Conformal field theory

$\underbrace{X^\mu(\sigma^1, \sigma^2)}_{\text{world-sheet.}}$: D free scalar fields

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu \right) \qquad (\text{Polyakov action with } \gamma_{ab} \mapsto \delta_{ab}).$$

Let $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$ $\Rightarrow$ $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. $g_{zz} = g_{\bar{z}\bar{z}} = 0$.

$\Rightarrow S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu$. vary X^μ $\Rightarrow$ eq of motion : $\partial\bar{\partial} X^\mu = 0$.

Define $\delta^2(z,\bar{z}) = \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2)$ s.t. $\int d^2z\, \delta^2(z,\bar{z}) = 1$.

$$0 = \int dX \frac{\delta}{\delta X_\mu(z)} \left(e^{-S} X^\nu(z') \right) \mathcal{A}(z'') = \int dX\, e^{-S} \left(\eta^{\mu\nu} \delta^2(z-z') + \frac{1}{\pi\alpha'} \partial\bar{\partial} X^\mu(z) \cdot X^\nu(z') \right) \mathcal{A}(z'')$$

$$= \eta^{\mu\nu} \langle \delta^2(z-z') \rangle \mathcal{A}(z'') + \frac{1}{\pi\alpha'} \partial\bar{\partial} \langle X^\mu(z) X^\nu(z') \rangle \mathcal{A}(z'')$$

$\Rightarrow \frac{1}{\pi\alpha'} \partial\bar{\partial} X^\mu(z) \cdot X^\nu(z') = -\eta^{\mu\nu} \delta^2(z-z')$ as an operator eq.

$\because \partial\bar{\partial} \log|z|^2 = 2\pi\delta^2(z)$ $\Rightarrow$ $:X^\mu(z) X^\nu(z'): = X^\mu(z) X^\nu(z') + \frac{\alpha'}{2} \eta^{\mu\nu} \log|z-z'|^2$.

s.t. $\partial\bar{\partial} :X^\mu(z) X^\nu(z'): = 0$

Then $X^\mu(z) X^\nu(z') = -\frac{\alpha'}{2} \eta^{\mu\nu} \log|z-z'|^2 + :X^\mu X^\nu(z'):$

$$+ \sum_{k\geq 1} \frac{1}{k!} \left((z-z')^k :(\partial^k X^\mu) X^\nu(z'): + (\overline{z-z'})^k :(\bar{\partial}^k X^\mu) X^\nu(z'): \right). + \underbrace{\partial\bar{\partial} \cdots}_{\substack{\shortparallel \\ 0}}$$

Define $:X^{\mu_1}(z_1) \cdots X^{\mu_n}(z_n): = X^{\mu_1}(z_1) \cdots X^{\mu_n}(z_n) + \sum$ subtractions

$\uparrow$ sum over all ways of choosing some pairs of fields from the product and replacing each pair with $\frac{\alpha'}{2} \eta^{\mu_i\mu_j} \log|z_{ij}|^2$ ($z_{ij} = z_i - z_j$)

In general $:\mathcal{A}: = \exp\left(\frac{\alpha'}{4} \int d^2z_1 d^2z_2 \log|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1)} \frac{\delta}{\delta X_\mu(z_2)} \right) \mathcal{A}$

$\rightsquigarrow \mathcal{A} = \exp(- \quad \cdots \quad) :\mathcal{A}: = :\mathcal{A}: + \sum$ contractions. ✓ $X^{\mu_i} X^{\mu_j} \mapsto -(\quad)$

Then $:\mathcal{A}::\mathcal{A}': = \exp\left(-\frac{\alpha'}{2} \int d^2z_1 d^2z_2 \log|z_{12}|^2 \frac{\delta}{\delta X^\mu(z_1)} \frac{\delta}{\delta X'_\mu(z_2)} \right) :\mathcal{A}\mathcal{A}':$

$= :\mathcal{A}\mathcal{A}': + \sum$ cross-contractions.

$\rightsquigarrow$ operator product expansion (OPE).

For example $:\partial X^\mu(z)\,\partial X_\mu(z)::\partial' X^\nu(z')\,\partial' X_\nu(z'):$

$$= :\partial X^\mu(z)\,\partial X_\mu(z)\,\partial' X^\nu(z')\,\partial' X_\nu(z'): -4\cdot\frac{\alpha'}{2}\underbrace{(\partial\partial'\log|z-z'|^2)}_{\overset{\shortparallel}{\frac{1}{(z-z')^2}}}:\underbrace{\partial X^\mu(z)}_{\overset{\shortparallel}{\partial' X^\mu(z') + (\partial')^2 X^\mu(z')(z-z') + \cdots}}\partial' X_\mu(z'):$$

$$+ 2\cdot\underbrace{\eta^\mu_\mu}_{\overset{\shortparallel}{D}}\left(-\frac{\alpha'}{2}\partial\partial'\log|z-z'|^2\right)^2$$

singular terms

$$\sim \frac{D\alpha'^2}{2(z-z')^4} - \frac{2\alpha'}{(z-z')^2}:\partial' X^\mu(z')\,\partial' X_\mu(z'): - \frac{2\alpha'}{z-z'}:(\partial')^2 X^\mu(z')\,X_\mu(z'):$$

Ward identity.

Consider the variation $\delta\phi_\alpha(\sigma) = \varepsilon(\cdots)$ of some field ϕ_α s.t. $[d\phi']e^{-S[\overset{\phi+\delta\phi}{\overset{\shortparallel}{\phi'}}]} = [d\phi]e^{-S[\phi]}$.

Noether procedure : $\phi' = \phi + \rho\,\delta\phi \Rightarrow [d\phi']e^{-S[\phi']} = [d\phi]e^{-S[\phi]}\left(1 + \frac{i\varepsilon}{2\pi}\int d^D\sigma\sqrt{g}\,\underbrace{j^a(\sigma)}_{\text{some coeff.}}\partial_a\rho(\sigma)\right.$

$+\,O(\varepsilon^2)$

ρ cpt supported $\Rightarrow$ $0 = \int[d\phi']e^{-S[\phi']}A - \int[d\phi]e^{-S[\phi]}A$ for A outside supp ρ.

$$= \frac{\varepsilon}{2\pi i}\int d^D\sigma\sqrt{g}\,\rho(\sigma)\langle\nabla_a j^a(\sigma)\,A\rangle + O(\varepsilon^2) \quad (\text{IBP} + \rho \text{ cpt supp}).$$

So $\nabla_a j^a = 0$ as an operator eq.

$\underset{\text{some region}}{\rho = \chi_\Omega}$ $\quad\sigma_0\in\Omega.$ $\Rightarrow$

$$\delta A(\sigma_0) + \frac{\varepsilon}{2\pi i}\underbrace{\int_\Omega d^D\sigma\sqrt{g}\,\nabla_a j^a(\sigma)\,A(\sigma_0)}_{\shortparallel\ \text{divergence thm.}} = 0$$

$$\int_{\partial\Omega} dA\cdot n_a j^a A(\sigma_0) = \frac{2\pi}{i\varepsilon}\delta A(\sigma_0)$$

For $(z,\bar z)\in\mathbb{C}$ ← flat.

$$\rightsquigarrow \int_{\partial\Omega}(\underset{j_z}{\underset{\shortparallel}{j}}\,dz - \underset{j_{\bar z}}{\underset{\shortparallel}{\tilde j}}\,d\bar z)A(z_0,\bar z_0) = \frac{2\pi}{\varepsilon}\delta A(z_0,\bar z_0)$$

If j holomorphic, $\tilde j$ anti-holomorphic $\Rightarrow$ $\delta A(z_0,\bar z_0) = \underline{i\varepsilon(\mathrm{Res}_{z_0}}\, j(z)A(z_0,\bar z_0) + \overline{\mathrm{Res}}_{\bar z_0}\tilde j(\bar z)A(z_0,\bar z_0))$

(usually true) Ward identity.

