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Schlessinger criterion

Λ: complete noetherian local ring $\mu \subset \Lambda$ maximal ideal

$k = \Lambda/\mu$ residue field

$\hat{C}$: Category of (complete) Artin local Λ-algebra with residue field k

$F: C \to Sets$ is called prorepresentable if

$$F(-) \cong \mathrm{Hom}_{\text{local } \Lambda\text{-alg.}}(R, -)$$

for some complete local Λ-alg. $R \supset \mathfrak{m}$ (max. ideal) s.t. $R/\mathfrak{m}^n \in C$ $\forall n$

A couple $(R, \xi \in F(R))$ is called a hull of F if

$h_R \to F$ induced by $f \in \mathrm{Hom}(R, A) \to f_*(\xi)$

is smooth (define later), and $h_R(k[\varepsilon]) \xrightarrow{\cong} F(k[\varepsilon])$ $(\cong t_F$ tangent space of $F)$

Thm. (Schlessinger) Let $F: C \to Sets$ with $F(k) = \{e\}$ a point, consider $\star$ $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ and ①

(H_1) $\star$ is surj. as $A'' \to A$ is small ($\ker(A'' \to A) = (t)$ with $\mathfrak{m}_{A''} \cdot t = (0)$)

(H_2) $\star$ is bij. as $A = k$, $A'' = k[\varepsilon]$

(H_3) $\dim_k(t_F) < \infty$ (the v.s. structure define later)

(H_4) $\star$ is bij. as $A' = A'' \to A$ is small.

Then $(H_1)(H_2)(H_3) \Leftrightarrow$ F has a hull

$(H_1)(H_2)(H_3)(H_4) \Leftrightarrow$ F is prorepresentable

Smooth. A morphism of functor $F \to G$ is smooth if

$$F(B) \to F(A) \times_{G(A)} G(B) \quad \text{surj.}$$

for any surjection $B \to A$ in C

Prop. If $R \to S$ in $\hat{C}$, then $h_S \to h_R$ is smooth iff S is a power series ring / R

Lemma. Denote $t^*_{A/\Lambda} = \mathfrak{m}/(\mathfrak{m}^2 + \mu A)$ the Zariski cotangent space then $B \to A$ in $\hat{C}$ is surj. iff $t^*_{B/\Lambda} \to t^*_{A/\Lambda}$ is surj.

① : $A' \times_A A'' := \{(a', a'') \in A' \times A'' \mid (A' \to A)a' = (A'' \to A)a''\}$

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Pf. Consider the diagram

$$\begin{array}{ccccccc} 0 \to & \mu A/(\mu A \cap m^2) & \to & m/m^2 & \to & t^*_A & \to 0 \quad \text{exact} \\ & \uparrow & & \uparrow & & \uparrow & (m = m_A,\ n = m_B) \\ 0 \to & \mu B/(\mu B \cap n) & \to & n/n^2 & \to & t^*_B & \to 0 \quad \text{exact} \end{array}$$

Since A is generated by image of Λ and m,

$\mu/\mu^2 \to \mu A/(\mu A \cap m^2)$ is surj. (Since $\Lambda/\mu = A/m = k$)

hence the left $\uparrow$ of the diagram is surj.

if right $\uparrow$ is also surj., then

$n/n^2 \to m/m^2$ is surj.

$\| \qquad\qquad \| \qquad\qquad \|$

$An/mn \overset{\text{Nakayama}}{\Rightarrow} A_n \to m$ surj.

$\Rightarrow B/n \to A/A_n$ surj. $(B/n = A/m = k)$

$\overset{\text{Nakayama}}{\Rightarrow} B \to A$ surj.

Now suppose $h_S \to h_R$ is smooth $S \supset R \supset r$ max ideal

then choose $x_1, \cdots, x_n \in S$ basis of $t^*_{S/R} = S/(S^2 + rS)$

set $T = R[[x_1, \cdots, x_n]]$, then we have homomorphism

$u_1 : S \to t^*_{T/R} = T/(t^2 + rT)$

by smoothness this can be lift to $u_2 : S \to T/t^2$ (R-alg.)

hence lift to $u_3 : S \to T/t^3$, etc.

thus we get $u : S \to T$ which induces $t^*_{S/R} \cong t^*_{T/R}$

$\Rightarrow u$ is surj. by lemma

Choose $y_1, \cdots, y_n$ s.t. $uy_i = x_i$, set $v : T \to S$ R-alg.

s.t. $vx_i = y_i$, then $uv = 1_T \Rightarrow v$ induce $t^*_{T/R} \cong t^*_{S/R}$

