Critical Dimension of Bosonic String Theory is 26. 2021. 12. 27.

Reference : J. Polchinski, String Theory vol. 1, chapter 1.

$\eta_{\mu\nu} = \mathrm{diag}(-,+,\dots,+)$: metric on D-dimensional spacetime, $X^0, X^1, \dots, X^{D-1}$.

0-dim object : particle position given by D-1 functions of time $\mathbf{X}(X^0)$.

τ : world-line parameter (time) $\rightsquigarrow$ motion of particle $\mathbf{X}^\mu(\tau)$.

translation & Lorentz invariant.

The simplest Poincaré-invariant and world-line reparameterization-invariant action is

$S_{pp} = -m \int d\tau \, (-\dot{X}^\mu \dot{X}_\mu)^{1/2}$, m : particle mass.

$\rightsquigarrow$ variation $\delta S_{pp} = -m \int d\tau \, \dot{u}_\mu \, \delta X^\mu$, where $u^\mu = \dot{X}^\mu (-\dot{X}^\nu \dot{X}_\nu)^{-1/2}$: normalized D-velocity.

Equation of motion : $\dot{u}^\mu = 0$.

Introduce a world-line matric $\gamma_{\tau\tau}(\tau)$ and tetrad $\eta(\tau) = (-\gamma_{\tau\tau}(\tau))^{1/2} > 0$.

Consider new action $S'_{pp} = \frac{1}{2} \int d\tau \left(\eta^{-1} \dot{X}^\mu \dot{X}_\mu - \eta m^2 \right)$: also Poincaré & reparameterization invariant. $\eta(\tau)d\tau = \eta'(\tau')d\tau'$

$\xrightarrow{\text{varying tetrad}}$ Equation of motion $\eta^2 = -\frac{1}{m^2} \dot{X}^\mu \dot{X}_\mu$. ($\Rightarrow S'_{pp}$ recover S_{pp}.)

S'_{pp} is better than S_{pp} : ① make sense for massless ($m=0$) particles.

② make sense for path integral formulation.

1-dim object World-sheet parameter τ, σ $\rightsquigarrow$ $X^\mu(\tau,\sigma)$.

Simplest reparameterization invariant action : Nambu-Goto action. $h_{ab} = \partial_a X^\mu \partial_b X_\mu$.

$$S_{NG} := \int d\tau d\sigma \, \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} := -\frac{1}{2\pi\alpha'} (-\det h_{ab})^{1/2}.$$

α' : constant called Regge slope. The tension of the string $T = \frac{1}{2\pi\alpha'}$.

S_{NG} is Poincaré invariant : $S_{NG}[X'] = S_{NG}[X]$ for all $X'^\mu(\tau,\sigma) = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz transform}} X^\nu(\tau,\sigma) + \underbrace{a^\mu}_{\text{translation}}$.

and reparameterization invariant $X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma)$.

To simplify S_{NG}, we introduce world-sheet metric $\gamma_{ab}(\tau,\sigma)$ s.t. γ_{ab} has signature $(-,+)$.

(From now on, all the indices raised/lowered by this)

The Polyakov action is

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \, (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \quad \gamma := \det \gamma_{ab}.$$

S_P is equivalent to S_{NG} : Take variation on metric γ,

$$\delta_\gamma S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma (-\gamma)^{1/2}\, \delta\gamma^{ab}\left(h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}\right).$$

$$\delta_\gamma S_P = 0 \Rightarrow h_{ab} = \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} \Rightarrow h_{ab}\cdot(-h)^{-1/2} = \gamma_{ab}\cdot(-\gamma)^{-1/2}.$$

$$\longrightarrow S_P[X,\gamma] \text{ becomes } -\frac{1}{2\pi\alpha'}\int d\tau d\sigma (-h)^{1/2} = S_{NG}[X]. \qquad \square$$

Now, S_P has symmetries :

① D-dimensional Poincaré invariance $X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu$, $\gamma'_{ab}(\tau,\sigma) = \gamma_{ab}(\tau,\sigma)$

② Diff invariance $X'^\mu(\tau',\sigma') = X(\tau,\sigma)$, $\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma)$

③ Weyl invariance : $X'^\mu(\tau,\sigma) = X(\tau,\sigma)$, $\gamma'_{ab}(\tau,\sigma) = \exp(2\omega(\tau,\sigma))\,\gamma_{ab}(\tau,\sigma)$.