Consider world-sheet translation $\delta\sigma^a = \rho(\sigma)\,\varepsilon v^a$

$$\delta S = \underset{\text{change metric} = -\text{change coor.}}{-}\int d^2\sigma\frac{\partial S}{\partial g^{ab}}\delta g^{ab} = \frac{1}{2\pi}\int d^2\sigma\,\underbrace{T_{ab}}_{\text{energy momentum tensor}\ \shortparallel}\,(\underbrace{\partial^a\delta\sigma^b}_{\overset{\shortparallel}{\varepsilon\cdot\partial^a\rho(\sigma)v^b}}) = \frac{\varepsilon}{2\pi i}\int d^2\sigma\sqrt{g}\,j^a(\sigma)\,\partial_a\rho(\sigma).$$

$$-\frac{1}{\alpha'}\left(\partial_a X^\mu\partial_b X_\mu - \frac{1}{2}\delta_{ab}\partial_c X^\mu\partial^c X_\mu\right)$$

$\Rightarrow j_a = i v^b T_{ab}$

T is traceless : $T_a{}^a = 0 \Rightarrow T_{z\bar z} = 0$.

$$\text{conservation} \quad \partial^a T_{ab} = 0 \xrightarrow{\qquad} \underset{\underset{T}{\|}}{\bar\partial T_{zz}} = \underset{\underset{\tilde T}{\|}}{\partial T_{\bar z\bar z}} = 0 \Rightarrow \begin{cases} T = T(z) & \text{holo} \\ \tilde T = \tilde T(\bar z) & \text{anti-holo} \end{cases}$$

$T(z) = -\frac{1}{\alpha}:\partial X^\mu \partial X_\mu:$, $\tilde T(\bar z) = -\frac{1}{\alpha'}:\bar\partial X^\mu \bar\partial X_\mu:$

$j(z) = iv(z)\,T(z)$, $\tilde j(\bar z) = iv(z)^*\,\tilde T(z)$ are conserved for any v

$$\text{So}\quad T(z)\underset{\underset{X^\mu(z')|_{z'=0}}{/\!/}}{X^\mu(0)} = -\frac{1}{\alpha}:\partial X^\mu(z)\,\partial X_\mu(z)::X^\mu(0):$$

$$= -\frac{1}{\alpha}:\partial X^\mu(z)\,\partial X_\mu(z)\,X^\mu(0):$$

$$-\frac{1}{\alpha}\cdot 2\left(-\frac{\alpha'}{2}\,\underset{\underset{\frac{1}{z}}{\|}}{\underline{\partial \log|z|^2}}\right)\underset{\underset{\partial X^\mu(0)+z(\cdots)}{\|}}{\underline{:\partial X^\mu(z):}} \sim \frac{1}{z}\,\partial X^\mu(0).$$

Similarly, $\tilde T(\bar z)\,X^\mu(0) \sim \frac{1}{\bar z}\,\bar\partial X^\mu(0)$

$$\text{Ward identity} \Rightarrow \delta X^\mu(0) = i\varepsilon\left(\mathrm{Res}_0\,\underset{\underset{iv(z)\,T(z)}{\|}}{j(z)}\,X^\mu(0) + \overline{\mathrm{Res}}_0\,\underset{\underset{iv(z)^*\tilde T(\bar z)}{\|}}{\tilde j(z)}\,X^\mu(0)\right)$$

$$= -\varepsilon v(0)\,\partial X^\mu(0) - \varepsilon v(0)^*\,\bar\partial X^\mu(0)$$

0 to general z' $\Rightarrow \delta X^\mu = -\varepsilon v\,\partial X^\mu - \varepsilon v^*\,\bar\partial X^\mu$.

For a general operator $\mathcal{A}$. $T(z)\,\mathcal{A}(0) \sim \sum_{n\geq 0}\frac{1}{z^{n+1}}\,\mathcal{A}^{(n)}(0)$

$$\tilde T(\bar z)\,A(0) \sim \sum_{n\geq 0}\frac{1}{\bar z^{n+1}}\,\tilde{\mathcal{A}}^{(n)}(0)$$

Write $v(z) = v_0 + v_1 z + \cdots$.

$$\text{Ward identity} \Rightarrow \delta A(0) = -\varepsilon\left(\mathrm{Res}_0\left(\sum_{m\geq 0} v_m z^m \sum_{n\geq 0}\frac{1}{z^{n+1}}\,\mathcal{A}^{(n)}(0)\right) + \text{c.c.}\right)$$

$$= -\varepsilon\sum_{n\geq 0}^{\infty} v_n\,\mathcal{A}^{(n)}(0) + \text{c.c.}$$

$$\rightsquigarrow \delta A(z,\bar z) = -\varepsilon\sum_{n\geq 0}^{\infty}\frac{1}{n!}\left(\partial^n v(z)\,\mathcal{A}^{(n)}(z,\bar z) + \bar\partial^n v(z)^*\,\tilde{\mathcal{A}}^{(n)}(z,\bar z)\right). \quad (*)$$

The infinite transformation $z' = z + \varepsilon v(z)$ comes from some $z' = \underset{\substack{\uparrow \\ \text{holomorphic}}}{f_\varepsilon(z)}$

For $z' = \zeta z$. $\mathcal{A}$: eigenstate, say $\mathcal{A}'(z', \bar z') = \zeta^{-h} \bar\zeta^{-\tilde h} \mathcal{A}(z, \bar z)$

$(h, \tilde h)$: weight of $\mathcal{A}$. $h + \tilde h$: dim of $\mathcal{A}$, $h - \tilde h$: spin of $\mathcal{A}$.

wt of ∂A, $\bar\partial A$ are $(h+1, \tilde h)$, $(h, \tilde h + 1)$, resp.

$$\mathcal{A}'(z) - \mathcal{A}(z) = \underbrace{\mathcal{A}'(z')}_{\zeta^{-h}\bar\zeta^{-\tilde h}\mathcal{A}(z)} - \delta z\, \underbrace{\partial \mathcal{A}(z')}_{\partial\mathcal{A}(z) + \cdots} - \delta\bar z\, \underbrace{\bar\partial \mathcal{A}(z')}_{\bar\partial\mathcal{A}(z) + \cdots} - \mathcal{A}(z) \qquad z' = \zeta z = z + \underbrace{\delta z}_{\varepsilon z}$$

$$= -(h\varepsilon - \tilde h \bar\varepsilon - \varepsilon z \partial - \bar\varepsilon \bar z \bar\partial)\, A(z) + O(\varepsilon^2)$$

$\Rightarrow \delta A = -\varepsilon(h + z\partial)\mathcal{A} - \bar\varepsilon(\tilde h + \bar z \bar\partial) A$. under $\delta z = \varepsilon z$, $\delta \bar z = \bar\varepsilon \bar z$.

(*) $\Rightarrow$ $A^{(0)}(0.0) = \tilde A^{(0)}(0.0) = 1$, $A^{(1)}(0.0) = h$, $\tilde A^{(1)}(0.0) = \tilde h$.

So $T(z) A(0.0) = \cdots + \frac{h}{z^2} A(0.0) + \frac{1}{z} \partial A(0.0) + \cdots$, $\tilde T$ similarly.

Special case : $\mathcal{O}'(z', \bar z') = (\partial z')^{-h} (\bar\partial \bar z')^{-\tilde h} \mathcal{O}(z, \bar z)$ ⟵ primary field, or tensor operator.

$\delta \mathcal{O} = -\varepsilon(h v'(z) + z\partial)\mathcal{O} -$ c.c.

(*) $\Rightarrow$ $T(z)\mathcal{O}(0.0) = \frac{h}{z^2}\mathcal{O}(0.0) + \frac{1}{z}\mathcal{O}(0.0) + \cdots$

OPE of T : $T(z)T(0) = \frac{1}{(\alpha')^2} :\partial X^\mu(z)\partial X_\mu(z)::\partial X^\nu(0)\partial X_\nu(0):$

$$\sim \frac{1}{(\alpha')^2}\left(\frac{D\alpha'^2}{2z^4} - \frac{2\alpha'}{z^2}:\partial X^\mu(0)\partial X_\mu(0): - \frac{2\alpha'}{z}:\partial^2 X^\mu(0)\partial X_\mu(0):\right)$$

$$= \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0).$$

$T(z)\tilde T(\bar 0)$ is nonsingular ($z - z'$ is not holomorphic in $\bar z'$).

$\Rightarrow$ $\varepsilon^{-1}\delta T(z) = -\frac{D}{12}\partial^3 v - 2(\partial v)T - v\partial T$. (✩)

The finite form is $(\partial z')^2 T'(z') = T(z) - \frac{D}{12}\{z', z\}$, $\underset{\substack{\uparrow \\ \text{Schwarzian derivative}}}{\{z', z\}} = \frac{\partial^3 z'}{\partial z'} - \frac{3}{2}\left(\frac{\partial^2 z'}{\partial z'}\right)^2$

$$\left(\begin{aligned} & z' = z + \varepsilon v \Rightarrow \{z', z\} = \frac{\varepsilon\partial^3 v}{1 + \varepsilon\partial v} - \frac{3}{2}\left(\frac{\varepsilon\partial^2 v}{1 + \varepsilon\partial v}\right)^2 = \varepsilon\partial^3 v + O(\varepsilon^2) \\ & T'(z) = (1 + \varepsilon\partial v)^{-2}\left(T(z) - \frac{D}{12}\varepsilon\partial^3 v\right) - \varepsilon v\partial T + O(\varepsilon^2) \Rightarrow (✩) \end{aligned}\right)$$

bc CFT.