$\Rightarrow v$ is surj., in summary $S \cong T = R[[x_1, \cdots, x_n]]$

over R

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vector space structure of t_F:

Prop. If $F(k[\varepsilon]) \times_k k[\varepsilon]) \xrightarrow{\sim} F(k[\varepsilon]) \times F(k[\varepsilon])$

then $t_F = F(k[\varepsilon])$ has k-v.s. structure

pf. The addition on t_F is given by

$$k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon], \quad \begin{matrix} (\varepsilon, 0) \to \varepsilon \\ (0, \varepsilon) \to \varepsilon \end{matrix}$$

The scalar multiplication

is given by the endomorphism $k[\varepsilon] \to k[\varepsilon] \quad \varepsilon \to a\varepsilon$

(apply F)

Now t_F has v.s.$_F$ structure if (H_2) holds

Also we define $t_R = t_{h_R} = \mathrm{Hom}(R, k[\varepsilon])$

Proof of Thm.

$(H_1)(H_2)(H_3) \Rightarrow$ hull:

Let $t_1, \dots, t_r$ be a dual basis of t_F (H_3)

$S = \Lambda[[T_1, \dots, T_r]]$ with max ideal m_S

$R_2 \cong S/(m_S^2 + (f)S) \cong k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$ $\quad t_F \otimes t_F^* = t_F \otimes t_F$

$(H_2) \Rightarrow F(R_2) \xrightarrow{\sim} F(k[\varepsilon]) \times \cdots \times F(k[\varepsilon]) \cong t_F \times \cdots \times t_F \cong \mathrm{End}(t_F)$

Let $\xi_2 \in F(R_2)$ be the element corr. to $id \in \mathrm{End}(t_F)$ $\quad \xi_2 \leftrightarrow (t_i^*)$

then $\forall f_i \in t_{R_2} = \mathrm{Hom}(R_2, k[\varepsilon])$ send to the ith comp.

$(f_i)_*(\xi_2) = t_i^*$

so ξ_2 induces $t_{R_2} \cong t_F$

Suppose we've found (R_q, ξ_q) with $R_q \cong S/J_q$ (J_q: ideal in S)

we choose J_{q+1} minimal among ideal J s.t. $m_S J_q \subset J \subset J_q$

Note that J_q satisfies such condition, and ξ_q lift to S/J (so nonempty)

These J are stable under pairwise $\cap$:

If J, K are two such ideals, enlarge J so that

$J + K = J_q$ and $J \cap K$ unchange, then

$S/J \times_{S/J_q} S/K \cong S/J \cap K$ (since $S/K \twoheadrightarrow S/J_q$)

By (H_1), $F(S/J \cap K) \to F(S/J) \times_{F(S/J_q)} F(S/K)$ is surj.

so ξ_q can be lifted to $F(S/J \cap K)$.

Therefore we have a unique choice of J_{q+1}

Now let $J = \bigcap_q J_q$ and $R = S/J$

we have $t_R \cong t_{R_2} \xrightarrow{\sim} t_F$

remains to show $h_R \to F$ is smooth

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$h_R \to F$ is smooth $\Leftrightarrow$ For any $A' \xrightarrow{p} A$ small extension

$h_R(A') \to F(A') \times_{F(A)} h_R(A)$ is surj.

i.e., $\forall \eta \in F(A)$ $f: R \to A$ with $F(f)(\xi) = \eta$ can be lifted to

$\exists \eta' \in F(A')$ $f: R \to A'$ with $F(f)(\xi) = \eta'$

Observation:

$\forall A \in \mathcal{C}$, $I \subset A$ with $m_A \cdot I = (0)$, we have

$A \times_{A/I} A \xrightarrow{\simeq} A \times_k (k \oplus I)$ ($I^2 = 0$)

$(x, y) \mapsto (x, \bar{x} + y - x)$ ($y - x \in I$; $\bar{x}$ = residue of x)

If $p: A' \to A'/I = A$ is small, then by (H_2),

$F(A' \times_A A') = F(A' \times_k k[I]) = F(A') \times_{F(k)} F(k[\varepsilon]) \otimes I$

so we have a map $F(A') \times t_F \otimes I \to F(A') \times_{F(A)} F(A')$

This defines a gp action of $t_F \otimes I$ on the fiber

$F(p)^{-1}(\eta)$

With (H_1) the map is surj. $\Rightarrow$ the action is transitive

(If (H_4) holds, then the map is bij. $\Rightarrow$ the action is trivial)

