Toric Construction of Mirror Manifolds (GTFT Final Report)

— Part I : Reflexive polytopes and Fano varietes (12/30)

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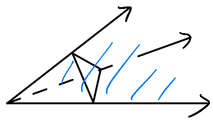
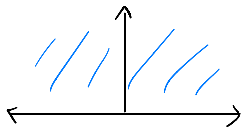
⓪ Definitions and basic properties.

Fix a lattice $N \simeq \mathbb{Z}^n$ and its dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$.

· A convex polyhedral cone (c.p.c.) σ is a set $\mathbb{R}_{\geq 0} \cdot v_1 + \cdots + \mathbb{R}_{\geq 0} \cdot v_s$ for some $v_1, \cdots, v_s \in N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.





The dual $\sigma^\vee$ of σ is $\{ u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \ \forall v \in \sigma \}$.

Fundamental fact : $(\sigma^\vee)^\vee = \sigma$.

· A face τ of σ is $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$. $\longrightarrow \tau \prec \sigma$.

(facet = face of codim 1)

· σ is rational if its generators $\in N$.

σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$

Facts (1) σ rational c.p.c. $\Rightarrow S_\sigma := \sigma^\vee \cap M$ f.g. semigroup.

(2) σ rat'l c.p.c. and $\tau = \sigma \cap u^\perp \Rightarrow \tau$ rat'l c.p.c. with $S_\tau = S_\sigma - \mathbb{Z}_{\geq 0} \cdot u$.

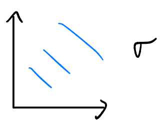
Therefore, $\underset{\displaystyle U_\tau}{\underset{\shortparallel}{\mathrm{Spec}\,\mathbb{C}[S_\tau]}} \hookrightarrow \underset{\displaystyle U_\sigma}{\underset{\shortparallel}{\mathrm{Spec}\,\mathbb{C}[S_\sigma]}}$ embeds U_τ as $D(u) \subset U_\sigma$.

· A fan Σ in N is a set of cones (rational strongly convex polyhedral cones)

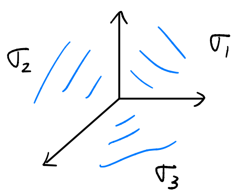
s.t (1) $\tau \in \Sigma$ if $\tau \prec \sigma \in \Sigma$

(2) $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$ if $\sigma, \sigma' \in \Sigma$.

$\longrightarrow$ Toric variety X_Σ by gluing along common faces.



$\mathbb{C}^2$



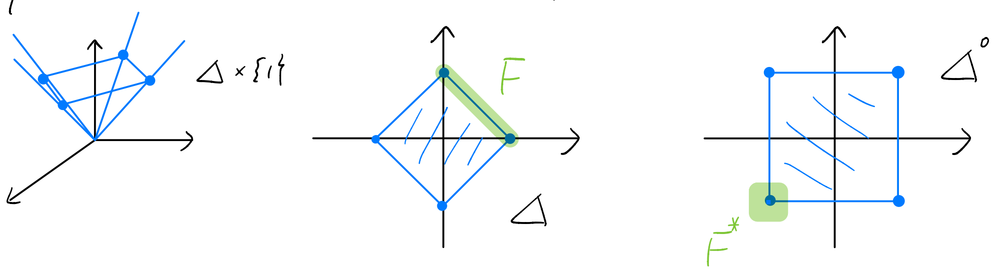
$\mathbb{P}^2$.

⓪ Polytopes

· A (rational convex) polytope Δ in $M_{\mathbb{R}}$ is the convex hull of a finite of points in M. Assume $0 \in \operatorname{int}\Delta$ and consider the cone over $\Delta \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$ $\longrightarrow$ similar definitions and properties :

A (proper) face F of Δ is $\{ u \in \Delta \mid \langle u, v\rangle = r \}$ for some v, r with $\langle u, v \rangle \geq r \quad \forall\, u \in \Delta$.

The polar / dual Δ° of Δ is $\{ v \in N_{\mathbb{R}} \mid \langle u, v\rangle \geq -1 \quad \forall u \in \Delta \}$.



· Assume $\dim \Delta = n$. The normal fan $\Sigma(\Delta)$ of Δ is $\{ \sigma_F \mid F < \Delta \}$ where $\sigma_F := \{ v \in N_{\mathbb{R}} \mid \langle u' - u, v \rangle \geq 0 \quad \forall u \in F, u' \in \Delta \}$.

$$\Updownarrow$$
$$\langle u, v\rangle \leq \langle u', v\rangle$$

If $0 \in \operatorname{int}\Delta$, σ_F is the cone over $F^* = \{ v \in \Delta^\circ \mid \langle u, v \rangle = -1 \quad \forall u \in F \}$.

$\longrightarrow X_{\Sigma(\Delta)} = X_{\Delta^\circ}$.

<u>Def</u> A n-dim'l polytope Δ is reflexive if

(1) All facets Γ of Δ are of the form $\{ u \in \Delta \mid \langle u, \underset{\in N}{v_\Gamma} \rangle = -1 \}$

(2) $\operatorname{int}\Delta \cap M = \{0\}$.