$S = \frac{1}{2\pi}\int d^2z \; b\bar\partial c$. $\quad bc = -cb$ (anti-commuting).

S is conformally invaint if $b.c$ are primary fields with weights $(\lambda.0)$, $(1-\lambda.0)$

($d^2z\, b\bar\partial c$ has wt $(-1+\lambda+(1-\lambda), -1+0+1) = (0.0)$).

eq. of motion : $\bar\partial b = \bar\partial c = 0$, $\bar\partial c(z)\, b(0) = 2\pi\delta^2(z)$, $\bar\partial b(z)\, b(0) = \bar\partial c(z)\, c(w) = 0$.

$\Rightarrow \; :b(z)\,c(z'): = b(z)\,c(z') - \frac{1}{z-z'} \quad (\because \bar\partial z^{-1} = 2\pi\delta^2(z))$.

So $b(z)c(z') \sim c(z)\,b(z') \sim \frac{1}{z-z'}$. $\quad b(z)\,b(z') = O(z-z')$, $c(z)\,c(z') = O(z-z')$

Energy momentum tensor : Under $\delta z = \varepsilon\rho(z)$.

$$b \mapsto (1+\varepsilon\partial\rho)^{\lambda}(b+\varepsilon\rho\,\partial b) , \quad c \mapsto (1+\varepsilon\partial\rho)^{1-\lambda}(c+\varepsilon\rho\,\partial c)$$

$$\Rightarrow \delta S = \frac{\varepsilon}{2\pi}\int d^2z\, (\lambda\partial\rho + \rho\partial)b\cdot\bar\partial c + b\bar\partial((1-\lambda)\partial\rho + \rho\partial)c .$$

$$= \frac{\varepsilon}{2\pi}\int d^2z\, (1-\lambda)\, b(\bar\partial\partial\rho)c + b(\bar\partial\rho)(\partial c) + \underbrace{\partial(\rho b\bar\partial c)}_{\overset{\|}{0}} \text{ total der.}$$

$$= \frac{\varepsilon}{2\pi}\int d^2z\, \rho\,\bar\partial\Big(\underbrace{(1-\lambda)\partial(bc) - b\partial c}_{\overset{\|}{-\lambda b(\partial c) + (1-\lambda)(\partial b)c}}\Big)$$

So $T = -\lambda :b(\partial c): + (1-\lambda) :(\partial b)c: \; = \; :(\partial b)c: - \lambda\partial :bc:$

$\tilde T = 0$ is trivial by same method.

OPE $\quad T(z)T(0) = \; :\partial b(z)\cdot c(z): \, :\partial' b(0)\cdot c(0): - \lambda :\partial b(z)\cdot c(z): \partial' :b(0)c(0):$

$\qquad\qquad - \lambda\, \partial :b(z)c(z): \, :\partial' b(0)c(0): + \lambda^2 \partial :b(z)c(z): \partial' :b(0)c(0):$

$$\sim \underline{\left(-\frac{1}{z^2}\right)\left(\frac{1}{z^2}\right)} - \frac{1}{z^2} :c(z)\,\partial' b(0): + \frac{1}{z^2} :\partial b(z)\, c(0):$$

$$- \lambda\partial'\Big(\underline{-\frac{1}{z^2}\cdot\frac{1}{z}} - \frac{1}{z^2} :c(z)\,b(0): + \frac{1}{z} :\partial b(z)\, c(0):\Big)$$

$$- \lambda\partial\Big(\underline{\frac{1}{z}\cdot\frac{1}{z^2}} + \frac{1}{z} :c(z)\,\partial' b(0): + \frac{1}{z^2} :b(z)\,c(0):\Big) + \underline{\lambda^2\partial\partial'\left(\frac{1}{z}\cdot\frac{1}{z}\right)}$$

$$\sim \frac{1}{z^4}\underbrace{\underline{(-6\lambda^2 + 6\lambda - 1)}}_{\overset{\|}{c/2}} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial' T(0) \quad \Rightarrow \quad c = -3(2\lambda-1)^2 + 1 .$$

c : central charge ($= D$ in X^μ case) $\qquad \tilde c = 0$

$\beta\gamma$ CFT.

$$S = \frac{1}{2\pi}\int d^2z\ \beta\bar\partial\gamma \qquad \underset{\text{wt}:(\lambda.0),(1-\lambda.0)}{\beta.\gamma} \text{ commuting} \rightarrow \beta(z)\gamma(z') = -\frac{1}{z-z'} .\ \gamma(z)\beta(z') = \frac{1}{z-z'}$$

$$T = :(\partial\beta)\gamma: - \lambda\partial :\beta\gamma: ,\ \tilde{T} = 0 \text{ and } c = 3(2\lambda-1)^2 - 1 .\ \tilde{c} = 0.$$

Virasoro algebra.

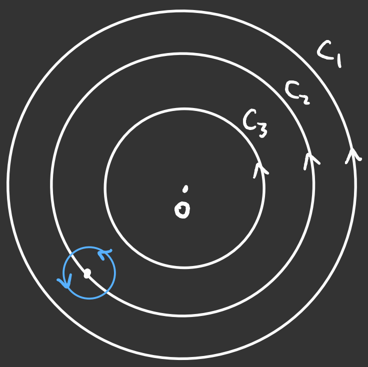
Closed string : $\sigma \sim \sigma + 2\pi \qquad w = \sigma + i\tau . \quad z = e^{-iw}$

$$T(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} ,\ \tilde{T}(\bar z) = \sum_{m\in\mathbb{Z}} \frac{\tilde{L}_m}{\bar z^{m+2}} \qquad L_m.\tilde{L}_m : \text{Virasora generator} \quad L_m = \int_C \frac{dz}{2\pi i z} z^{m+2}\, T(z) .$$

$$T_w = \underbrace{(\partial_w z)^2}_{\overset{\shortparallel}{-z^2}} T + \frac{c}{12}\underbrace{\{z.w\}}_{\overset{\shortparallel}{\frac{1}{2}}} = -\sum_{m\in\mathbb{Z}} e^{im\sigma - m\tau} \underset{\overset{\shortparallel}{L_m - \delta_{m0}\frac{c}{24}}}{T_m} ,\quad \tilde{T}_w(w) = -\sum_{m\in\mathbb{Z}} e^{-im\sigma - m\tau} \underset{\overset{\shortparallel}{\tilde{L}_m - \delta_{m.0}\frac{\tilde c}{24}}}{\tilde{T}_m} .$$

Hamiltonian $H = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\ T_{\tau\tau} = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\,(T_w + \tilde{T}_w) = L_0 + \tilde{L}_0 - \frac{c+\tilde{c}}{24}$

$Q\{C\} := \int_C \frac{dz}{2\pi i} j \leftarrow$ Noether current.



$Q_1\{C_1\}\,Q_2\{C_2\} - Q_1\{C_3\}\,Q_2\{C_2\}$

$\xrightarrow{\text{path integral}} \quad [\hat{Q}_1.\hat{Q}_2] = \hat{Q}_1\hat{Q}_2 - \hat{Q}_2\hat{Q}_1$

For $z_2 \in C_2$, we deform $C_1 - C_3$ to $C_{z_2}(p)$

$\Rightarrow [Q_1.Q_2]\{C_2\} = \int_{C_2}\frac{dz_2}{2\pi i}\ \text{Res}_{z_1\to z_2}\ j_1(z_1)\,j_2(z_2)$

Similarly, $[Q_1.\mathcal{A}(z_2.\bar z_2)] = \text{Res}_{z_1\to z_2}\ j(z_1)\,\mathcal{A}(z_2.\bar z_2) \underset{\uparrow}{=} \frac{1}{i\varepsilon}\delta\mathcal{A}(z_2,\bar z_2)$

Ward identity.

For $j_m(z) = z^{n+1}\,T(z)$, we get

$$\text{Res}_{z_1\to z_2}\ j_m(z_1)\,j_n(z_2) = \text{Res}_{z_1\to z_2}\ z_1^{m+1} z_2^{n+1}\left(\frac{c}{2z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2)\right)$$
$$= \frac{c}{12}(\partial^3 z_2^{m+1})\,z_2^{n+1} + 2(\partial z_2^{m+1})\,z_2^{n+1}\,T(z_2) + z_2^{m+n+2}\,\partial T(z_2)$$
$$= \frac{c}{12}(m^3-m)\,z_2^{m+n-1} + (m-n)\,z_2^{m+n+1}\,T(z_2) + \partial(\cdots).$$

$$\Rightarrow [L_m.L_n] = (m-n)\,L_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m.-n}.$$

Mode expansions.

