot 2i(\Sigma\,\bar\varepsilon_{\mp}\theta^{\pm}\partial_{\pm})\psi_{\pm}}_{\pm 2i\theta^{2}\bar\varepsilon_{\pm}\partial_{\mp}\psi_{\pm}} + \underbrace{(\delta\theta^{2})}_{\theta^{\pm}\varepsilon_{\pm}}F + \theta^{2}\underbrace{\delta F}_{0}$$
$$= \underbrace{(\pm\varepsilon_{\pm}\psi_{\mp})}_{\delta\phi} + \theta^{\pm}\underbrace{(\pm 2i\bar\varepsilon_{\mp}\partial_{\pm}\phi + \varepsilon_{\pm}F)}_{\delta\psi_{\pm}} + \theta^{2}\underbrace{(-2i\bar\varepsilon_{\pm}\partial_{\mp}\psi_{\pm})}_{\delta F}.$$

$$\Rightarrow \delta S = \delta\int d^2x\,\left(|\partial_0\phi|^2 - |\partial_1\phi|^2 - |W'(\phi)|^2 + i\bar\psi_{\pm}(\partial_0\mp\partial_1)\psi_{\pm} - W''(\phi)\psi_{+-} - \bar{W}''(\bar\phi)\bar\psi_{-+}\right)$$
$$= \int d^2x\,\Big(\partial_0(\varepsilon_{+}\psi_{-})\partial_0\bar\phi - \partial_1(\varepsilon_{+}\psi_{-})\partial_1\bar\phi - \overbrace{\underbrace{W''(\phi)\varepsilon_{+}\psi_{-}\bar{W}'(\bar\phi)}_{} + i(2i\varepsilon_{+}\partial_{-}\bar\phi)(\partial_0+\partial_1)\psi_{-}}$$
$$)=0$$
$$+ i\bar\psi_{+}(\partial_0-\partial_1)(\varepsilon_{+}F) - W'''(\phi)\varepsilon_{+}\underbrace{\psi_{-}\psi_{+-}}_{0} - \overbrace{W''(\phi)\varepsilon_{+}F\psi_{-}}$$
$$- \bar{W}''(\bar\phi)(2i\varepsilon_{+}\partial_{-}\bar\phi)\bar\psi_{+}\Big) \quad \text{with } F = -\bar{W}'(\bar\phi)$$

$+$ (terms involve $\varepsilon_{-}, \bar\varepsilon_{+}, \bar\varepsilon_{-}$)

$$= \int d^2x \Big(\varepsilon_+ \big(\partial_0\psi_- \partial_0\bar\phi - \partial_1\psi_- \partial_1\bar\phi \big) + (\partial_0\varepsilon_+)\psi_-(\partial_0\bar\phi) - (\partial_1\varepsilon_+)\psi_-(\partial_1\bar\phi) - \varepsilon_+\big((\partial_0-\partial_1)\bar\phi\,(\partial_0+\partial_1)\psi_-\big)$$

$$+ i(\partial_0-\partial_1)(\varepsilon_+\underbrace{\overline{W}'(\bar\phi))\bar\psi_+ - i\,\overline{W}''(\bar\phi)\,\varepsilon_+(\partial_0-\partial_1)(\bar\phi)\bar\psi_+}_{=0} \Big) + \cdots$$

$$= \int d^2x \Big((\partial_0\varepsilon_+)\psi_-(\partial_0\bar\phi) - (\partial_1\varepsilon_+)\psi_-(\partial_1\bar\phi) + \varepsilon_+\big(\partial_1\bar\phi\,\partial_0\psi_- - \partial_0\bar\phi\,\partial_1\psi_-\big) + i(\partial_0-\partial_1)(\varepsilon_+)\,\overline{W}'(\bar\phi)\bar\psi_+ \Big) + \cdots$$

zBP

$$= \int d^2x \Big((\partial_0\varepsilon_+)\big(2\partial_-\bar\phi\psi_- + i\overline{W}'(\bar\phi)\bar\psi_+\big) + (\partial_1\varepsilon_+)(\cdots) \Big) + \cdots$$

$$\Rightarrow Q_- = \int dx^1\, G^0_- \quad \text{with } G^0_- = 2\partial_-\bar\phi\psi_- + i\overline{W}'(\bar\phi)\bar\psi_+ \quad , Q_+, \bar Q_-, \bar Q_+ = \cdots$$