<u>Lemma</u> Δ reflexive $\Rightarrow$ Δ° reflexive.

pf: · $\Delta = \{ u \in M_{\mathbb{R}} \mid \langle u, v_\Gamma \rangle \geq -1 \}$ $\longrightarrow$ vertices of $\Delta^\circ = \{ \underset{\in N}{v_\Gamma} \}$

· facets of $\Delta^\circ = \{ v \in \Delta^\circ \mid \langle v, p \rangle = -1 \}$ for p vertex of Δ

· $\operatorname{int}\Delta^\circ \cap N = \{0\}$ (For $v \in N$ s.t. $\langle v, p\rangle > -1 \quad \forall p$ vertex, $\langle v, p \rangle = 0$)

$\square$

⓪ T-Divisors.

- $\{0\} \subset \sigma \longrightarrow T = T_N = (\mathbb{C}^\times)^n \hookrightarrow U_\sigma \longrightarrow T$-action on X_Σ

Let $\tau_1, \cdots, \tau_d$ be the edges of $X_\Sigma \longrightarrow D_i :=$ orbit closure corr. to τ_i.

T-Weil divisor : $\sum a_i D_i$ e.g. $D_i|_{U_{\tau_i}} = \{0\} \times (\mathbb{C}^\times)^{n-1}$

(Orbit closure $V(\tau)$:

Let N_τ be the lattice gen. by $N \cap \tau$ and $\bar{\sigma}$ = the image of σ via $N \longrightarrow N/N_\tau$.

Then $V(\tau) := X_{Star(\tau)}$ where $Star(\tau) = \{\bar{\sigma} \mid \tau < \sigma, \sigma \in \Sigma\}$)

- T-Cartier divisor : $(M(\sigma) := \sigma^\perp \cap M)$

On U_σ, T-Cartier divisors $= div(X^u)$ for some $u \in M/M(\sigma)$ and

$[div(X^u)] = \sum \langle u, v_i \rangle D_i$ where v_i is the first lattice pt on τ_i.

(D. T-inv. $\Rightarrow \Gamma(U_\sigma, \mathcal{O}(D)) = \oplus\, \mathbb{C} \cdot X^u$ for some $u \in M$.

locally principal $\Rightarrow I = \mathbb{C}[S_\sigma] \cdot X^u$ for some $u \longrightarrow D = div(X^u)$.

Further, $div(X^u) = div(X^{u'}) \Leftrightarrow u - u' \in M(\sigma)$

and $ord_{D_i}(div(X^u)) = \langle u, v \rangle$)

On X_Σ, T-Cartier divisors $= \{ div(X^{-u(\sigma)}) \text{ or } u(\sigma) \mid \text{agree on overlaps} \}_{\sigma:max.}$

- For a T-Cartier divisors $D = \{u(\sigma)\} = \sum a_i D_i$, we have

the support function of D : $\psi_D(v) = \langle u(\sigma), v \rangle$ for $v \in \sigma$

Note $\Gamma(U_\sigma, \mathcal{O}(D)) = \bigoplus_{u \in P_D(\sigma)} \mathbb{C} \cdot X^u$ with $P_D(\sigma) = \{ u \in M \mid \langle u, v_i \rangle \geq -a_i \ \forall v_i \in \sigma \}$

$\longrightarrow \Gamma(X_\Sigma, \mathcal{O}(D)) = \cap \Gamma(U_\sigma, \mathcal{O}(D)) = \bigoplus_{u \in P_D} \mathbb{C} \cdot X^u$

with $P_D = \{ u \in M \mid \langle u, v_i \rangle \geq -a_i \ \forall i \}$ $\quad u \geq \psi_D$

Now, assume $\bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$.

Prop D is g.b.g.s. $\Leftrightarrow \psi_D$ is convex.

pf: They are equivalent to $[\langle u(\sigma), v_i \rangle \geq -a_i \ \forall i$ with $v_i \in \sigma \Rightarrow$ "$=$" . $]$

□

Prop D is ample $\Leftrightarrow$ ψ_D is strictly convex i.e. $u(\sigma) = u(\sigma')$ if $\sigma \neq \sigma'$
i.e. $\langle u(\sigma), v_i \rangle > -a_i$ for $v_i \notin \sigma$.

pf: (kD v.a. $\Leftrightarrow$ $X_\Sigma \xrightarrow{\varphi} \mathbb{P}^{|P_{kD}|-1}$, $x \longmapsto (X^u(x))_{u \in P_{kD}}$ is an immersion.)

"$\Rightarrow$" May assume D v.a. by $\psi_{kD} = k\psi_D$. Suppose $u(\sigma) = u(\sigma')$.

Note $\varphi|_{U_\sigma} : U_\sigma \longrightarrow \{ u(\sigma)\text{-th coordinate} \neq 0 \}$ ($X^{u(\sigma)}$ gen. $\mathcal{O}(D)|_{U_\sigma}$)

$\longrightarrow \varphi|_{U_\sigma \cup U_{\sigma'}} : U_\sigma \cup U_{\sigma'} \longrightarrow \{ \quad \}$ $\times$

"$\Leftarrow$" $\varphi|_{U_\sigma} : U_\sigma \longrightarrow \{ u(\sigma)\text{-th coordinate} \neq 0 \}$ is given by $(X^{u-u(\sigma)})_{u \in P_{kD} \setminus \{u(\sigma)\}}$

$\longrightarrow$ closed embedding if S_σ is gen. by $P_{kD} - u(\sigma)$, which holds for large k. $\square$

⊙ Fano toric varieties.

Def X_Σ is (Gorenstein) Fano if the anti-canonical divisor $-K_{X_\Sigma}$ is Cartier and ample.

rmk All toric varieties are normal and Cohen-Macaulay.

Lemma $K_X = -\sum D_i$.

pf: Since X_Σ is normal, we may assume X_Σ is smooth.