$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{\alpha_m^\mu}{z^{m+1}}$, $\bar\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{\tilde\alpha_m^\mu}{\bar z^{m+1}}$ $\Rightarrow \alpha_0^\mu - \tilde\alpha_0^\mu = \frac{1}{2\pi}\sqrt{\frac{2}{\alpha'}}\oint \underbrace{\partial X^\mu(z)\,dz + \bar\partial X^\mu(\bar z)\,d\bar z}_{\substack{\parallel\\ dX^\mu(z)}}$

$(z = e^{-iw})$ $\qquad\Rightarrow \alpha_0^\mu = \tilde\alpha_0^\mu$

Under the spacetime translation $\delta X^\mu = \varepsilon a^\mu$, $j_a = a_\mu \underbrace{\left(\frac{i}{\alpha'}\partial_a X^\mu\right)}_{=:\, j_a^\mu}$.

$\left(\delta X^\mu = \varepsilon\rho a^\mu \Rightarrow \delta S = \frac{\varepsilon a_\mu}{2\pi\alpha'}\int d^2\sigma\,\partial^a X^\mu\partial_a\rho = \frac{\varepsilon}{2\pi i}\int d^2\sigma\sqrt{g}\,j^a(\sigma)\,\partial_a\rho\right)$

$\Rightarrow$ spacetime momentum $p^\mu = \frac{1}{2\pi}\int_0^{2\pi} j_\tau^\mu\,d\sigma = \frac{1}{2\pi i}\oint j^\mu dz - \tilde{j}^\mu d\bar z = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\tilde\alpha_0^\mu$

So $X^\mu(z,\bar z) = x^\mu - \frac{i\alpha'}{2}p^\mu\log|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m}+\frac{\tilde\alpha_m^\mu}{\bar z^m}\right)$

$\mathrm{Res}_{z\to z'}\underbrace{z^m\partial X^\mu(z)\,z'^n\partial X^\nu(z')}_{\substack{\wr\\ -\frac{\alpha'}{2}\eta^{\mu\nu}z^m(z')^n\underbrace{\partial_1\partial_2\log|z-z'|^2}_{\parallel\ \frac{1}{(z-z')^2}}}} = -\frac{\alpha'}{2}\eta^{\mu\nu}m(z')^{m+n-1} \Rightarrow [\alpha_m^\mu,\alpha_n^\nu] = (2\pi i)^2\cdot\left(\frac{1}{2\pi}\sqrt{\frac{2}{\alpha'}}\right)^2$

$\cdot\oint -\frac{\alpha'}{2}\eta^{\mu\nu}m\,z^{m+n-1}\,dz$

$= m\eta^{\mu\nu}\delta_{m,-n}$.

Similarly, $[\tilde\alpha_m^\mu,\tilde\alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n}$, $[x^\mu,p^\nu] = i\eta^{\mu\nu}$.

$T(z) = -\frac{1}{\alpha'}:\partial X^\mu\partial X_\mu: = \frac{1}{2}\sum_{m,n}\frac{:\alpha_m^\mu\alpha_{\mu n}:}{z^{m+n+2}} \Rightarrow L_m \sim \frac{1}{2}\sum_n\alpha_{m-n}^\mu\alpha_{\mu n}$.

$\parallel\ \sum_m\frac{L_m}{z^{m+2}}$

↑ ignored operator ordering.

For $m\neq0$, $\alpha_{m-n}^\mu\alpha_{\mu n}$ commutes $\Rightarrow$ "$=$". For $m=0$, $L_0 = \frac{\alpha' p^2}{4} + \sum_{n\geq1}\alpha_{-n}^\mu\alpha_{\mu n} + a$

↑ some const.

$2L_0|0;0\rangle = (L_1L_{-1}-L_{-1}L_1)|0;0\rangle = 0 \Rightarrow a = 0$

↑

So write $L_m = \frac{1}{2}\sum_n {}^\circ_\circ\alpha_{m-n}^\mu\alpha_{\mu n}{}^\circ_\circ$ $\quad\because p^\mu, \alpha_n^\mu$ annihilate $|0;0\rangle$

↑ creation - annihilation normal ordering (same as $:\ :$ in this case).

For bc CFT, $b(z) = \sum\frac{b_m}{z^{m+\lambda}}$, $c(z) = \sum\frac{c_m}{z^{m+1-\lambda}}$ $\qquad\lambda$ = holo wt of $b\in\mathbb{Z}$

↑ assume.

$\mathrm{Res}_{z\to z'}\underbrace{z^{m+\lambda-1}b(z)(z')^{n-\lambda}c(z')}_{\substack{\wr\\ z^{m+\lambda-1}(z')^{n-\lambda}\frac{1}{z-z'}}} = (z')^{m+n-1} \Rightarrow \{b_m,c_n\} = \oint z^{m+n-1}dz = \delta_{m,-n}$.

$|\downarrow\rangle$, $|\uparrow\rangle$ ground states ann. by b_n, c_n, $n>0$ s.t. $b_0|\downarrow\rangle = c_0|\uparrow\rangle = 0$, $b_0|\uparrow\rangle = |\downarrow\rangle$, $c_0|\downarrow\rangle = |\uparrow\rangle$.

$T(z) = :(\partial b)c: - \lambda\partial:bc: = -\sum(m+\lambda)\frac{:b_mc_n:}{z^{m+n+2}} + \lambda\sum(m+n+1)\frac{:b_mc_n:}{z^{m+n+2}}$

$\Rightarrow L_m = \sum_n(m\lambda-n){}^\circ_\circ b_nc_{m-n}{}^\circ_\circ + \delta_{m,0}\,a$

$2L_0|\downarrow\rangle = (L_1L_{-1}-L_{-1}\underbrace{L_1)|\downarrow\rangle}_{\substack{\parallel\\0}} = L_1(1-\lambda)b_{-1}c_0|\downarrow\rangle = \lambda(1-\lambda)b_0\underbrace{c_1b_{-1}|\uparrow\rangle}_{\substack{\parallel\\|\uparrow\rangle}} = \lambda(1-\lambda)|\downarrow\rangle$.

So $a = \frac{1}{2}\lambda(1-\lambda)$

Polyakov path integral.

Intergate over all metrics g : $\int dX\, dg\, e^{-S}$

$$S = S_X + \lambda\chi . \quad S_X = \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, \partial_a X^\mu \partial^a X_\mu \ldots \overset{\text{Euler number.}}{\overset{\swarrow}{\chi}} = \frac{1}{4\pi}\int_M d^2\sigma\sqrt{g}\, R + \frac{1}{2\pi}\int_{\partial M} ds\, k .$$

Then the weight $e^{-\lambda\chi}$ only depends on the topology.

and S is (diff × Weyl)- invariant

(for open string, we assume 

$\Rightarrow \chi = \underset{\int_{(\partial M)_{sm}} ds\, k}{\underset{\uparrow}{\tilde\chi}} + \frac{1}{4}\underset{\#\text{ of corner}}{\underset{\uparrow}{n_c}}$)

Diff: $X'^\mu(\sigma') = X^\mu(\sigma) . \quad \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b} g'_{cd} = g_{ab}$

Weyl: $X'^\mu(\sigma') = X^\mu(\sigma) , \quad g'_{ab} = e^{2w} g_{ab} .$

So we consider $Z = \int \frac{dX\, dg}{V_{\text{diff}\times\text{Weyl}}} e^{-S}$. freedom of metric $= 3 = \overset{\text{coor.}}{\overset{\downarrow}{2}} + \overset{\text{local scale.}}{\overset{\downarrow}{1}}$. Fix $g \mapsto \overset{\text{fiducial metric}}{\overset{\downarrow}{\hat g}}$

Check: any metric can be brought to the flat form (locally)

$\Rightarrow$ coor-equiv to the unit metric δ_{ab}.

(Under Weyl, $\sqrt{g'}R' = \sqrt{g}\,(R - 2\nabla^2 w)$

$\Rightarrow$ solving $2\nabla^2 w = R$ (elliptic PDE))

But a conformal transformation $z' = f(z)$ with $w = \log|f'|$ leaves the metric in the unit gauge.

Let $\zeta : g \mapsto g^\zeta$ be a coor + Weyl trans. : $g^\zeta_{ab}(\sigma') = e^{2w(\sigma)}\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}(\sigma)$.