(later) is simply transitive)

Now f factors as $(R, \xi) \to (R_q, \xi_q) \to (A, \eta)$ for some q

So we want to complete the diagram (chosen s.t. diagram ☐)

$$\begin{array}{ccc} R_{q+1} & \dashrightarrow & A' \\ \downarrow & & \downarrow p \\ R_q & \longrightarrow & A \end{array} \quad (\Leftrightarrow) \quad \begin{array}{ccc} S & \xrightarrow{w} & R_q \times_A A' \\ \downarrow & & \downarrow pr_1 \leftarrow \text{small} \\ R_{q+1} & \longrightarrow & R_q \end{array}$$

If pr_1 have a section then $\exists$ is obvious

otherwise we show w is surj. ~~Suppose not, then derive~~

~~TCB~~ ~~Hint~~

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Suppose w is not surj., then consider the subring $w(S) \subsetneq R_q \times_A A'$

we have $w(S) \xrightarrow{pr_1} R_q$ surj.

But since $R_q \times_A A' \xrightarrow{pr_1} R_q$ is small,

$$\text{length}(R_q \times_A A') = \text{length}(R_q) + 1$$

$$\Rightarrow \text{length}(w(S)) \leq \text{length}(R_q)$$

Combine these we must have $w(S) \cong R_q$, hence $w(S)$ is a section $\star$

Now by (H_1), $F(R_q \times_A A') \to F(R_q) \times_{F(A)} F(A')$ is surj.

So $\xi_q \in F(R_q)$ lifts to $F(R_q \times_A A')$

hence $J_{q+1} \subset \ker w$ by minimality

(Note that $\ker w \supseteq m J_q$ since $m \ker p = (0)$)

therefore there exists $R_{q+1} \longrightarrow R_q \times_A A'$ section.

From above, we can always lift $f : R \to A$ to $f' : R \to A'$

By Observation, we can replace f' by $g' = \sigma \cdot f'$

for some $\sigma \in t_F \otimes I$, s.t. $F(f')(\xi) = \eta'$

(Note that $F(p) \circ F(f')(\xi) = F(f)(\xi) = \eta$

$\Rightarrow F(f')(\xi)$ is in the fiber $F(p)^{-1}(\eta)$)

thus we're done.

$(H_1) \sim (H_4) \Rightarrow$ pro-representable: Induction on length of A

If $h_R(A) \cong F(A)$ true for A,

then $h_R(A') \longrightarrow F(A')$ is surj. by smoothness

and on fibers $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$

$t_F \otimes I$ act simply transitively on both

so in fact $h_R(A') \cong F(A')$

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Deformation of Schemes, Formal moduli

X scheme / k locally of finite type, $A \in C$

a deformation of X/k to A is a product diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

i closed immersion

$X \cong Y \times_{\operatorname{Spec} A} \operatorname{Spec} k$

with Y flat / A

$D_{X/k}$

$D(A)$ = Set of isom. classes of def. of X/k to A

Prop. If X is proper / k (and $H^0(X, T_X) = 0$)

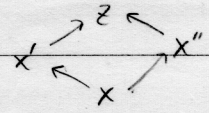
then $D_{X/k}$ has a hull (is prorepresentable, resp.ly)

Proof (Sketch) (H_1): Given $X' \leftarrow^{u'} Y \xrightarrow{u''} X''$ deformations of X

corr. to $(A', \eta') \to (A, \eta) \leftarrow (A'', \eta'')$ with $A'' \to A$ surj.

then consider the scheme given by $(|X|, \mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''})$

(H_2): Show the uniqueness of the deformation Z over $A' \times_k A''$



(The canonical morphism $Z \to X' \times_X X''$ is an isom.)

(H_3): There is a sequence

$0 \to H^1(X, T_X) \to t_D \to H^0(X, T^1_X) \to \dots$

($T^1_X = 0$ if X smooth) where $T^1_X = T^1(X/k, \mathcal{O}_X)$ is some coh. sheave / X

(H_4): Note that $D_\eta(A' \times_A A') \xrightarrow{\sim} D_\eta(A') \times D_\eta(A')$

(D_η denotes those elements sent to $\eta \in D(A)$)

iff every autom. of the def. $Y = Y' \otimes_A A$

is induced from an autom. of the def. Y'

IMAGE

(1)

S/J

Prop. Suppose $R = k[[x_1, \dots x_r]]/J$ with $J \subset m_S^2$

then $T^2 = (J/mJ)^\vee$ s.t. there is an exact seq.