$\longrightarrow U_\sigma = \mathbb{C}[X_1, \ldots, X_k, X_{k+1}^{\pm}, \ldots, X_n^{\pm}]$ (σ is gen. by part of a basis of N)

$\longrightarrow \mathrm{div}(\omega)$ and $-\sum D_i$ are the same on U_σ.

$\omega = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_n}{X_n}$ (up to ± 1 dep. on the choice of coordinate) $\square$

Prop Δ reflexive $\Leftrightarrow$ $X_{\Sigma(\Delta)}$ Fano.

pf: $X_{\Sigma(\Delta)} = X_{\Delta^\circ}$ and facets of Δ° $= \{ v \in \Delta^\circ \mid \langle v, v_i \rangle = -1 \}$

$\updownarrow$
max. cone.

$\square$

— Part 2: Calabi-Yau hypersurfaces and MPCP desingularization. (1/6)

⊙ Resolution of singularities and projective subdivisions.

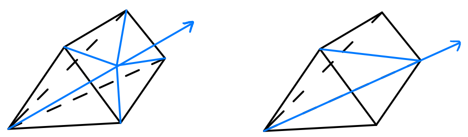
· Let Σ' be a refinement of Σ i.e. $\bigcup_{\sigma\in\Sigma}\sigma = \bigcup_{\sigma'\in\Sigma'}\sigma'$ and $\sigma = \bigcup(\text{some } \sigma'\in\Sigma')$.

$\rightsquigarrow X_{\Sigma'} \longrightarrow X_\Sigma$ is birational and proper.

iso. on T_N

Prop $\exists$ refinement $\tilde{\Sigma}$ s.t. $X_{\tilde{\Sigma}} \longrightarrow X_\Sigma$ resol. of sing.

pf: 1° Add vectors and cones s.t. each cone is gen. by lin. indep. vectors.



2° Add vectors s.t. $N_\sigma = \sum_{v:\text{ edge of }\sigma}\mathbb{Z}\cdot v$

(Adding $\overset{0}{\underset{v}{\neq}} \in N_\sigma / \sum_{v:\text{ edge of }\sigma}\mathbb{Z}\cdot v$ decreases the index.)

Then $\tilde{\Sigma}$ created by 1°, 2° is non-singular. □

Lemma Gorenstein ⇒ canonical sing.

pf: $K_\Sigma = -\sum_{v_i\in\Sigma(1)} D_i$ is Cartier ⇒ $\exists u\in M$ s.t. $\langle u, v_i\rangle = 1$.

Let $X_{\tilde{\Sigma}} \xrightarrow{\varphi} X_\Sigma$ resol. of sing.

$\rightsquigarrow K_{\tilde{\Sigma}} - \varphi^* K_\Sigma = \sum_{v_i\in\tilde{\Sigma}(1)} (-1+\psi(v_i))D_i = \sum_{v_i\in\tilde{\Sigma}(1)\setminus\Sigma(1)} \overset{\geq 0}{(\psi(v_i)-1)}D_i$ □

supp. func. of $K_\Sigma/\varphi^*K_\Sigma$

Def Let $\Delta\subset M_{\mathbb{R}}$ reflexive. A fan in Σ in $N_{\mathbb{R}}$ is a projective subdivision if

(1) Σ refines $\Sigma(\Delta)$.

(2) $\Sigma(1) := \{\text{1-dim'l cones in } \Sigma\} \subset \Delta^\circ \cap N\setminus\{0\}$

(3) X_Σ is projective and simplicial (i.e. every cone in Σ can be gen. by lin. indep. vect)

Further, Σ is a maximal projective subdivision if $\Sigma(1) = \Delta^\circ\cap N\setminus\{0\}$.

Fact Max. proj. subdivisions exist.

⊙ Calabi-Yau hypersurfaces

Def A Calabi-Yau variety is an n-dim'l normal compact variety V s.t.

(1) V has at most Gorenstein canonical singularieties.

(2) $\hat{\Omega}^d_V = \mathcal{O}_V$

(3) $H^1(V, \mathcal{O}_V) = \cdots = H^{d-1}(V, \mathcal{O}_V) = 0$

Further, V is a minimal Calabi-Yau variety if it has at most Gorenstein $\mathbb{Q}$-factorial terminal singularieties.

Prop Let Δ. reflexive of dim n. Then general $V_\Delta \in |-K_\Delta|$ is CY of dim $(n-1)$, where $-K_\Delta = -K_{X_{\Sigma(\Delta)}}$.

Further, if Σ. proj. subdivision, (i) general $V \in |-K_\Sigma|$ is a CY orbifold.

(ii) Σ. max. $\Rightarrow$ V. min.

↘ MPCP-desingularization of V_Δ

pf: · V_Δ. CY:

(1) $X_{\Sigma(\Delta)}$ Fano $\Rightarrow$ at most canonical sing. $\xrightarrow[\text{Reid}]{\text{Bertini}}$ So does V_Δ.