Define the Faddeev-Popov measure by $1 = \Delta_{FP}(g)\int d\zeta\, \delta(g - \hat g^\zeta)$ ($d\zeta$: (diff × Weyl) -inv measure)

Then
$$Z[\hat g] = \int \frac{d\zeta\, dX\, dg}{V_{\text{diff}\times\text{Weyl}}}\Delta_{FP}(g)\,\delta(g - \hat g^\zeta)\, e^{-S[X.g]} = \int \frac{d\zeta\, dX^\zeta}{V_{d\times W}}\Delta_{FP}(\hat g^\zeta)\, e^{-S[X^\zeta, \hat g^\zeta]} \qquad (*)$$

$$\because \Delta_{FP}(g^\zeta)^{-1} = \int d\zeta'\, \delta(g^\zeta - \hat g^{\zeta'}) = \int d\zeta'\, \delta(g - \hat g^{\zeta^{-1}\cdot\zeta'}) = \int d\zeta''\, \delta(g - \hat g^{\zeta''}) = \Delta_{FP}(g)^{-1}.$$

We get
$$(*) = \int \frac{d\zeta\, dX}{V_{d\times W}}\Delta_{FP}(\hat g)\, e^{-S[X,\hat g]} = \int dX\, \Delta_{FP}(\hat g)\, e^{-S[X.\hat g]}.$$

For ζ near the identity. $\delta g_{ab} = 2\delta w\, g_{ab} - \nabla_a\delta\sigma_b - \nabla_b\delta\sigma_a = (2\delta w - \nabla_c\delta\sigma^c)g_{ab} - 2(P_1\delta\sigma)_{ab}$

where $(P_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - g_{ab}\nabla_c\delta\sigma^c)$ is a traceless symm 2-tensors.

$$\Rightarrow \Delta_{FP}(\hat g)^{-1} = \int d\delta w\, d\delta\sigma\ \delta\left(-(2\delta w - \hat\nabla\cdot\delta\sigma)\hat g + 2\hat P_1\delta\sigma\right)$$

Integral representation for δ.

$$= \int d\delta w\, d\beta\, d\delta\sigma\ \exp\left[2\pi i\int d^2\sigma\sqrt{\hat g}\, \beta^{ab}\left(-(2\delta w - \hat\nabla\cdot\delta\sigma)\hat g + 2\hat P_1\delta\sigma\right)_{ab}\right]$$

$$= \int \underset{\text{traceless}}{\underset{\uparrow}{d\beta'}}\, d\delta\sigma\ \exp\left[4\pi i\int d^2\sigma\sqrt{\hat g}\, \beta'^{ab}(\hat P_1\delta\sigma)_{ab}\right]$$

$$\left(\int d\delta w\ \exp\left(2\pi i\int d^2\sigma\sqrt{\hat g}\, \beta^{ab}(-2\delta w)\hat g_{ab}\right) = \delta_{\beta^{ab}\hat g_{ab}, 0}\right)$$

For a general action $S = \int d^dx\, \phi^1(x)\, \Delta \phi^2(x)$

Expand $\phi^1(x) = \sum \phi_i^1 \Phi_i^1(x)$, $\Phi_i^1(x)$ eigenfunction of Δ^* with ev. λ_i s.t. $\int d^dx\, \Phi_i^1(x) \Phi_i^2(x) = 1$

$\phi^2(x) = \sum \phi_i^2 \Phi_i^2(x)$, $\Phi_i^2(x)$ " Δ.

$$\Rightarrow \int d\phi^1 d\phi^2 \exp\left(i \int d^dx\, \phi^1 \Delta \phi^2\right) = \prod_i \int d\phi_i^1 \underbrace{d\phi_i^2 \exp(i \lambda_i \phi_i^1 \phi_i^2)}_{\parallel \text{ after integration } \; 2\pi\delta(\lambda\phi_i^1)} = \prod_i \frac{2\pi}{\lambda_i} = \left(\det \frac{\Delta}{2\pi}\right)^{-1}$$

$$= (\det \Delta)^{-1}$$

renormalized

But for Grassmann varibles ψ^1, ψ^2,

$$\int d\psi^1 d\psi^2 \exp\left(\int d^dx\, \psi^1 \Delta \psi^2\right) = \prod_i \int d\psi_i^1 d\psi_i^2 \exp(\lambda_i \psi_i^1 \psi_i^2) = \prod_i \lambda_i = \det \Delta$$

So we get $\Delta_{FP}(\hat{g}) = \int db\, dc\, e^{-S_g}$ by $\beta'_{ab} \mapsto b_{ab}$, $\delta\sigma^a \mapsto c^a$. (ghost field).

where the ghost action $S_g = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab} \hat{\nabla}^a c^b = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab} (\hat{P}_1 c)^{ab}$

$$\Rightarrow Z[\hat{g}] = \int dX\, db\, dc\; e^{-S_X - S_g} = \underbrace{(\det \hat{\nabla}^2)^{-\frac{D}{2}}}_{\int dX} \underbrace{\det \hat{P}_1}_{\int db\, dc}.$$

In conformal gauge $\hat{g}_{ab}(\sigma) = e^{2\omega(\sigma)} \delta_{ab}$, $S_g = \frac{1}{2\pi} \int d^2z \left(b_{zz} \nabla_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \nabla_z c^{\bar{z}}\right)$

$$= \frac{1}{2\pi} \int d^2z \left(\underline{b_{zz} \bar{\partial} c^z} + \underline{b_{\bar{z}\bar{z}} \partial c^{\bar{z}}}\right)$$

⇝ bc CFT with $\lambda = 2$.

⇝ $\tilde{b}\tilde{c}$ CFT with $\lambda = 2$.

Weyl anomaly.

Is $Z[g]$ independent of the choice of fiducial metric, $Z[g] = Z[g^3]$?.

We should also require $\langle \cdot \rangle_g = \langle \cdot \rangle_{g^3}$, where $\langle A \rangle_g = \int dx\, db\, dc\; e^{-S[X,b,c,g]} A$.

Recall. $T^{ab} = \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}}$ s.t. $\delta \langle A \rangle_g = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g}\, \delta g_{ab}(\sigma) \langle T^{ab}(\sigma) A \rangle_g$.

Under Weyl trans, $\delta_W \langle A \rangle_g = -\frac{1}{2\pi} \int d^2\sigma \sqrt{g}\, \delta\omega(\sigma) \langle T^a_a(\sigma) A \rangle_g \overset{?}{=} 0$

The trace T^a_a must be intrinsic, and <u>vanish in the flat case</u>.

$\Rightarrow T^a_a = a_1 R + \cancel{a_2}^{\,0}$ for some const a_1. (higher derivatives will be forbidden)

$a_1 = ?$

metric is cov-const.

In conformal gauge, $T_{z\bar z} = \frac{a_1}{2} g_{z\bar z} R \Rightarrow \nabla^{\bar z} T_{z\bar z} = \frac{a_1}{2} \nabla^{\bar z}(g_{z\bar z} R) \overset{\downarrow}{=} \frac{a_1}{2} \partial R$.

$$\nabla^{z} T_{zz} + \nabla^{\bar z} T_{\bar z z} = 0 \Rightarrow \nabla^{z} T_{zz} = -\frac{a_1}{2} \partial R \quad (*)$$

Under Weyl transformation, $\sqrt{g'} R' = \sqrt{g}(R - 2\nabla^2 w) \Rightarrow \overset{\overset{0}{\|}}{\delta\sqrt{g}} \cdot R + \overset{\overset{1}{\|}}{\sqrt{g}}\, \delta R = -2\nabla^2 \delta w$ at the flat metric δ

$\Rightarrow$ RHS of $\delta(*) = a_1 \partial \nabla^2 \delta w = 4 a_1 \partial^2 \bar\partial \delta w$. (at δ)

Under conformal trans. $z' = f(z) = z + \varepsilon v(z)$, $w = \log|\partial f|$ $\left(\Rightarrow ds'^2 = e^{2w} |\partial f|^{-2} dz' d\bar z' = dz' d\bar z' \right)$

$$\varepsilon^{-1} \delta T_{zz} = -\frac{c}{12} \partial^3 v \boxed{-2\partial v \cdot T_{zz} - v \partial T_{zz}}$$

coor. trans.

$$\delta w = \frac{1}{2} \log(1 + \varepsilon \partial v) + \frac{1}{2} \log(1 + \bar\varepsilon \bar\partial v^*) = \varepsilon \partial v + \bar\varepsilon \bar\partial v^*.$$

$\Rightarrow \delta_w T_{zz} = -\frac{c}{6} \partial^2 \delta w$. $\Rightarrow$ LHS of $\delta(*) = 2\bar\partial\left(-\frac{c}{6} \partial^2 \delta w\right) = -\frac{c}{3} \partial^2 \bar\partial \delta w$.

So $a_1 = -\frac{c}{12}$.