$$D(A') \xrightarrow{D(\phi)} D(A) \xrightarrow{\delta} T^2 \otimes I \quad , \ I = \ker \phi$$

as $A' \xrightarrow{\phi} A$ small extension

moreover δ is functorial, i.e., if we have the $\square$ diagram of small extension

$$\begin{array}{ccc} A' & \xrightarrow{\phi} & A \\ \downarrow \beta' & & \downarrow \beta \\ B' & \xrightarrow{\phi'} & B \end{array}$$

then there is a $\square$ diagram

$$\begin{array}{ccccc} D(A') & \xrightarrow{D(\phi)} & D(A) & \xrightarrow{\delta} & T^2 \otimes I \\ \downarrow D(\beta') & & \downarrow D(\beta) & & \downarrow 1 \otimes \beta' \\ D(B') & \xrightarrow{D(\phi')} & D(B) & \xrightarrow{\delta'} & T^2 \otimes J' \end{array}$$

Proof. Given $f \in D(A) = \mathrm{Hom}(S/J, A)$, let $f(x_i) = a_i \in A$,

$a_i' \in A'$ any lifting of a_i, then $f' : S \to A'$ given by

$f'(x_i) = a_i'$ is a lifting of f,

- $\forall \beta \in J$, $f'(\beta) \in I$
- $\forall \beta \in mJ$, $f'(\beta) = 0$ by smallness

so f' induces $J/mJ \to I$, i.e. an element of $T^2 \otimes I$ $(=: \delta(f))$

$\delta(f)$ does not depend on choice of f', and

$f' : S \to A'$ factor through S/J iff $\delta(f) = 0$

hence the seq is exact.

(2)

T^1-lifting property

$D: C \to Sets$, C cat. of local artin. alg. $/k$ (of char 0) covariant with $D(k)$ one element and satisfies $(H_1) \sim (H_4)$, hence pro-representable by R

$A_n := k[t]/t^{n+1}$, $B_n := k[x,y]/(x^{n+1}, y^2)$, $\alpha_n : A_{n+1} \to A_n$

$\beta_n : B_n \to A_n$ given by $\beta_n(x) = t$, $\beta_n(y) = 0$

$\forall\ X_n \in D(A_n)$ define $T^1(X_n/A_n) = \{Y_n \in D(B_n) \mid D(\beta_n) Y_n = X_n\}$

Thm. If the following T^1-lifting property holds:

$\forall n\ \forall\ X_{n+1} \in D(A_{n+1})$, the natural map

$T^1(\alpha_n) : T^1(X_{n+1}/A_{n+1}) \to T^1(X_n/A_n)$

is surj. ($X_n = D(\alpha_n) X_{n+1}$), then D is unobstructed

(R is a power series)

Proof: We only have to show $D(A_{n+1}) \xrightarrow{D(\alpha_n)} D(A_n)$ surj.:

If we write $R = k[[x_1, \dots, x_r]]/I$ with $I \subset m^2$

m max. ideal of $k[[x_1, \dots, x_r]]$

$f = f_d + f_{d+1} \dots \in I$ nonzero, then choose $a_1, \dots, a_r \in k$ s.t. $f_d(a_1, \dots, a_r) \neq 0$, and consider $R \to k(t)/t^2$ given by $x_i \mapsto a_i t$, then this can be extended to $R \to k(t)/t^n$, hence $R \to k[[t]]$, but then $f \mapsto t^d f_d(a_1, \dots, a_r) + \dots \neq 0$ $\lightning$, thus $I = 0$, and R is a power series

IMAGE

③

Define $C_n = k[x,y]/(x^{n+1}, x^n y, y^2)$ $(\cong B_{n-1} \times_{A_{n-1}} A_n)$

$\gamma_n : B_n \to C_n$ $\quad \epsilon_n : A_{n+1} \to B_n$ by $t \mapsto x+y$

$\delta_n : B_n \to B_{n-1}$ $\quad \epsilon'_n : A_n \to C_n$ by $t \mapsto x+y$

Consider the following diagram

$$\begin{array}{ccccc} D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) & \xrightarrow{\delta_1} & T^2 \otimes (t^{n+1}) \\ \downarrow D(\epsilon_n) & & \downarrow D(\epsilon'_n) & & \downarrow \varphi \\ D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) & \xrightarrow{\delta_2} & T^2 \otimes (x^n y) \end{array}$$

φ is an isom. sending t^{n+1} to $(n+1)x^n y$

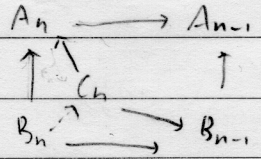
To show that $D(\alpha_n)$ surj., or $\delta_1 = 0$,

it suffices to show $\delta_2 = 0$, or $D(\gamma_n)$ surj.