(2) Adjunction formula $\Rightarrow$ $\hat{\Omega}^{n-1}_{V_\Delta} \simeq \hat{\Omega}^n_{X_{\Sigma(\Delta)}}(-K_\Delta) \otimes \mathcal{O}_{V_\Delta} \simeq \mathcal{O}_{V_\Delta}$

↖ Cartier

(3) Consider $0 \to \hat{\Omega}^n_{X_{\Sigma(\Delta)}} \to \mathcal{O}_{X_{\Sigma(\Delta)}} \to \mathcal{O}_{V_\Delta} \to 0$

$$\longrightarrow \quad \cdots \to \underset{\substack{\| \\ 0 \text{ for } k>0}}{H^k(\mathcal{O}_{X_{\Sigma(\Delta)}})} \to H^k(\mathcal{O}_{V_\Delta}) \to \underset{\substack{\| \text{ Serre} \\ H^{n-(k+1)}(\mathcal{O}_{X_{\Sigma(\Delta)}})^* = 0 \text{ for } k<n-1}}{H^{k+1}(\hat{\Omega}^n_{X_{\Sigma(\Delta)}})} \to \cdots$$

· V. CY: simplicial $\Rightarrow$ orbifold.

X_Σ. Gorenstein since $-K_\Sigma = \sum_{v_i \in \Sigma(1)} D_i$ and $\Sigma(1) \subset \Delta^\circ \cap N \setminus \{0\}$.

(Δ°. reflexive $\Rightarrow$ $\langle m_F, F \rangle = -1$ for some $m_F \in M$.)

The rest of proof is similar.

· Σ. max. $\Rightarrow$ V. min.:

It suffices to show V has terminal sing.

Indeed, $K_{\tilde{\Sigma}} - \varphi^* K_\Sigma = \sum_{v_i \in \tilde{\Sigma}(1) \setminus \Sigma(1)} (\underset{\substack{\vee \\ 0}}{\psi_{K_\Sigma}(v_i) - 1}) D_i \xrightarrow[\text{Reid}]{\text{Bertini}}$ terminal.

□

Prop $(X_\Sigma)_{sing} = \bigcup_{\sigma:\, sing.} V(\sigma)$ and $X_\Sigma \setminus (X_\Sigma)_{sing} = \bigcup_{\sigma:\, sm.} U_\sigma$.

pf: Note σ: smooth $\Rightarrow$ τ: smooth for $\tau \prec \sigma$.

Since $\bigcup_{\sigma:\, sm.} U_\sigma$ is smooth and $\underbrace{X_\Sigma \setminus \bigcup_{\sigma:\, sm.} U_\sigma}_{\shortparallel \ \bigcup_{\sigma:\, sm.} O(\sigma)} = \bigcup_{\sigma:\, sing} O(\sigma) = \bigcup_{\sigma:\, sing} V(\sigma)$,

it suffices to show σ: sing $\Rightarrow$ $O(\sigma)$ sing.

Indeed, γ_σ is sing if σ: sing. $\left(U_{\sigma, N} = \underset{\overset{\Downarrow}{\gamma_\sigma}}{U_{\sigma, N_\sigma}} \times T_{N'} \right)$ $\square$

Lemma $\operatorname{codim} (X_\Sigma)_{sing} \geq 4$

pf: It suffices to show "$\operatorname{codim} V(\tau) < 4 \Rightarrow \tau$ smooth"

i.e. $\tau \in \Sigma(k), k \leq 3$

Let $\tau = \operatorname{cone}(v_1, v_2, v_3)$ with v_1, v_2, v_3 lin. indep.

We will prove v_1, v_2, v_3 gen. N_τ.

Let $v \in \{ a_1 v_1 + a_2 v_2 + a_3 v_3 \mid 0 \leq a_i \leq 1 \} \cap N$, then

$\langle u, v \rangle = -(a_1 + a_2 + a_3) \in \mathbb{Z}$ where $u \in M$ is the vector defining the facet containing v_1, v_2, v_3. $\longrightarrow \langle u, v \rangle = 0, -1, -2, -3$

- $\langle u, v \rangle = 0 \Rightarrow v = 0$
- $\langle u, v \rangle = -1 \overset{\Sigma:\, max.}{\Rightarrow} v = v_1, v_2, v_3$
- $\langle u, v \rangle = -2 \overset{\Sigma:\, max.}{\Rightarrow} (v_1 + v_2 + v_3) - v = v_1, v_2, v_3$ i.e. $v = v_i + v_j$, $i, j = 1, 2, 3$
- $\langle u, v \rangle = -3 \Rightarrow v = v_1 + v_2 + v_3$

In conclusion, $[N_\tau : \mathbb{Z} v_1 + \mathbb{Z} v_2 + \mathbb{Z} v_3] = 1$ $\square$

rmk A Gorenstein orbifold has at most terminal sing. $\Rightarrow$ sing locus has codim ≥ 4

— Part 3 : Batyrev mirrors.

· reflexive polytope $\Delta \longrightarrow$ normal fan $\Sigma(\Delta) \longrightarrow$ max. proj. subdivision Σ

$\longrightarrow$ CY hypersurface V

Similarly, $\Delta^\circ \longrightarrow V^\circ$ the Batyrev mirror of V

<u>Thm</u> $h^{1,1}(V) = h^{n-2,1}(V^\circ)$ and $h^{n-2,1}(V) = h^{1,1}(V^\circ)$. Precisely,

$$\begin{cases} h^{1,1}(V) = \underbrace{\left[\ell(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ \text{ codim } 1} \ell^*(\Gamma^\circ) \right]}_{h^{1,1}_{toric}(V)} + \sum_{\theta^\circ \text{ codim } 2} \ell^*(\theta^\circ)\,\ell^*(\hat{\theta}^\circ) \quad \leftarrow \text{dual face} \\ h^{n-2,1}(V) = \underbrace{\left[\ell(\Delta) - n - 1 - \sum_{\Gamma \text{ codim } 1} \ell^*(\Gamma) \right]}_{h^{n-2,1}_{poly}(V)} + \underbrace{\sum_{\theta \text{ codim } 2} \ell^*(\theta)\,\ell^*(\hat{\theta})}_{\text{correction term}} \end{cases}$$

where $\ell(\Delta) := \# \Delta \cap M$ and $\ell^*(\Delta) := \# \{ m \in \Delta \cap M \mid m \notin \text{facet} \}$.

pf: "$h^{1,1}(V)$" Let $D_i \subset X_\Sigma$ corr. to $v_i \in \Sigma(1)$. Let $f : X_\Sigma \longrightarrow X_{\Sigma(\Delta)} = X_{\Delta^\circ}$

· $v_i \in \text{int}(\text{facet of } \Delta^\circ) \longrightarrow f(D_i) = pt \longrightarrow$ such D_i contribute nothing.