For axial rotation $\psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm$. $(\Rightarrow \delta\psi_\pm = \mp i\alpha\cdot\psi_\pm,\ \delta\bar\psi_\pm = \pm i\alpha\bar\psi_\pm)$

$$\delta S = \int d^2x \Big(i(\pm i\alpha)\bar\psi_\pm(\partial_0\mp\partial_1)\psi_\pm + i\bar\psi_\pm(\partial_0\mp\partial_1)(\mp i\alpha\psi_\pm) - W''(\phi)\underbrace{(-i\alpha+i\alpha)}_{0}\psi_{+-} - \overline{W}''(\bar\phi)\underbrace{(i\alpha-i\alpha)}_{0}\bar\psi_{-+} \Big)$$

$$= \int d^2x \big(\mp\alpha\bar\psi_\pm(\partial_0\mp\partial_1)\psi_\pm \pm\alpha\bar\psi_\pm(\partial_0\mp\partial_1)\psi_\pm \pm \bar\psi_\pm\psi_\pm(\partial_0\mp\partial_1)(\alpha) \big)$$

$$= \int d^2x \Big((\partial_0\alpha)\underbrace{(\bar\psi_+\psi_+ - \bar\psi_-\psi_-)}_{J^0_A} + (\partial_1\alpha)\underbrace{(-\bar\psi_+\psi_+ - \bar\psi_-\psi_-)}_{J^1_A} \Big)$$

$W(\Phi) = c\Phi^k$

For vector rotation $\phi \mapsto e^{\frac{2}{k}i\alpha}\phi$. $\psi_\pm \mapsto e^{(\frac{2}{k}-1)i\alpha}\psi_\pm$ $(\Rightarrow \delta\phi = \frac{2}{k}i\alpha\phi,\ \delta\psi_\pm = (\frac{2}{k}-1)i\alpha\psi_\pm)$

$$\delta S = \int d^2x \Big(\partial_0(\tfrac{2}{k}i\alpha\phi)\partial_0\bar\phi + \partial_0\phi\,\partial_0(-\tfrac{2}{k}i\alpha\bar\phi) - \partial_1(\tfrac{2}{k}i\alpha\phi)\partial_1\bar\phi - \partial_1\phi\,\partial_1(-\tfrac{2}{k}i\alpha\bar\phi)$$

$$+ \underbrace{(ck(k-1)\phi^{k-2})\tfrac{2}{k}i\alpha\phi\cdot(\bar c k\bar\phi^{k-1}) - (ck\phi^{k-1})\tfrac{2}{k}i\alpha\bar\phi\,(\bar c k(k-1)\bar\phi^{k-2})}_{0}$$

$$+ i\big(-(\tfrac{2}{k}-1)i\alpha\bar\psi_\pm\big)(\partial_0\mp\partial_1)\psi_\pm + i\bar\psi_\pm(\partial_0\mp\partial_1)\big((\tfrac{2}{k}-1)i\alpha\psi_\pm\big)$$

$$- \Big(\underbrace{(ck(k-1)(k-2)\phi^{k-3})(\tfrac{2}{k}i\alpha\phi) + 2(ck(k-1)\phi^{k-2})(\tfrac{2}{k}-1)i\alpha}_{0}\Big)\psi_{+-} - \underbrace{(\ )}_{0}\bar\psi_{+-} \Big)$$

$$= \int d^2x \Big((\partial_0\alpha)\underbrace{\big(\tfrac{2}{k}i(\phi\,\partial_0\bar\phi - \partial_0\phi\cdot\bar\phi) - (\tfrac{2}{k}-1)\bar\psi_\pm\psi_\pm\big)}_{J^0_V} + (\partial_1\alpha)\underbrace{(\cdots)}_{J^1_V} \Big)$$

Let $g_{i\bar j} = \partial_i \bar\partial_{\bar j} K(\Phi, \bar\Phi)$. Check $L_{kin} = \int d^4\theta\, K(\Phi,\bar\Phi) = -g_{i\bar j}\partial^\mu \phi^i \partial_\mu \bar\phi^{\bar j} + 2i g_{i\bar j} \bar\psi^{\bar j}_\mp D_\pm \psi^i_\mp + R_{i\bar j k \bar l}\psi^{ik\bar j \bar l}_{+--+}$

up to a total der. $\quad + g_{i\bar j}(F^i - \Gamma^i_{pq}\psi^{pq}_{+-})(\bar F^{\bar j} - \bar\Gamma^{\bar j}_{\bar r\bar s}\psi^{\bar r\bar s}_{-+})$