In bc CFT with $\lambda = 2$, central charge $c^g = -3(2\lambda - 1)^2 + 1 = -26$

Total change $= c^m + c^g = 0 \Rightarrow D = 26$.
(matter + ghost)

Remark. In conformal gauge, $R = -2\nabla^2 w = -2 e^{-2w} \partial_a \partial_a w$

$$\Rightarrow \delta_w Z[e^{2w}\delta] = \frac{a_1}{\pi} Z[e^{2w}\delta] \int d^2\sigma\, \delta w(\sigma) \partial_a \partial_a w$$

IBP

$$\Rightarrow Z[e^{2w}\delta] = Z[\delta] \exp\left(-\frac{a_1}{2\pi} \int d^2\sigma\, \partial_a w \partial_a w\right)$$

So for a general metric g,

$$Z[g] = Z[\delta] \exp\left(\frac{a_1}{8\pi} \int d^2\sigma \int d^2\sigma' \underbrace{(\sqrt{g} R)(\sigma) G(\sigma, \sigma') (\sqrt{g} R)(\sigma')}_{\text{diff-inv. expression.}}\right),$$

where G is defined by $\sqrt{g(\sigma)} \nabla^2 G(\sigma.\sigma') = \delta^2(\sigma - \sigma')$

$$\left(\because\ (\sqrt{g} R)(\sigma) \int d^2\sigma'\, G(\sigma, \sigma') (\sqrt{g} R)(\sigma') = -2 (\sqrt{g} R)(\sigma) \int d^2\sigma' \sqrt{g(\sigma')} \nabla^2 G(\sigma.\sigma') w(\sigma') \right.$$
$$\left. = -2\sqrt{g} R w(\sigma) = 4 w \cdot \partial_a \partial_a w \overset{\text{IBP}}{=} -4 \partial_a w \partial_a w \right)$$

Expand $g = \delta + h$ also gives $c = -12 a_1$.

ψ theory.

In bc CFT, $S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c$. with b weight $(\lambda, 0)$, c weight $(1-\lambda, 0)$.

If we take $\lambda = \frac{1}{2}$, then $c = -3(2\lambda-1)^2+1 = 1$.

Let $\psi = b$. $\bar{\psi} = c$. $\psi_1 = \frac{1}{\sqrt{2}}(\psi+\bar{\psi})$. $\psi_2 = \frac{i}{\sqrt{2}}(\bar{\psi}-\psi)$

$$\Rightarrow S = \frac{1}{2\pi}\int d^2z\, \frac{1}{\sqrt{2}}(\psi_1+i\psi_2)\frac{1}{\sqrt{2}}\bar{\partial}(\psi_1-i\psi_2) = \frac{1}{4\pi}\int d^2z\, \psi_1\bar{\partial}\psi_1+\psi_2\bar{\partial}\psi_2.$$

$$T = :(\partial b)c: - \lambda\partial :bc: = \frac{1}{2}:\partial(\psi_1+i\psi_2)\cdot(\psi_1-i\psi_2): - \frac{1}{4}\partial :\underbrace{(\psi_1+i\psi_2)(\psi_1-i\psi_2)}_{-2i\psi_1\psi_2}:$$

$$= -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2$$

OPE:

$\psi_1(z)\psi_1(0) \sim \frac{1}{z}$

$\psi_1(z)\psi_2(0) \sim 0$

$\psi_2(z)\psi_2(0) \sim \frac{1}{z}$

Superstrings.

mass-shell eq: $p_\mu p^\mu + m^2 = 0$.

Generalize to the Dirac eq. $ip_\mu\Gamma^\mu + m = 0$ $\left(\begin{array}{l} m^2 = -(p_\mu\Gamma^\mu)(p_\nu\Gamma^\nu) = -p_\mu p_\nu\Gamma^\mu\Gamma^\nu = -p_\mu p^\mu \\ \Rightarrow \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\ ? \end{array}\right)$

where Γ^μ is the center-of-mass mode of ψ^μ

$(p^\mu$ 〃 $(\partial X^\mu, \bar{\partial}X^\mu))$

$(\{b_m, c_n\} = \delta_{m,-n} \Rightarrow \{b_0, c_0\} = 1)$

Consider the action $S = \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right)$.

OPE: $X^\mu(z,\bar{z})X^\nu(0,0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z|^2$. $\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}$, $\tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{\bar{z}}$

$T_B = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu$, $\tilde{T}_B = -\frac{1}{\alpha'}\bar{\partial}X^\mu\bar{\partial}X_\mu - \frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu$

bosonic

supercurrents: $T_F = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu$, $\tilde{T}_F = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu\bar{\partial}X_\mu$. $(\bar{\partial}T_F = \partial\tilde{T}_F = 0)$.

OPE: $T_F(z)X^\mu(0) \sim i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial\left(-\frac{\alpha'}{2}\log|z|^2\right) = -i\sqrt{\frac{\alpha'}{2}}\frac{\psi^\mu}{z}$

$T_F(z)\psi^\mu(0) \sim i\sqrt{\frac{2}{\alpha'}}\frac{\eta^{\mu\nu}}{z}\partial X_\nu = i\sqrt{\frac{2}{\alpha'}}\frac{\partial X^\mu}{z}$.

Let $j(z) = \eta(z)T_F(z)$, $\tilde{j}(\bar{z}) = \eta(z)^*\tilde{T}_F(\bar{z})$.

Then Ward identity

$$\Rightarrow \delta X^\mu(z) = i\varepsilon\, \mathrm{Res}_z\, j(z')X^\mu(z) + i\bar{\varepsilon}\,\overline{\mathrm{Res}}_{\bar{z}}\,\tilde{j}(\bar{z}')X^\mu(\bar{z}))$$

$$= \sqrt{\frac{\alpha'}{2}}\left(\varepsilon\eta(z)\psi^\mu(z) + \bar{\varepsilon}\eta(z)^*\tilde{\psi}^\mu(\bar{z})\right)$$

$$\delta\psi^\mu(z) = -\varepsilon\sqrt{\frac{2}{\alpha'}}\eta(z)\partial X^\mu(z),\quad \delta\tilde{\psi}^\mu(\bar{z}) = -\bar{\varepsilon}\sqrt{\frac{2}{\alpha'}}\eta(z)^*\bar{\partial}X^\mu(\bar{z}).$$

ψ theory.

In bc CFT, $S = \frac{1}{2\pi}\int d^2z\, b\bar\partial c$. with b weight $(\lambda, 0)$, c weight $(1-\lambda, 0)$.

If we take $\lambda = \frac{1}{2}$, then $c = -3(2\lambda-1)^2+1 = 1$.

Let $\psi = b$. $\bar\psi = c$. $\psi_1 = \frac{1}{\sqrt2}(\psi+\bar\psi)$. $\psi_2 = \frac{i}{\sqrt2}(\bar\psi - \psi)$

$$\Rightarrow\ S = \frac{1}{2\pi}\int d^2z\ \frac{1}{\sqrt2}(\psi_1+i\psi_2)\frac{1}{\sqrt2}\bar\partial(\psi_1 - i\psi_2) = \frac{1}{4\pi}\int d^2z\ \psi_1\bar\partial\psi_1 + \psi_2\bar\partial\psi_2 .$$

$$T = :(\partial b)c: - \lambda\partial :bc: = \frac{1}{2}:\partial(\psi_1+i\psi_2)\cdot(\psi_1-i\psi_2): - \frac{1}{4}\partial:\underbrace{(\psi_1+i\psi_2)(\psi_1-i\psi_2)}_{-2i\psi_1\psi_2}:$$

$$= -\frac{1}{2}\psi_1\partial\psi_1 - \frac{1}{2}\psi_2\partial\psi_2$$

OPE:

$\psi_1(z)\psi_1(0) \sim \frac{1}{z}$

$\psi_1(z)\psi_2(0) \sim 0$

$\psi_2(z)\psi_2(0) \sim \frac{1}{z}$

Superstrings.

mass-shell eq: $p_\mu p^\mu + m^2 = 0$.

Generalize to the Dirac eq. $ip_\mu\Gamma^\mu + m = 0$ $\left(\begin{array}{l} m^2 = -(p_\mu\Gamma^\mu)(p_\nu\Gamma^\nu) = -p_\mu p_\nu\Gamma^\mu\Gamma^\nu = -p_\mu p^\mu \\ \Rightarrow \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\ ? \end{array}\right)$

where Γ^μ is the center-of-mass mode of ψ^μ $\rightarrow$ $(\{b_m, c_n\} = \delta_{m,-n} \Rightarrow \{b_0, c_0\} = 1)$

$(p^\mu$ 〃 $(\partial X^\mu, \bar\partial X^\mu))$

Consider the action $S = \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar\partial X_\mu + \psi^\mu\bar\partial\psi_\mu + \tilde\psi^\mu\partial\tilde\psi_\mu\right)$.