· $v_i \in \text{int}(\theta^\circ : \text{face of } \Delta^\circ) \longrightarrow f(D_i) = X_{\Sigma(\hat{\theta}^\circ)} \subset X_{\Delta^\circ}$, and

$V \cap D_i$ and $f(V) \cap X_{\Sigma(\hat{\theta}^\circ)}$ has the same number of irr. comp.

$\therefore$ codim $\theta^\circ \geq 3 \longrightarrow \dim \hat{\theta}^\circ \geq 2 \xrightarrow[\dim X_{\Sigma(\hat{\theta}^\circ)} \geq 2]{\text{Bertini}} f(V) \cap X_{\Sigma(\hat{\theta}^\circ)}$ irr.

$\therefore$ codim $\theta^\circ = 2 \longrightarrow X_{\Sigma(\hat{\theta}^\circ)}$ curve $\longrightarrow f(V) . X_{\Sigma(\hat{\theta}^\circ)} = \text{vol}(\hat{\theta}^\circ) = \ell^*(\hat{\theta}^\circ) + 1$.

Note $\text{div}(X^m)$, $m \in M$ are the only relations between these divisors.

Therefore, $h^{1,1}(V) = \underbrace{\# \{ v_i \in \Sigma(1) \mid v_i \notin \text{int}(\text{facet}) \}}_{\parallel \; \ell(\Delta^\circ) - 1 - \sum_{\Gamma^\circ \text{ codim } 1} \ell^*(\Gamma^\circ)} - n + \sum_{\theta^\circ \text{ codim } 2} \ell^*(\theta^\circ)\,\ell^*(\hat{\theta}^\circ)$

⊙ The Hodge-Deligne numbers of hypersurfaces in $T = T^n$.

· Let D_Δ be the divisor corr. to Δ i.e. on U_{σ_P} ($P \in \Delta(1)$), $\underset{\substack{\| \\ \mathcal{O}(\Delta)}}{\mathcal{O}_{X_{\Sigma(\Delta)}}(D_\Delta)}$ is gen. by x^P.

In particular, if Δ reflexive, $\mathcal{O}(D_\Delta) = -K_{X_{\Sigma(\Delta)}}$.

($L(\Delta) = \Gamma(X_{\Sigma(\Delta)}, \mathcal{O}(D_\Delta))$ the space of all Laurent polynomials with supp $\subset \Delta$.)

· For $f \in L(\Delta)$, let $\bar{Z} = \bar{Z}_{(\Delta,f)}$ be the hypersurface of $X_{\Sigma(\Delta)}$ defined by f.

i.e. on U_{σ_P} ($P \in \Delta(1)$), $\bar{Z}$ is given by $x^{-P} f = 0$.

$\longrightarrow$ $\bar{Z}$ is a compactification of $Z = Z_f = \bar{Z} \cap T^n$ the hypersurface in T^n defined by f.

· $f \in L(\Delta)$ is Δ-regular if $\bar{Z}_{(\Delta,f)}$ transversally intersects all strata of $X_{\Sigma(\Delta)}$

i.e. $\bar{Z} \cap T_\Gamma$ is smooth and of codim 1 in T_Γ for $\Gamma \prec \Delta$.

Note a general element of $L(\Delta)$ is Δ-regular by Bertini.

Now, we assume f is Δ-reg.

Prop $(-1)^{n-1} e^P(Z_f) = (-1)^P \binom{n}{p+1} + (-1)^{n-p} \sum_{k \geq 1} (-1)^k \binom{n+1}{p+k+1} \ell^*(k\Delta)$

pf (sketch) Assume Δ smooth. Let $X = X_{\Sigma(\Delta)}$, $D = X \setminus T$ and $D_{\bar{Z}} = D \cap \bar{Z}$.

Consider $\left(0 \longrightarrow \Omega^p_{(\bar{Z}, D_{\bar{Z}})} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\bar{Z}) \longrightarrow \Omega^{p+1}_{(X,D)} \otimes_{\mathcal{O}_X} \mathcal{O}_{\bar{Z}} \longrightarrow \Omega^{p+1}_{(\bar{Z}, D_{\bar{Z}})} \longrightarrow 0\right) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)\Delta)$

Then $(-1)^{n-1} e^P(Z_f) = (-1)^{n-1+p} \chi(\bar{Z}, \Omega^p_{(\bar{Z}, D_{\bar{Z}})})$

$$= (-1)^{n-1+p} \sum_{k \geq 0} (-1)^k \underset{\substack{\| \\ \chi(X, \Omega^{p+1+k}_{(X,D)}((k+1)\Delta)) - \chi(X, \Omega^{p+1+k}_{(X,D)}(k\Delta))}}{\chi(X, \Omega^{p+1+k}_{(X,D)}((k+1)\Delta) \otimes_{\mathcal{O}_X} \mathcal{O}_{\bar{Z}})}$$

by $0 \to \mathcal{O}_X(-\Delta) \to \mathcal{O}_X \to \mathcal{O}_{\bar{Z}} \to 0$

$$= (-1)^p \binom{n}{p+1} + (-1)^{n-p} \sum_{k \geq 1} (-1)^k \binom{n+1}{p+k+1} \ell^*(k\Delta)$$