$\Phi^i = \phi^i(y) + \theta^\pm \psi^i_\pm(y) + \theta^2 F^i(y) = \phi^i - i\theta^{+\mp}\partial_+\phi^i - i\theta^{--}\partial_-\phi^i - \theta^4\partial_+\partial_-\phi^i + \theta^\pm\psi^i_\pm - i\theta^{\pm\mp\mp}\partial_\mp\psi^i_\pm + \theta^+\theta^- F^i$. Define $\varepsilon^i = \Phi^i - \phi^i$

$\Rightarrow K(\Phi,\bar\Phi) = K + \partial_i K\cdot\varepsilon^i + \bar\partial_{\bar i} K\cdot\bar\varepsilon^{\bar i} + \partial_i\partial_j K\cdot\varepsilon^{ij} + \partial_i\bar\partial_{\bar j}K\,\varepsilon^{i\bar j} + \bar\partial_{\bar i}\bar\partial_{\bar j}K\,\varepsilon^{\bar i\bar j} + \partial_{i\bar j k}K\,\varepsilon^{i\bar j k} + \partial_{\bar j k\bar l}\,\varepsilon^{\bar j k\bar l} + \partial_{i\bar j k\bar l}\,\varepsilon^{i\bar j k\bar l}$

$\Rightarrow \int d^4\theta\, K(\Phi,\bar\Phi) = \Big(-(\partial_i K)\partial_2\phi^i - (\bar\partial_{\bar j}K)\partial_2\bar\phi^{\bar j} - \frac{1}{2}(\partial_i\partial_j K)\big(\underbrace{(\partial_+\phi^i)(\partial_-\phi^j)}_{(i\leftrightarrow j)\ \times 2} + (\partial_-\phi^i)(\partial_+\phi^j)\big) - \frac{1}{2}(\bar\partial_{\bar i}\bar\partial_{\bar j}K)\big((\partial_+\bar\phi^{\bar i})(\partial_-\bar\phi^{\bar j}) + (\partial_-\bar\phi^{\bar i})(\partial_+\bar\phi^{\bar j})\big) \Big)$ ①

$+ \underbrace{(\partial_i\bar\partial_{\bar j}K)}_{g_{i\bar j}}\Big(\underbrace{(\partial_+\phi^i)(\partial_-\bar\phi^{\bar j}) + (\partial_-\phi^i)(\partial_+\bar\phi^{\bar j})}_{①} + \underbrace{F^i\bar F^{\bar j}}_{②} + \underbrace{i\psi^i_\pm\partial_\mp\bar\psi^{\bar j}_\pm - i\bar\psi^{\bar j}_\pm\partial_\mp\psi^i_\pm}_{③}\Big)$ (IBP)

$+ \frac{1}{2}\underbrace{(\partial_{i\bar j k}K)}_{g_{p\bar j}\Gamma^p_{ik}}\Big(\underbrace{\pm\psi^i_\mp\psi^k_\pm\bar F^{\bar j}}_{②} + \underbrace{i\bar\psi^{\bar j}_\pm\psi^i_\pm\partial_\mp\phi^k + i\bar\psi^{\bar j}_\pm\psi^k_\pm\partial_\mp\phi^i}_{③}\Big)$ ($\times 2$, $(i\leftrightarrow k)$)

$+ \frac{1}{2}\underbrace{(\partial_{i\bar j\bar k}K)}_{g_{j\bar p}\bar\Gamma^{\bar p}_{\bar i\bar k}}\Big(\underbrace{\mp\bar\psi^{\bar i}_\mp\bar\psi^{\bar k}_\pm F^{j}}_{②} + \underbrace{i\psi^j_\pm\bar\psi^{\bar i}_\pm\partial_\mp\bar\phi^{\bar k} + i\psi^j_\pm\bar\psi^{\bar k}_\pm\partial_\mp\bar\phi^{\bar i}}_{③}\Big) + \underbrace{(\partial_{i\bar j k\bar l}K)\,\psi^{i\bar j k\bar l}_{+-=\mp}}_{④}$ ($\times 2$, $(i\leftrightarrow k)$)