OPE: $X^\mu(z,\bar z)X^\nu(0,0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z|^2$. $\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}$, $\tilde\psi^\mu(\bar z)\tilde\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{\bar z}$

$T_B = -\frac{1}{\alpha'}\partial X^\mu\partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu$, $\tilde T_B = -\frac{1}{\alpha'}\bar\partial X^\mu\bar\partial X_\mu - \frac{1}{2}\tilde\psi^\mu\bar\partial\tilde\psi_\mu$

$\uparrow$ bosonic

$$T_B(z)T_B(0) = T_X(z)T_X(0) + T_\psi(z)T_\psi(0)$$

$$\sim \frac{D}{2z^4} + \frac{2}{z^2}T_X(0) + \frac{1}{z}\partial T_X(0) + \frac{D/2}{2z^4} + \frac{2}{z^2}T_\psi(0) + \frac{1}{z}\partial T_\psi(0)$$

$$= \frac{1}{2z^4}\left(\frac{3}{2}D\right) + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0).$$

Let $\Psi^\mu = \begin{pmatrix}\psi^\mu \\ \tilde\psi^\mu\end{pmatrix}$, $\underbrace{\rho^1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \rho^2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}}_{\text{Dirac matrices}}$ $\Rightarrow \begin{cases}\rho^z = \rho^1 + i\rho^2 = \begin{pmatrix}0 & 2 \\ 0 & 0\end{pmatrix} \\ \rho^{\bar z} = \rho^1 - i\rho^2 = \begin{pmatrix}0 & 0 \\ 2 & 0\end{pmatrix}\end{cases}$.

So $\psi^\mu\bar\partial\psi_\mu + \tilde\psi^\mu\partial\tilde\psi_\mu = \frac{1}{2}(\Psi^\mu)^*(\rho^z\partial + \rho^{\bar z}\bar\partial)\Psi_\mu$.

$$\Rightarrow S = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(\frac{1}{\alpha'}\partial_a X^\mu\partial^a X_\mu + i(\Psi^\mu)^*\rho^a\partial_a\Psi_\mu\right),$$

where $\rho^a = e^a_k\rho^k$, e : zweibein satisfies $g_{ab} = e^k_a e^l_b\delta_{kl}$ $\left((e^k_a)^{-1} = (e^a_k)\right)$

$e'^k_a = \frac{\partial\sigma^b}{\partial\sigma'^a} e^k_b$, $e = \det e^k_a = \sqrt{g}$, $\{\rho^a, \rho^b\} = e^a_k e^b_l \underbrace{\{\rho^k, \rho^l\}}_{\shortparallel\; 2\delta^{kl}} = 2g^{ab}$.

Eq of motion: $\partial_a\partial^a X^\mu = 0$, $\rho^a\partial_a \Psi^\mu = 0$.

S is invariant under $\delta X^\mu = i\sqrt{\frac{\alpha'}{2}}\,\varepsilon^* \Psi^\mu$, $\delta\Psi^\mu = \sqrt{\frac{2}{\alpha'}}\cdot\frac{1}{2}\rho^a\partial_a X^\mu \underset{\shortparallel}{\varepsilon}$,

but is not local SUSY. (ε being a function)

$\begin{pmatrix}\varepsilon\\ \tilde{\varepsilon}\end{pmatrix}$ constant.

Introduce grawitino χ, $\chi_a = \begin{pmatrix}\chi_a\\ \tilde{\chi}_a\end{pmatrix}$

Add $S' = \frac{1}{8\pi i}\int d^2\sigma\, e\, \chi^*_a \rho^b\rho^a \Psi^\mu \cdot \sqrt{\frac{2}{\alpha'}}\,\partial_b X_\mu$.

Get

SUSY: $\delta X^\mu = i\sqrt{\frac{\alpha'}{2}}\,\xi^* \Psi^\mu$ $\qquad \delta e^k_a = \frac{i}{2}\xi^*\rho^k \chi_a$

$\delta\Psi^\mu = \frac{1}{2}\rho^a\left(\sqrt{\frac{2}{\alpha'}}\,\partial_a X^\mu - \frac{i}{2}\chi^*_a\Psi^\mu\right)$ $\qquad \delta\chi_a = 2\nabla_a\xi$

Weyl $(g \mapsto e^{2\Lambda} g)$: $\delta_\Lambda X^\mu = 0$ $\qquad \delta_\Lambda e^k_a = \Lambda e^k_a \leftarrow$ dim -1

$\delta_\Lambda \Psi^\mu = \underline{-\frac{1}{2}}\Lambda\Psi^\mu$ $\qquad \delta_\Lambda \chi_a = \frac{1}{2}\Lambda\chi_a \leftarrow$ dim $-\frac{1}{2}$

$\because$ dim of $\Psi = \frac{1}{2}$

Super-Weyl $(\chi \mapsto \chi + \rho\eta)$: $\delta_\eta \chi_a = \underset{\shortparallel}{\rho_a}\eta$ $\qquad \delta_\eta(\text{others}) = 0$

Poincaré invariance, reparametrization $\qquad g_{ab}\rho^b = e^k_a \rho^l \delta_{kl}$

Gauge fixing: $g.\chi \overset{?}{\longmapsto} \delta, 0$. (locally).

Write $\chi_a = \underbrace{\delta^b_a}_{\shortparallel\;\frac{1}{2}(\rho_a\rho^b + \rho^b\rho_a)} \chi_b = \underbrace{\frac{1}{2}\rho^b\rho_a\chi_b}_{\shortparallel\;\chi^0_a} + \frac{1}{2}\rho_a \underbrace{\rho^b\chi_b}_{\shortparallel\;\lambda}$

For flat case, $\begin{pmatrix}\tilde{\chi}^0_1\\ \chi^0_1\end{pmatrix} = \begin{pmatrix}i\tilde{\chi}^0_2\\ -i\chi^0_2\end{pmatrix}$

$\Rightarrow \begin{pmatrix}\chi^0_1\\ \tilde{\chi}^0_1\end{pmatrix} = \begin{pmatrix}(\nabla_1 - i\nabla_2)\kappa\\ (\nabla_1 + i\nabla_2)\kappa\end{pmatrix}, \begin{pmatrix}\chi^0_2\\ \tilde{\chi}^0_2\end{pmatrix} = \begin{pmatrix}(i\nabla_1 + \nabla_2)\kappa\\ (-i\nabla_1 + \nabla_2)\kappa\end{pmatrix}$

$\Rightarrow \chi^0$ is ρ-traceless: $\rho^a\chi^0_a = \frac{1}{2}\underbrace{\rho^a\rho^b\rho_a}_{\shortparallel}\chi_b = 0 \Rightarrow \chi^0_a = \rho^b\rho_a\nabla_b\kappa$ (locally)

$\underbrace{2g^{ab}\rho_a}_{\shortparallel\;\rho^b} - \rho^b \underbrace{\rho^a\rho_a}_{\shortparallel\;2} = 0$

SUSY $\Rightarrow \delta_\xi\chi_a = 2\nabla_a\xi = \rho^b\rho_a\nabla_b\xi + \rho_a\rho^b\nabla_b\xi \Rightarrow$ eliminate κ and $\chi \mapsto \frac{1}{2}\rho\lambda$

reparametrization $\Rightarrow g = e^{2w}\delta$ (and $\chi = \frac{1}{2}\rho\lambda \mapsto \frac{1}{2}\rho\lambda$ since ρ_a is coor. trans.-inv)

Weyl trans $\Rightarrow g \mapsto \delta$, $\chi \mapsto \frac{1}{2}e^{-w}\rho\lambda = \frac{1}{2}\rho\lambda'$.