($\Lambda^p(M) \otimes_{\mathbb{Z}} \mathcal{O}_X(-D) \simeq \Omega^p_{(X,D)}$ $\longrightarrow$ $\begin{cases} \mathcal{O}(-D) \simeq \Omega^n_X \\ \Gamma(X, \Omega^p_{(X,D)}(\Delta)) \simeq \Lambda^p(M) \otimes_{\mathbb{Z}} \underset{\substack{L^*(\Delta) \\ \|}}{\mathcal{O}_X(\Delta - D)} \end{cases}$

$\longrightarrow$ $\chi(X, \Omega^p_{(X,D)}) = (-1)^n \binom{n}{p}$, $\chi(X, \Omega^p_{(X,D)}(\Delta)) = \binom{n}{p} \cdot \ell^*(\Delta)$.) □

rmk In general, one should replace Δ by Δ' s.t. Δ' majorizes (refines) Δ and $\bar{Z}_{(\Delta',\Delta,f)}$ is a smooth compactification of Z_f.

Prop The Gysin homomorphism $H^i(\bar{Z}, \mathbb{C}) \longrightarrow H^{i+2}(X_{\Sigma(\Delta)}, \mathbb{C})$ is $\begin{cases} \text{iso. for } i > n-1 \\ \text{surj. for } i = n-1 \end{cases}$.

pf: Use the exact seq. $0 \to \Omega^{\cdot}_{X_{\Sigma(\Delta)}} \to \Omega^{\cdot}_{X_{\Sigma(\Delta)}}(\log D) \to \Omega^{\cdot-1}_{\bar{Z}} \to 0$.

Cor. $H^i_c(Z, \mathbb{C}) \longrightarrow H^{i+2}_c(T^n, \mathbb{C})$ is $\begin{cases} \text{iso. for } i > n-1 \\ \text{surj. for } i = n-1 \end{cases}$.

pf:
$$\begin{tikzcd}
\arrow[r] & H^{i-1}_c((X_{\Sigma(\Delta)} \setminus T) \cap \bar{Z}) \arrow[r] \arrow[d, "\wr \text{ or surj. induction}"] & H^i(T \cap \bar{Z}) \arrow[r] \arrow[d] & H^i(X_{\Sigma(\Delta)} \cap \bar{Z}) \arrow[r] \arrow[d, "\wr"] & H^i((X_{\Sigma(\Delta)} \setminus T) \cap \bar{Z}) \arrow[r] \arrow[d, "\wr \text{ induction}"] & {} \\
\arrow[r] & H^{i-1}_c(X_{\Sigma(\Delta)} \setminus T) \arrow[r] & H^i(T) \arrow[r] & H^i(X_{\Sigma(\Delta)}) \arrow[r] & H^i(X_{\Sigma(\Delta)} \setminus T) \arrow[r] & {}
\end{tikzcd}$$
□

Prop $e^{p,0}(Z) = (-1)^{n-1} \sum_{\dim \Gamma = p+1} \ell^*(\Gamma)$ for $p > 0$.

pf: · $e^{n-1,0}(Z) = e^p(Z) - e^{n-1,n-1}(Z) = (1 + (-1)^{n-1}\ell^*(\Delta)) - 1 = (-1)^{n-1} \sum_{\dim \Gamma = n} \ell^*(\Gamma)$

· Let $0 < p < n-1$. Assume Δ is smooth.

(the general case can be proved by taking Δ' majorizing/refining Δ. see Appedix E)

By Poicaré duality, $0 = e^{p,0}(Z) = \sum_{\Gamma \in \Delta} e^{p,0}(Z \cap T_{\Gamma'})$.

It suffices to show $\sum_{\Gamma \in \Delta} (-1)^{\dim \Gamma - 1} \sum_{\substack{\Gamma' \prec \Gamma \\ \dim \Gamma' = p+1}} \ell^*(\Gamma') = 0$ by induction.

We reduce to prove $\sum_{\Gamma' \prec \Gamma} (-1)^{\dim \Gamma} = 0$ for fixed Γ'

i.e. $\sum_{\sigma_\Gamma \subset \sigma_{\Gamma'}} (-1)^{\dim \sigma_\Gamma} = 0$, which is true. □

Cor $(-1)^{n-1} e^{n-2,1}(Z) = \ell^*(2\Delta) - (n+1)\ell^*(\Delta) - \sum_{\Gamma \text{ codim } 1} \ell^*(\Gamma)$.

pf: $(-1)^{n-1} e^{n-2,1}(Z) = (-1)^{n-1}[e^{n-2}(Z) - e^{n-2,0}(Z) - e^{n-2,n-2}(Z)]$

$= (-1)^{n-2} n + (-(n+1)\ell^*(\Delta) + \ell^*(2\Delta)) - \sum_{\Gamma \text{ codim } 1} \ell^*(\Gamma) - (-1)^{n-2} n$