$① = \underbrace{(\partial_+\partial_i K)\partial_-\phi^i + (\partial_+\bar\partial_{\bar j}K)\partial_-\bar\phi^{\bar j}}_{\partial_j\partial_i K\,\partial_+\phi^j + \bar\partial_{\bar j}\partial_i K\,\partial_+\bar\phi^{\bar j}} - (\partial_i\partial_j K)(\partial_-\phi^i)(\partial_+\phi^j) - (\bar\partial_{\bar i}\bar\partial_{\bar j}K)(\partial_-\bar\phi^{\bar i})(\partial_+\bar\phi^{\bar j}) + g_{i\bar j}\big((\partial_+\phi^i)(\partial_-\bar\phi^{\bar j}) + (\partial_-\phi^i)(\partial_+\bar\phi^{\bar j})\big)$

$\quad = 2g_{i\bar j}(\partial_-\phi^i\partial_+\bar\phi^{\bar j} + \partial_+\phi^i\partial_-\bar\phi^{\bar j}) = g_{i\bar j}(\partial_0\phi^i\partial_0\bar\phi^{\bar j} - \partial_1\phi^i\partial_1\bar\phi^{\bar j})$

$② = g_{i\bar j}\big(F^i\bar F^{\bar j} \pm \frac{1}{2}\Gamma^i_{pq}\underbrace{\psi^p_\mp\psi^q_\pm}_{(p\leftrightarrow q)}\bar F^{\bar j} \mp \frac{1}{2}\bar\Gamma^{\bar j}_{\bar r\bar s}\underbrace{\bar\psi^{\bar r}_\mp\bar\psi^{\bar s}_\pm}_{(r\leftrightarrow s)}F^i\big) = g_{i\bar j}(F^i - \Gamma^i_{pq}\psi^{pq}_{+-})(\bar F^{\bar j} - \bar\Gamma^{\bar j}_{\bar r\bar s}\psi^{\bar r\bar s}_{-+})$

$\quad - g_{i\bar j}\Gamma^i_{pq}\bar\Gamma^{\bar j}_{\bar r\bar s}\psi^{pq\bar r\bar s}_{+--+}$

$③ = i g_{i\bar j}\big(\underbrace{2\psi^i_\pm\partial_\mp\bar\psi^{\bar j}_\pm}_{\text{IBP}:\ \bar\psi^{\bar j}_\pm \boxed{\partial_\mp\psi^i_\pm}} + \bar\psi^{\bar j}_\pm\Gamma^i_{pq}\psi^p_\pm\partial_\mp\phi^q + \psi^i_\pm\bar\Gamma^{\bar j}_{\bar p\bar q}\bar\psi^{\bar p}_\pm\partial_\mp\bar\phi^{\bar q}\big) + i\underbrace{\partial_\mp g_{i\bar j}}_{g_{p\bar j}\Gamma^p_{iq}\partial_\mp\phi^q + g_{i\bar p}\bar\Gamma^{\bar p}_{\bar j\bar q}\partial_\mp\bar\phi^{\bar q}}\psi^{i\bar j}_{\pm\pm}$

$\partial_\mp\psi^i_\pm = D_\mp\psi^i_\pm - \partial_\mp\phi^q\Gamma^i_{pq}\psi^p_\pm$

$\quad = 2i g_{i\bar j}\bar\psi^{\bar j}_\pm D_\mp\psi^i_\pm$

$④ = \underbrace{\partial_i\partial_{\bar j}g_{k\bar l}}_{g_{p\bar q}\Gamma^p_{ik}\bar\Gamma^{\bar q}_{\bar j\bar l} + R_{i\bar j k\bar l}}\psi^{i\bar j k\bar l}_{+--+} = R_{i\bar j k\bar l}\psi^{i\bar j k\bar l}_{+--+} + g_{i\bar j}\Gamma^i_{pq}\bar\Gamma^{\bar j}_{\bar r\bar s}\psi^{pq\bar r\bar s}_{+--+}.$

Compute $\langle X_i X_j \rangle_{(1)}$ and $\langle X_i X_j X_k X_l \rangle_{(1)}$.