Super-Weyl $\Rightarrow \chi = \frac{1}{2}\rho\lambda' \mapsto 0$.

$$Z = \int \frac{dX\, d\Psi\, dg\, d\chi}{V} e^{-S} = \int dX\, d\Psi\, \Delta_{FP}(\hat{g})\, \Delta_{FP}(\hat{\chi})\, e^{-S}$$

$\Delta_{FP}(\hat{g}) = e^{-S_{bc}}$, where $S_{bc} = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c$ (in conformal gauge).

$$\delta\chi_a = \delta_\xi \chi_a + \delta_\eta \chi_a = 2\nabla_a\xi + \rho_a\eta = \rho_a\eta + \rho_a\rho^b\nabla_b\xi + \underbrace{(P_{\frac{1}{2}}\xi)_a}_{\substack{2\nabla_a\xi - \rho_a\rho^b\nabla_b\xi \\ \rho\text{-traceless.}}}$$

$$\Delta_{FP}(\hat{\chi})^{-1} = \int d\xi\, \delta(\hat{\chi} - \hat{\chi}^\xi)$$

$$= \int d\eta\, d\xi\, \left(-\hat{\rho}_a\eta - \hat{\rho}_a\hat{\rho}^b\hat{\nabla}_b\xi - (\hat{P}_{\frac{1}{2}}\xi)_a\right)$$

$$= \int d\tau\, d\eta\, d\xi\, \exp\left(2\pi\int d^2\sigma\sqrt{g}\,(\tau^a)^*\left(-\hat{\rho}_a\eta - \hat{\rho}_a\hat{\rho}^b\hat{\nabla}_b\xi - (\hat{P}_{\frac{1}{2}}\xi)_a\right)\right)$$

$$= \int \underset{\substack{\uparrow \\ (\tau^a)^*\hat{\rho}_a = 0}}{d\tau'}\, d\xi\, \exp\left(-2\pi\int d^2\sigma\sqrt{g}\,(\tau^a)^*(\hat{P}_{\frac{1}{2}}\xi)_a\right)$$

$\tau' \mapsto \beta,\ \xi \mapsto \bar{\gamma} \Rightarrow \Delta_{FP}(\hat{\chi}) = \int d\beta\, d\gamma\, e^{-S_{\beta\gamma}}$,

where $S_{\beta\gamma} = \frac{i}{2\pi}\int d^2\sigma\sqrt{g}\,(\beta^a)^*(\hat{P}_{\frac{1}{2}}\bar{\gamma})_a$

In conformal gauge. $g(z,\bar{z}) = e^{2w}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow e = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix}$.

$$\Rightarrow (P_{\frac{1}{2}}\gamma)_z = 2\underbrace{\nabla_z}_{e^{-w}\partial}\bar{\gamma} - \underbrace{\rho_z\rho^b\nabla_b}_{(\frac{1}{2}\rho^{\bar{z}})(\rho^z\nabla_z + \rho^{\bar{z}}\nabla_{\bar{z}}) = \begin{pmatrix} 0 & 0 \\ e^{-w} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2\partial \\ 2\bar{\partial} & 0 \end{pmatrix}}\bar{\gamma} = 2e^{-w}\begin{pmatrix} \partial\tilde{\gamma} \\ 0 \end{pmatrix}$$

$$(P_{\frac{1}{2}}\gamma)_{\bar{z}} = 2e^{-w}\begin{pmatrix} 0 \\ \bar{\partial}\gamma \end{pmatrix}$$

$$\Rightarrow S_{\beta\gamma} = \frac{1}{2\pi}\int d^2z\,(\tilde{\beta}^z\partial\tilde{\gamma} + \beta^{\bar{z}}\bar{\partial}\gamma)$$

ξ, and hence $\gamma, \tilde{\gamma}$ has dim $-\frac{1}{2}$ $\Rightarrow$ $S_{\beta\gamma}$ is a $\beta\gamma$ CFT with $\lambda = \frac{3}{2}$.

Weyl anomaly $\Rightarrow$ total central charge $\underbrace{c^X + c^\psi}_{\frac{3}{2}D} + \underbrace{c^{bc}}_{-26} + \underbrace{c^{\beta\gamma}}_{3(2\cdot\frac{3}{2}-1)^2 - 1 = 11} = 0$

$\Rightarrow D = 10.$

Energy momentum tensor $(T_B)_{ab} = \frac{2\pi}{e}\frac{\delta S}{\delta e^b_k} e^l_a \delta_{kl} = \frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{ab}}$ (as before)

(Using $\frac{\partial g^{ab}}{\partial e^c_k} = \frac{\partial}{\partial e^c_k}(e^a_l e^b_m \delta^{lm}) = \delta^a_c e^{bk} + e^{ak}\delta^b_c$, we get

$(T_B)_{ab} = \frac{2\pi}{e}\frac{\delta S}{\delta g^{cd}}\frac{\partial g^{cd}}{\partial e^b_k} e^l_a \delta_{kl} = \frac{2\pi}{e}\frac{\delta S}{\delta g^{cd}}(\delta^c_b\delta^d_a + \delta^c_a\delta^d_b) = \frac{4\pi}{e}\frac{\delta S}{\delta g^{ab}}$)

Analogously, we define the supercurrent

$$(T_F)_a = \frac{2\pi}{e}\frac{\delta S}{i\delta(\chi^a)^*} = -\frac{1}{4}\frac{\delta}{\delta(\chi^a)^*}\left(\chi^*_a\rho^b\rho^a\Psi^\mu\sqrt{\frac{2}{\alpha'}}\partial_b X_\mu\right) = -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^b\rho_a\Psi^\mu\partial_b X_\mu.$$

In flat metric, $(T_F)_z = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\begin{pmatrix}\psi^\mu\partial X_\mu\\ 0\end{pmatrix}$, $(T_F)_{\bar z} = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\begin{pmatrix}0\\ \tilde\psi^\mu\bar\partial X_\mu\end{pmatrix}$

Define $T_F = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial X_\mu$, $\tilde T_F = i\sqrt{\frac{2}{\alpha'}}\tilde\psi^\mu\bar\partial X_\mu$ $(\bar\partial T_F = \partial\tilde T_F = 0)$.

OPE: $T_F(z)X^\mu(0) \sim i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial\left(-\frac{\alpha'}{2}\log|z|^2\right) \sim -i\sqrt{\frac{\alpha'}{2}}\frac{\psi^\mu(0)}{z}$

$T_F(z)\psi^\mu(0) \sim i\sqrt{\frac{2}{\alpha'}}\frac{\eta^{\mu\nu}}{z}\partial X_\nu(z) \sim i\sqrt{\frac{2}{\alpha'}}\frac{\partial X^\mu(0)}{z}$

Let $j(z) = \eta(z)T_F(z)$, $\tilde j(\bar z) = \eta(z)^*\tilde T_F(\bar z)$.

Then Word identity $\Rightarrow \delta X^\mu(z) = i\varepsilon\,\mathrm{Res}_z\, j(z')X^\mu(z) + i\bar\varepsilon\,\overline{\mathrm{Res}}_{\bar z}\,\tilde j(\bar z')X^\mu(\bar z)$

$= \sqrt{\frac{\alpha'}{2}}\left(\varepsilon\eta(z)\psi^\mu(z) + \bar\varepsilon\eta(z)^*\tilde\psi^\mu(\bar z)\right)$

$\delta\psi^\mu(z) = -\varepsilon\sqrt{\frac{2}{\alpha'}}\eta(z)\partial X^\mu(z)$, $\delta\tilde\psi^\mu(\bar z) = -\bar\varepsilon\sqrt{\frac{2}{\alpha'}}\eta(z)^*\bar\partial X^\mu(\bar z)$

$T_B(z)T_F(0) \sim \frac{3}{2z^3}T_F(0) + \frac{1}{z}\partial T_F(0) \Rightarrow T_F$ is a tensor of wt $(\frac{3}{2}, 0)$

The matter fermion action $\frac{1}{4\pi}\int d^2w\,(\psi^\mu\bar\partial\psi_\mu + \tilde\psi^\mu\partial\tilde\psi_\mu)$ is inv. under $w \sim w + 2\pi$

eq of motion $\Rightarrow \psi^\mu(w+2\pi) = \underbrace{\psi^\mu(w)}_{R}$ or $\underbrace{-\psi^\mu(w)}_{NS} = e^{2\pi i\nu}\psi^\mu(w)$

$(\nu = 0$ or $\frac{1}{2})$

So $\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu}\frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}$, $\tilde\psi^\mu(\bar z) = \sum_{r\in\mathbb{Z}+\tilde\nu}\frac{\tilde\psi^\mu_r}{\bar z^{r+\frac{1}{2}}}$

$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{\alpha^\mu_m}{z^{m+1}}$, $\bar\partial X^\mu(\bar z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{\tilde\alpha^\mu_m}{\bar z^{m+1}}$.

OPE $\Rightarrow \{\psi^\mu_r, \psi^\nu_s\} = \{\tilde\psi^\mu_r, \tilde\psi^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}$, $[\alpha^\mu_n, \alpha^\nu_n] = [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m,-n}$

Let $T_F(z) = \sum_{r\in\mathbb{Z}+v} \frac{G_r}{z^{r+\frac{3}{2}}}$, $\tilde{T}_F(z) = \sum_{r\in\mathbb{Z}+\tilde{v}} \frac{\tilde{G}_r}{\bar{z}^{r+\frac{3}{2}}}$

Then
$$\left.\begin{aligned}
[L_m, L_n] &= (m-n) L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r,-s} \\
[L_m, G_r] &= \frac{m-2r}{2} G_{m+r}.
\end{aligned}\right) \text{super Virasoro algebra.}$$