Let $z_n \in D(C_n)$

x_n, x_{n-1}, y_{n-1} the image of z_n

in A_n, A_{n-1}, B_{n-1} resply.



then by T^1 lifting property,

y_{n-1} lift to $y_n \in T^1(x_n / A_n) \subset D(B_n)$

Note that $C_n \cong B_{n-1} \times_{A_{n-1}} A_n$, so $D(\gamma_n)(y_n) = z_n$

Hence $D(\gamma_n)$ is surj. ∎

IMAGE

char 0

6.

(Deligne - Illusie) Let S be a k-scheme and $f: X \to S$ smooth proper, then $R^q f_* \Omega^p_{X/S}$ is locally free and its formation commutes with base change

Proof. We only deal with the case $S = \operatorname{spec} A$ with A artinian over $k = \mathbb{C}$

(We can always reduce to this case)

The proof breaks into two step:

① length $H^q(X, \Omega^p_{X/A}) \leq$ length (A) length $(H^q(X_{red}, \Omega^p_{X_{red}}))$ with equality iff $H^q(X, \Omega^p_{X/S})$ is free

② $\sum_{p+q=n}$ length $H^q(X, \Omega^p_{X/A}) \geq$ length $\mathbb{H}^n(X, \Omega^\cdot_{X/A})$ $(= i H^n_{dR}(X/A))$ with equality holds iff the Hodge - de Rham spec. seq. $H^q(X, \Omega^p_{X/A}) \Rightarrow H^{p+q}_{dR}(X/A)$ degenerates

Combine these we have

$\sum_{p+q=n}$ length (A) length $H^q(X_{red}, \Omega^p_{X_{red}})$

$\geq$ length $\mathbb{H}^n(X, \Omega^\cdot_{X/A}) =$ length (A) length $(H^n_{dR}(X_{red}))$

with equality iff the conditions of ①, ② holds

but we have the equality by usual Hodge theory.

IMAGE

Application to Calabi-Yau mfld

Thm. Let X be a smooth var. with $k_X = \mathcal{O}_X$ and $H^0(X, T_X) = 0$ / $\mathbb{C}$
then X has unobstructed deformation

Note that $h^0(X, T_X) = h^n(X, \Omega^1) = h^1(X, \Omega^n) = h^1(X, \mathcal{O}_X) = 0$
if X is Calabi-Yau. (↑ hodge sym.)

Proof. Viewing $T^1(X_n/A_n)$ as first order deformation
of X_n over A_n, we have

$$T^1(X_A/A) \cong H^1(X_A, T_{X_A/A}), \quad A \in C, \ T_A \in D(C)$$

Note that $T_{X_A/A} \cong (\Omega^1_{X_A/A})^\vee = (\Omega^1_{X_A/A})^\vee \otimes \omega_{X/A} = \Omega^{n-1}_{X_A/A}$

here $\omega_{X/A}$ is trivial since $\omega_{X/k}$ is trivial
and the nonvanishing section of $\omega_{X/k}$ lifts to $\omega_{X/A}$
by Deligne-Illusie ($H^0(X_A, \omega_{X_A/A})$ free and surj. to
$H^0(X, \omega_X) = H^0(X_A, \omega_{X_A/A}) \otimes k$)

So we want to check the surjectivity of the natural map

$$H^1(X_{n+1}, \Omega^{n-1}_{X_{n+1}/A_{n+1}}) \to H^1(X_n, \Omega^{n-1}_{X_n/A_n})$$

which is again given by Deligne-Illusie.

IMAGE

Hodge - de Rham spectral seq.

Let $p: X \to S$, there is a complex

$$\Omega^{\cdot}_{X/S} = \mathcal{O}_{X/S} \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \to \cdots$$

with $\Omega^i_{X/S} = \wedge^i(\Omega^1_{X/S})$ and $d(g_0\, dg_1 \wedge \cdots \wedge dg_p) = dg_0 \wedge \cdots dg_p$

called de Rham complex

Given $F: \mathcal{A} \to \mathcal{B}$ left exact functor

$RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined by:

cpx bdd below / quasi-isom

$\forall\, \mathcal{F}^{\cdot} \in D^+(\mathcal{A})$, choose $\mathcal{I}^{\cdot}$ bdd cpx of injectives, then $RF(\mathcal{F}^{\cdot}) = F(\mathcal{I}^{\cdot})$ (algebraic)