$= \ell^*(2\Delta) - (n+1)\ell^*(\Delta) - \sum_{\Gamma \text{ codim } 1} \ell^*(\Gamma)$. □

proof of "$h^{n-2,1}(V) = \ell(\Delta) - n - 1 - \sum_{\Gamma \text{ codim} 1} \ell^*(\Gamma) + \sum_{\theta \text{ codim} 2} \ell^*(\theta)\ell^*(\hat{\theta})$":

Note $h^{n-2,1}(V) = (-1)^{n-1} e^{n-2,1}(V) = (-1)^{n-1} \sum_{\sigma \in \Sigma} e^{n-2,1}(V_\sigma)$ $\quad V_\sigma \overset{\shortparallel}{} V \cap O(\sigma)$

$= (-1)^{n-1} e^{n-2,1}(V_0) + (-1)^{n-1} \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma \geq 1}} e^{n-2,1}(V_\sigma)$

$\cdot\ (-1)^{n-1} e^{n-2,1}(V_0 = V\cap T^n) = \ell^*(2\Delta) - (n+1)\ell^*(\Delta) - \sum_{\Gamma\text{. codim }1} \ell^*(\Gamma)$.

$$\overset{\text{reflexive.}}{=} \ell(\Delta) - n - 1 - \sum_{\Gamma\text{. codim }1} \ell^*(\Gamma)$$

$(\dim\sigma \geq 1)$

$$\cdot\ e^{n-2,1}(V_\sigma = V\cap T_\sigma) = \cancel{e^{n-2,1}(V_\Delta\cap T_{\theta^*}) \cdot e^{0,0}(T^{\dim\theta^*+1-\dim\sigma})}\ ① +$$

$$e^{n-3,0}(V_\Delta\cap T_{\theta^*}) \cdot e^{1,1}(T^{\dim\theta^*+1-\dim\sigma})\ ②$$

$$\longrightarrow (-1)^{n-1} \sum_{\substack{\sigma\in\Sigma \\ \dim\sigma\geq 1}} e^{n-2,1}(V_\sigma) = \sum_{\theta\text{. codim }2} \ell^*(\theta)\ell^*(\hat{\theta})$$

① : $n-2 \leq \dim(V_\Delta\cap T_{\theta^*}) = n - (\dim\theta^* + 1) - 1 \Rightarrow \dim\theta^* = 0$

$\longrightarrow e^{n-2,1}(V_\Delta\cap T_{\theta^*}) = 0$.

② : $n-3 \leq \dim(V_\Delta\cap T_{\theta^*}) \Rightarrow \dim\theta^* = \underset{*}{0}$ or $1 \longrightarrow \dim\theta = n-2$ and

$$e^{n-3,0}(V_\Delta\cap T_{\theta^*}) = (-1)^{n-3}\ell^*(\hat{\theta})$$

$\square$

§ Appendix

This section contains more precise / general descriptions of some properties used above.

A. The Orbit-Cone Correspondence

- The distinguished point γ_σ of σ is defined by $S_\sigma \longrightarrow \mathbb{C}$, $u \longmapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$

$\longrightarrow$ γ_σ is T-fixed if $\dim \sigma = n$ and

- The torus orbit $O(\sigma)$ corr. to σ is $T_N \cdot \gamma_\sigma$.

Note that $M(\sigma) = \sigma^\perp \cap M$ is the dual lattice of $N(\sigma) = N/N_\sigma$.

$\longrightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^\times) = T_{N(\sigma)}$.

<u>Lemma</u> $O(\sigma) = \{\gamma : S_\sigma \longrightarrow \mathbb{C} \mid \gamma(u) \neq 0 \iff u \in \sigma^\perp \cap M\} \simeq T_{N(\sigma)}$.

<u>Thm</u> (orbit-cone)

(a) $\{\text{cones in } \Sigma\} \xleftrightarrow{\text{bij}} \{T_N\text{-orbits in } X_\Sigma\}$, $\sigma \longleftrightarrow O(\sigma)$

and $\dim \sigma + \dim O(\sigma) = n$.

(b) $U_\sigma = \bigcup_{\tau \prec \sigma} O(\tau)$

(c) $\tau \prec \sigma \iff O(\sigma) \subset V(\tau) = \overline{O(\tau)}$ and $V(\tau) = \bigcup_{\tau \prec \sigma} O(\sigma)$.

<u>Prop</u> $V(\tau) \simeq X_{\mathrm{Star}(\tau)}$

<u>Prop</u> $F \prec \Delta \Rightarrow V(\sigma_F) \simeq X_F$. $(\subset X_{\Sigma(\Delta)})$

B. Resolution of Singularities

<u>Def</u> For $v \in (\bigcup_{\sigma \in \Sigma} \sigma) \cap N$, the star subdivision $\Sigma^*(v)$ of Σ at v is the set consisting of $\begin{cases} \sigma & \text{if } v \notin \sigma \in \Sigma \\ \mathrm{cone}(\tau, \sigma) & \text{if } v \notin \tau \in \Sigma \text{ and } \{v\} \cup \tau \subset \sigma \in \Sigma. \end{cases}$

<u>Fact</u> Let $\sigma = \mathrm{cone}(\underbrace{u_1, \cdots, u_n}_{\text{basis for } N})$ and $u_0 = u_1 + \cdots + u_n$. Then $X_{\Sigma^*(u_0)} \longrightarrow X_\Sigma$ is the blow-up at γ_σ.

Prop $\exists$ refinement Σ' s.t. Σ' simplicial, $\Sigma'(1) = \Sigma(1)$, and Σ' is obtained from Σ by a sequence of star subdivisions.