$$\langle X_i X_j \rangle = \frac{1}{Z(M,C)} \int \underbrace{\prod_{i=1}^{n} dx^i}_{\substack{\shortparallel \\ d^n x}} \, e^{-\frac{1}{2}M_{kl}x^{kl} + C_{pqrs}x^{pqrs}} x^i x^j$$

$$\int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl} + C_{pqrs}x^{pqrs}} x^i x^j = \int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl}} \sum_{m=0}^{\infty} \frac{1}{m!}\left(C_{pqrs}x^{pqrs}\right)^m x^i x^j$$

$$= \sum \frac{1}{m!}\left(\prod_{t=1}^{m} C_{p_t q_t r_t s_t}\right) \int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl}} \prod_{t=1}^{m} x^{p_t q_t r_t s_t} \cdot x^i x^j \quad (1)$$

$$Z(M,C) = \int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl} + C_{pqrs}x^{pqrs}} = \sum \frac{1}{m!}\left(\prod_{t=1}^{m} C_{p_t q_t r_t s_t}\right) \int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl}} \prod_{t=1}^{m} x^{p_t q_t r_t s_t} \quad (2)$$

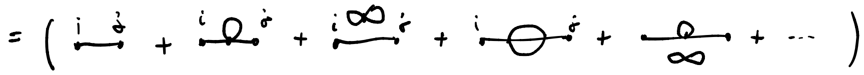
Let $\underset{\substack{\cap \\ \mathbb{R}^n}}{f(J)} = \int d^n x \, e^{-\frac{1}{2}M_{kl}x^{kl} + \langle J, x\rangle} = \int d^n y \; e^{-\frac{1}{2}|y|^2 + \langle (P^T)^{-1}J, y\rangle} \cdot (\det P)^{-1}$

$y = Px$, where $M = P^T P$

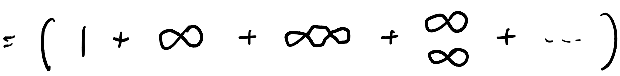
$$= (\det M)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}|(P^T)^{-1}J|^2} \underbrace{\int d^n y \; e^{-\frac{1}{2}|y|^2}}_{\substack{\shortparallel \\ (2\pi)^{\frac{n}{2}}}} = \underbrace{\sqrt{\frac{(2\pi)^n}{\det M}}}_{\substack{\shortparallel \\ B}} \; e^{-\frac{1}{2}J^T M^{-1} J}.$$

Then $\int d^n x \; e^{-\frac{1}{2}M_{kl}x^{kl}} \, x^\alpha = \partial_\alpha f(J)\Big|_{J=0} = \sqrt{\frac{(2\pi)^n}{\det M}} \; \sum M^{\binom{\alpha}{2}\binom{\alpha-2}{2}\cdots\binom{2}{2}}$ $\rightsquigarrow$ graph.

$$\Rightarrow \frac{(1)}{B} = \sum \frac{1}{m!}\left(\prod_{t=1}^{m} C_{p_t q_t r_t s_t}\right) \partial_{\Sigma\{p_t q_t r_t s_t\} + ij} f(J)\Big|_{J=0}$$

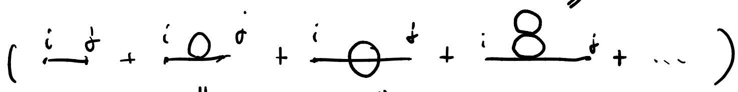


$$\frac{(2)}{B} = \sum \frac{1}{m!}\left(\prod_{t=1}^{m} C_{p_t q_t r_t s_t}\right) \partial_{\Sigma\{p_t q_t r_t s_t\}} f(J)\Big|_{J=0}$$



$C_{\alpha\beta\gamma\delta} C_{\xi\zeta\eta\theta} M^{(i\alpha)(\beta\xi)(\zeta\eta)(\theta\gamma)(\delta j)}$

$\Rightarrow \frac{(1)}{(2)} =$

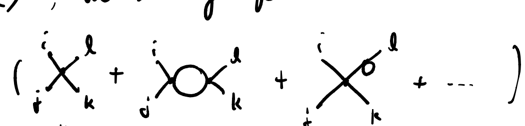


$C_{\alpha\beta\gamma\delta} M^{i\alpha} M^{\beta\gamma} M^{\delta j}$ $\qquad$ $C_{\alpha\beta\gamma\delta} C_{\xi\zeta\eta\theta} M^{(i\alpha)(\beta\gamma)(\delta\xi)(\zeta\eta)(\theta j)}$

For $\langle X_i X_j X_k X_l \rangle$, we similarly get

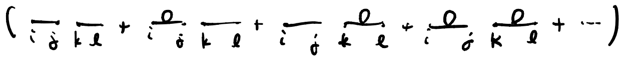
disconnected graph.