Define $H^i_{dR}(X/S) = \mathbb{H}^i(\mathbb{R}\Gamma(X, \Omega^{\cdot}_{X/S}))$ the de-Rham Cohomology

$(= \mathbb{H}^i(X, \Omega^{\cdot}_{X/S}))$

For $S = \operatorname{spec} \mathbb{C}$ we have the resolution

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{X/k} \xrightarrow{=\mathcal{O}_X} \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots \quad \text{(holom.)}$$

so there is a quasi-isom.

$$0 \to \underline{\mathbb{C}} \to 0 \to \cdots \quad \simeq \quad 0 \to \Omega^0_{X/k} \to \Omega^1_{X/k} \to \cdots$$

Thus $H^i_{sing}(X, \mathbb{C}) \cong H^i(X, \underline{\mathbb{C}}) \cong \mathbb{H}^i(X, \Omega^{\cdot}_{X/k}) \simeq H^i_{dR}(X/k)$

by de Rham thm.

Given $\mathcal{I}^\cdot$ bdd below cpx in $\mathcal{A}$, $F: \mathcal{A} \to \mathcal{B}$

$\mathrm{Fil}^\cdot \mathcal{I}^\cdot$ a decreasing filtration; i.e., $\mathrm{Fil}^i \mathcal{I}^\cdot \subseteq \mathcal{I}^\cdot$ s.t.

$\mathrm{Fil}^i \mathcal{I}^p \supseteq \mathrm{Fil}^j \mathcal{I}^p \quad \forall i<j$

moreover $\mathrm{Fil}^i \mathcal{I} = \mathcal{I}^\cdot \ \forall\ i \ll 0$

$\mathrm{Fil}^i \mathcal{I}^\cdot = 0 \quad \forall\ i \gg 0$ (biregular)

$\mathrm{gr}_i \mathrm{Fil}^\cdot \mathcal{I} := \mathrm{Fil}^i \mathcal{I} / \mathrm{Fil}^{i+1} \mathcal{I}$

Also there's a filtration on $\mathbb{R}^l F(\mathcal{I}^\cdot) := h^l(\mathbb{R}F(\mathcal{I}^\cdot))$

given by $\mathrm{Fil}^i \mathbb{R}^l F(\mathcal{I}) := \mathrm{im}(\mathbb{R}^l F(\mathrm{Fil}^i \mathcal{I}^\cdot) \to \mathbb{R}^l F(\mathcal{I}^\cdot))$

which is also biregular

Claim: There is a spectral sequence of filtered cpx

$E_1^{ab} := \mathbb{R}^{a+b} F(\mathrm{gr}_a \mathrm{Fil}^\cdot \mathcal{I}^\cdot) \Rightarrow \mathbb{R}^{a+b} F(\mathcal{I}^\cdot)$

so we know that $E_\infty^{ab} = \mathrm{gr}_a \mathrm{Fil}^\cdot \mathbb{R}^{a+b} F(\mathcal{I}^\cdot)$

Corollary. If F takes values in the category of fin. dim $\mathbb{C}$ v.s.

then $\dim \mathbb{R}^l F(\mathcal{I}^\cdot) = \dim \mathrm{gr}_\cdot \mathrm{Fil}^\cdot \mathbb{R}^l F(\mathcal{I}^\cdot)$

$= \sum_{a+b=l} \dim_k E_\infty^{ab} \leq \sum_{a+b=l} E_1^{ab}$

with $=$ holds iff the s.s. degenerates on the 1st page

(Note that E_∞^{ab} is a subquotient of E_1^{ab})

Now consider the Hodge Filtration

$$\mathrm{Fil}^i \mathcal{I}^j = \begin{cases} 0 & \text{if } j<i \\ \mathcal{I}^i & \text{if } j \geq i \end{cases}$$

only sheaf with index 0

then $\mathrm{gr}_a \mathrm{Fil}^\cdot \mathcal{I}^\cdot = \mathrm{Fil}^a \mathcal{I}^\cdot / \mathrm{Fil}^{a+1} \mathcal{I}^\cdot \simeq \mathcal{I}^a[-a]$

and $\mathbb{R}^{a+b} F(\mathcal{I}^a[-a]) = \mathbb{R}^b F(\mathcal{I}^a) = R^b(\mathcal{I}^a)$

thus the s.s. becomes $E_1^{ab} = R^b F(\mathcal{I}^a) \Rightarrow \mathbb{R}^{a+b} F(\mathcal{I}^\cdot)$