Prop $\exists$ refinement $\tilde{\Sigma}$ s.t. $X_{\tilde{\Sigma}} \longrightarrow X_\Sigma$ resol. of sing.

C. Images of Orbit Closure via Refinement

Let $f: X_{\Sigma'} \longrightarrow X_\Sigma$ be the refinement map. Given $\sigma' \in \Sigma'$, let σ be the minimal cone in Σ containing σ'.

Then
$$f(\gamma_{\sigma'}) = \left[\begin{array}{l} S_\sigma \hookrightarrow S_{\sigma'} \longrightarrow \mathbb{C} \\ \qquad\quad u \longmapsto \begin{cases} 1 & \text{if } u \in \sigma'^{\perp} \cap M \\ 0 & \text{otherwise.} \end{cases} \end{array} \right]$$

$$= \left[\begin{array}{l} S_\sigma \longrightarrow \mathbb{C} \\ u \longmapsto \begin{cases} 1 & \text{if } u \in \sigma^{\perp} \cap M \\ 0 & \text{otherwise.} \end{cases} \end{array} \right] = \gamma_\sigma .$$

Therefore, $f(O(\sigma')) = O(\sigma)$ and $f(V(\sigma')) = V(\sigma)$

(The T_N-actions on X_Σ and $X_{\Sigma'}$ are the same)

D. Vanishing Theorem for Sheaf Cohomology of Toric Varieties.

<u>Thm</u> (Demazure) D Cartier and g.b.g.s. $\Rightarrow H^{p>0}(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0$.

E. Hodge - Deligne Theory.

- For each cpx. alg. var. X, the cohomology $H^k(X)$ carries a (mix) Hodge str.

  Now, we assume X is cpt. quasi-smooth, which implies the Hodge str. on $H^k(X) = H_c^k(X)$ is pure of weight k and $e^{p,q}(X) := \sum_k (-1)^k h^{p,q}(H_c^k(X))$

  $$= (-1)^{p+q} h^{p,q}(X)$$

- $e(X) = e(X; x, \bar{x}) := \sum_{p,q} e^{p,q}(X) x^p \bar{x}^q$.

<u>Prp</u> (1) X is stratified by loc. closed $X_i \Rightarrow e(X) = \sum_i e(X_i)$

Therefore, if X is covered by loc. closed $X_i \Rightarrow e(X) = \sum_{i_0 < \cdots < i_k} (-1)^k e(X_{i_0, \cdots, i_k})$.

(2) $e(X \times Y) = e(X) \cdot e(Y)$. Hence, $e(X) = e(Y) \cdot e(F)$ if $f : X \longrightarrow Y$ is a bundle with fiber F.

(3) $e^p(X) := \sum_q e^{p,q}(X) = (-1)^p \chi(\bar{X}, \Omega^p_{(\bar{X}, D)})$ for a smooth compactification $\bar{X}$ s.t. $D = \bar{X} \setminus X$ has transversal intersections in $\bar{X}$.

<u>Examples</u>
- $e(\mathbb{C}^n) = x^n \bar{x}^n$.
- $e(\mathbb{P}^n) = 1 + x\bar{x} + \cdots + x^n \bar{x}^n$
- $e(T^n) = (x\bar{x} - 1)^n$.

<u>Def</u> Δ' majorizes Δ if $\exists\ \alpha : \Delta'(1) \longrightarrow \Delta(1)$ s.t. $\sigma_{P, \Delta'} \subset \sigma_{\alpha(P), \Delta}$

i.e. $\Sigma(\Delta')$ refines $\Sigma(\Delta)$. $\longrightarrow \rho_{\Delta', \Delta} : X_{\Sigma(\Delta)} \longrightarrow X_{\Sigma(\Delta')}$.

(My veiwpoint : Δ' might not be a polytope !)

$\longrightarrow$
- α can be extend to the set of all faces.
- Δ' smooth, f Δ-reg. $\Rightarrow$ f is Δ'-reg. and $\bar{Z}_{(\Delta', \Delta, f)} := \rho^*_{\Delta', \Delta}(\bar{Z}_{(\Delta, f)})$ is a smooth compactification of Z
- For $\Gamma' \prec \Delta'$, $\ell^*(\Gamma') := \sum_{\Gamma \prec \alpha(\Gamma')} \ell^*(\Gamma)$.

With these notations, we give a proof of "$e^{p,0}(Z) = (-1)^{n-1} \sum_{\dim \Gamma = p+1} \ell^*(\Gamma)$ for $0 < p < n-1$"

pf : By Poicaré duality, $0 = e^{p,0}(\bar{Z} = \bar{Z}_{(\Delta', \Delta, f)}) = \sum_{\Gamma' \in \Delta'} e^{p,0}(\bar{Z} \cap T_{\Gamma'})$.

It suffices to show $\sum_{\Gamma' \in \Delta'} (-1)^{\dim \Gamma' - 1} \sum_{\substack{\Gamma \prec \alpha(\Gamma') \\ \dim \Gamma = p+1}} \ell^*(\Gamma) = 0$ by induction.

We reduce to prove $\sum_{\substack{\Gamma' \in \Delta' \\ \underline{\Gamma \prec \alpha(\Gamma')}}} (-1)^{\dim \Gamma'} = 0$ i.e. $\sum_{\sigma_{\Gamma'} \subset \sigma_{\Gamma}} (-1)^{\dim \sigma_{\Gamma'}} = 0$, which is true.

$\Leftrightarrow \sigma_{\Gamma'} \subset \sigma_{\Gamma}$

□

Ref :

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