$\langle X_i X_j X_k X_l \rangle =$



$+ \sum_{cyc} \langle X_i X_j \rangle \langle X_k X_l \rangle$

$C_{\alpha\beta\gamma\delta} M^{(i\alpha)(j\beta)(k\gamma)(l\delta)}$ $\qquad$ $C_{\alpha\beta\gamma\delta} C_{\xi\zeta\eta\theta} M^{(i\alpha)(j\beta)(\gamma\xi)(\delta\zeta)(\eta k)(\theta l)}$



$m+ \rightsquigarrow k+$ and $(m-k)+$ $\Rightarrow C^m_k$ ways.
$$C^m_k = \frac{m!}{k!(m-k)!}$$

Wess-Zumino Gauge:

Write $V = \sum \theta^\alpha V_\alpha$. $A = \phi + \theta^\pm \psi_\pm + \theta^2 F = \phi - i\theta^{+\bar{+}}\partial_+\phi - i\theta^{-\bar{-}}\partial_-\phi - \theta^4 \partial_+\partial_-\phi$

$$+ \theta^\pm \psi_\pm - i\theta^{+-\bar{-}}\partial_-\psi_+ - i\theta^{-+\bar{+}}\partial_+\psi_- + \theta^2 F.$$

We want to eliminate V_α for $|\alpha| \leq 1$ under the gauge transformation $V \mapsto V + \underbrace{i(\bar{A} - A)}_{2\operatorname{Im} A}$.

$$\operatorname{Im} A \equiv \operatorname{Im}\phi - \frac{i}{2}\sum_{|\alpha|=1} \theta^\alpha \psi_\alpha \quad \bmod (\theta)^2.$$

Take ϕ s.t. $\operatorname{Im}\phi = -V_0$, $\psi_+ = -iV_+$, $\psi_- = -iV_-$.

V real $\Rightarrow$ $V_+ + \bar{V}_{\bar{+}} = 0$, $V_- + \bar{V}_{\bar{-}} = 0$ $\Rightarrow$ $\bar{\psi}_+ = i\bar{V}_+ = -iV_{\bar{+}}$. $\bar{\psi}_- = i\bar{V}_- = -iV_{\bar{-}}$.

$\Rightarrow$ $V' = V + i(\bar{A} - A) \equiv 0 \bmod (\theta)^2$.

So we may assume that V has no lower term.

V real $\Rightarrow$ $\overline{\theta^\alpha V_\alpha} = \theta^{\bar{\alpha}} V_{\bar{\alpha}}$ $\Rightarrow$ $\bar{V}_\alpha = (-1)^{|\alpha|} V_{\bar{\alpha}}$.

So we can write $V = \theta^{\pm\bar{\pm}}(v_0 \pm v_1) + \theta^4 D + \left(-\theta^{-\bar{+}}\sigma + i\theta^{-}\theta^{+}(\bar{\theta}^{-}\bar{\lambda}_- + \bar{\theta}^{+}\bar{\lambda}_+)\right) + (\text{c.c.})$

$\Rightarrow$ D real.

Let $w(x) = \dfrac{x^{k+2}}{k+2} - x$, $x_*^\alpha = e^{\frac{2\pi i \alpha}{k+1}}$, $w_*^\alpha = w(x_*^\alpha) = -\frac{k+1}{k+2} x_*^\alpha$.

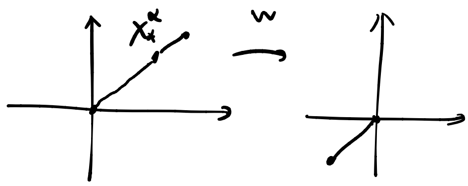
Soliton connecting x_*^α and x_*^β

Consider the paths $\gamma_\alpha : [0,1] \to \mathbb{C}$, γ_β defined similarly.
$$t \mapsto t w_*^\alpha$$

$$w^{-1}(0) = \{0, \sqrt[k+1]{k+2}\, x_*^\alpha, \ \alpha = 0, 1, \dots, k\}$$

Since $\mathrm{ord}_{w(x_*^\alpha)}(w(x) - w(x_*^\alpha)) = 2$, $\exists$ 2 $w^{-1}(\gamma_\alpha)$ that contains x_*^α

$\Rightarrow$ $w^{-1}(\gamma_\alpha) = [0,1] \cdot x_*^\alpha$ or $[1, \sqrt[k+1]{k+2}] \cdot x_*^\alpha$.



$\Rightarrow$ Soliton must be $[0,1] \cdot x_*^\alpha \cup [0,1] x_*^\beta$