Take $F = \Gamma(X, \cdot)$ and $\mathcal{I}^\cdot = \Omega^\cdot_{X/k}$ and we have

$$H^l_{\text{Hodge}}(X/k) \simeq \bigoplus_{a+b=l} H^b(X, \Omega^a_{X/k}) \simeq H^l_{dR}(X/k)$$

$\Leftrightarrow$ the s.s. degenerates on first page

$\Leftrightarrow \sum_{a+b=l} \dim H^b(X, \Omega^a_{X/k}) = \dim H^l_{dR}(X/k)$ (In general $\geq$)

if X is proper/k (to ensure dim are finite)

$$\begin{array}{ccccccccc}
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \cdots & \to & 0 \to 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \cdots & \to & \mathcal{I}^N \to 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \xrightarrow{d} & 0 & \xrightarrow{d} & \mathcal{I}^1 & \xrightarrow{d} & \cdots & \to & \mathcal{I}^N \to 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \xrightarrow{d} & \mathcal{I}^0 & \xrightarrow{d} & \mathcal{I}^1 & \xrightarrow{d} & \cdots & \to & \mathcal{I}^N \to 0
\end{array}$$

A similar result holds for general basis, e.g. ③ of Deligne - Illusie

logarithmic form, holomorphic log cpx

$Y \subset X$ divisor with normal crossing (locally $\prod_{i=1}^{v} z_i = 0$)

X: smooth, dim n. $\Omega_X^\cdot(\log Y)$ is defined as follow:

$\forall\, U \subset X$ holom. $\Omega_X^p(\log Y)$ consists of holom. p-form on $U \setminus Y$, s.t. w & dw has at most simple pole along Y, with d the exterior derivative.

Thus, $\forall p \in Y$, $\Omega_X^1(\log Y)$ is a free $\mathcal{O}_{X,p}$-module with generators $dz_1/z_1, \dots dz_v/z_v, dz_{v+1}, \dots dz_n$

and also, $\Omega_X^p(\log Y) \cong \bigwedge^p_{\mathcal{O}_X} \Omega_X^1(\log Y)$.

Let $f: X \to S$ holom. to a smooth curve (dim $n+1$)

$T \subset S$ finite set of pts s.t. $f^{-1}(T)$ is a divisor with normal crossing on X, and f smooth on $X \setminus Y$

then $\Omega^1_{X/S}(\log Y) \cong \Omega^1_X(\log Y) / f^*\Omega^1_S(\log T)$ is locally free. and $\Omega^p_{X/S}(\log Y) \cong \bigwedge^p \Omega^1_{X/S}(\log Y)$

$\forall\, p \in Y$ $\Omega^1_{X/S}(\log Y)_p$ is a free $\mathcal{O}_{X,p}$-module with generators $dz_0/z_0, \dots dz_v/z_v, dz_{v+1}, \dots dz_n$

and relation $\sum_{i=0}^{v} e_i\, dz_i/z_i = 0$ if f locally has the form

$$f(z_0, \dots, z_n) = z_0^{e_0} \cdots z_v^{e_v}$$

Deformation of CY with ordinary double pts

Let $f: V \to S$ given by $t = f(x_0, \cdots, x_d) = \sum_{j=0}^{d} x_j^2$

$\mathbb{C}^{d+1}$ $\quad$ $\mathbb{C}$

$\mu: W \to V$ blowup at the center

$g = f \circ \mu$, V_0' strict transform of $V_0 = f^{-1}(0)$

$E = \mu^{-1}(0) \cong \mathbb{P}^d$, $W_0 = V_0' \cup E = g^{-1}(0)$

Lemma 1. $\mathbb{R}\mu_* \Omega^i_{W/S}(\log W_0) \cong \Omega^i_{V/S}$ (on index 0.)

Let $\sigma: S^{(m)} = \mathbb{C} \to S$ given by $\sigma^*(t) = t^m$

$V^{(m)}, f^{(m)}, W^{(m)\#}$ — base change of V, f, W by σ

$W^{(m)}$ normalization of $W^{(m)\#}$

The projection $V^{(m)} \to V$ also denoted by σ

For a singular pt P on X, P has a neighborhood U_P in X_n s.t. $\pi_n: U_P \to \operatorname{Spec} A_n$ is equivalent to (for some m) $f^{(m)}_{A_n}: U_P' \to \operatorname{Spec} A_n$ base changed from $f^{(m)}$ by $\tau: \operatorname{Spec} A_n \to S^{(m)}$ the natural closed immersion

Lemma 2. $\mathbb{R}\mu^{(m)}_{A_n *} \Omega^i_{W^{(m)}_{A_n}/A_n}(\log W_0^{(m)}) \cong \Omega^i_{V^{(m)}_{A_n}/A_n}$

Lemma 3. $\omega_{X_n/A_n} \cong \mathcal{O}_{X_n}(\cdots((\mu^{(m)}_{A_n})^{-1}(U_P'))$

We glue these pieces into a morphism $Y_n \to A_n$

and define $\Omega^i_{Y_n/A_n}(\log Y_0)$ by pulling back

$\Omega^i_{W^{(m)}/S^{(m)}}(\log W_0^{(m)})$ on each P, i.e.

Lemma 3 $\cdots \omega_{Y_n} \cong \mathcal{O}_{Y_n}$

IMAGE

In general, we have

$$T^1(X_n/A_n) \cong \mathrm{Ext}^1(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n})$$

So we shall prove the natural map

$$\mathrm{Ext}^1(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}) \to \mathrm{Ext}^1(\Omega^1_{X_{n-1}/A_{n-1}}, \mathcal{O}_{X_{n-1}})$$

is surj. By Grothendieck duality, and Lemma 3, this is

$$\mathrm{Hom}(H^{d-1}(X_n, \Omega^1_{X_n/A_n}), A_n) \to \mathrm{Hom}(H^{d-1}(X_{n-1}, \Omega^1_{X_{n-1}/A_{n-1}}), A_{n-1})$$

is surj. Similar to the smooth case, we want to show $H^q(Y_n, \Omega^p_{Y_n/A_n}(\log Y_0))$ is A_n-free and commute with base change. (then by Lemma 2)

We need two more results:

Lemma 4. $H^i(\Omega^\cdot_{Y_n/A_n}(\log Y_0)) \cong H^i(\Omega^\cdot_{Y_n/A_n}(\log Y_0) \otimes \mathcal{O}_{Y_0}) \otimes_{\mathbb{C}} A_n$

Lemma 5. The specteral sequence

$$E_1^{p,q} = H^q(Y_0, \Omega^p_{Y_n/A_n}(\log Y_0) \otimes \mathcal{O}_{Y_0}) \Rightarrow H^{p+q}(\,\cdot\,,\cdot\,)$$

degenerate at first page.

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Proof of Lemma 1.

We want to show that

$$R^i \mu_* \Omega^1_{W/S}(\log W_0) = \Omega^1_{V/S} \quad i=0$$
$$= 0 \quad \text{o.w.}$$

Consider $0 \to \mathcal{O}_V \xrightarrow{d} \Omega^1_V \to \Omega^1_{V/S} \to 0$

and $0 \to \mathcal{O}_W \xrightarrow{d} \Omega^1_W(\log W_0) \to \Omega^1_{W/S}(\log W_0) \to 0$

Take μ_* on the second sequence, and note that

- $R^i \mu_* \mathcal{O}_W = \mathcal{O}_V \quad i=0$
  $= 0 \quad \text{o.w.}$
- sections of Ω^1_V can be lifted to $\Omega^1_W(\log W_0)$
- $\Omega^1_{V/S}$ and $\Omega^1_{W/S}(\log W_0)$ are locally free

We obtain $\mu_* \Omega^1_{W/S}(\log W_0) = \Omega^1_{V/S}$

and $R^i \mu_* \Omega^1_W(\log W_0) \cong R^i \mu_* \Omega^1_{W/S}(\log W_0)$

Consider $0 \to \Omega^1_W(\log E) \to \Omega^1_W(\log W_0) \xrightarrow{res} \mathcal{O}_{V_0'} \to 0$

again $R^i \mu_* \mathcal{O}_{V_0'} = 0$, $i>0$, so it suffices to show

$R^i \mu_* \Omega^1_W(\log E) = 0$, $i>0$

Let H_j, $j=0,\dots,d$ be the coordinate hyperplanes on V

$H = \bigcup H_j$, $H' = \bigcup_{j=0}^d H_j'$ the strict transform

Consider $0 \to \Omega^1_V \to \Omega^1_V(\log H) \xrightarrow{res} \bigoplus_{j=0}^d \mathcal{O}_{H_j} \to 0$

and $0 \to \Omega^1_W(\log E) \to \Omega^1_W(\log(H' \cup E)) \xrightarrow{res} \bigoplus_{j=0}^d \mathcal{O}_{H_j'} \to 0$

and note that $\Omega^1_W(\log(H' \cup E)) = \mu^* \Omega^1_V(\log H)$.

FROM IMAGE