(not appear in NG-action : up to local scaling, γ_{ab} gives "same" spacetime embedding.)

The energy-momentum tensor is

$$T^{ab}(\tau,\sigma) := -4\pi(-\gamma)^{-1/2}\frac{\delta}{\delta\gamma_{ab}}S_P$$
$$= -\frac{1}{\alpha'}\left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu\right)$$

diff invariance $\Rightarrow \nabla_a T^{ab} = 0$. Weyl invariance $\Rightarrow \gamma_{ab}\frac{\delta}{\delta\gamma_{ab}}S_P = 0 \Rightarrow T_a{}^a = 0$.

Varying γ_{ab} in S_P gives eq. of motion $T_{ab} = 0$.

Varying X^μ gives $\partial_a\left((-\gamma)^{1/2}\gamma^{ab}\gamma_b X^\mu\right) = (-\gamma)^{1/2}\nabla^2 X^\mu = 0$ under some boundary conditions :

assume $(\tau,\sigma)\in\mathbb{R}\times[0,\ell]$, then

$$\delta S_P = \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^\ell d\sigma (-\gamma)^{1/2}\delta X^\mu \nabla^2 X_\mu - \underbrace{\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau (-\gamma)^{1/2}\delta X^\mu \partial^\sigma X_\mu \Big|_{\sigma=0}^{\sigma=\ell}}_{(*)}.$$

- If $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\ell) = 0$, then $(*) = 0$. (open string condition)
- Alternatively, if $X^\mu(\tau,\ell) = X^\mu(\tau,0)$, $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\ell)$, $\gamma_{ab}(\tau,\ell) = \gamma(\tau,0)$, then $(*)$ vanishes as well. i.e. X is periodic. (closed string condition)

(Under Poincaré invariant, we must have these conditions!)

Open String Spectrum

Consider light-cone coordinate $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, x^i, $i = 2, 3 \cdots D-1$.

Under this coordinate, metric : $a^\mu b_\mu = -a^+ b^- - a^- b^+ + a^i b^i$

$a_- = -a^+$, $a_+ = -a^-$, $a_i = a^i$.

Particle case : Make a Gauge choice : $X^+(\tau) = \tau$.

The action $S'_{pp} = \frac{1}{2}\int d\tau \left(-2\eta^{-1}\dot{X}^- + \eta^{-1}\dot{X}^i\dot{X}^i - \eta m^2\right)$

Canonical momenta : $P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} \Rightarrow P_- = -\eta^{-1}$, $P_i = \eta^{-1}\dot{X}^i$

$\rightsquigarrow$ Hamiltonian $H = P_- \dot{X}^- + P_i \dot{X}^i - L = \frac{p^i p^i + m^2}{2p^+}$ $(**)$.

To quantize, we impose conditions $[P_i, X^j] = -i\delta_i^j$, $[P_-, X^-] = -i$.

Gauge choice relates τ and X^+ translations, so $H = -P_+ = P^-$.

$\rightsquigarrow$ $(**)$ becomes $-2p^+p^- + p^i p^i + m^2 = 0$, i.e. $P_\mu P^\mu + m^2 = 0$. (mass-shell equation)

Open string case : Make a Gauge choice
$$\begin{cases} X^+ = \tau & \cdots (a) \\ \partial_\sigma \gamma_{\sigma\sigma} = 0 & \cdots (b) \\ \det \gamma_{ab} = -1 & \cdots (c) \end{cases}$$

How to get these? Choose (a) by changing coordinates.

Now, $f = \gamma_{\sigma\sigma}(-\det \gamma_{ab})^{-1/2}$ transform as $f' d\sigma' = f d\sigma$ (reparameterize σ.)

$\rightsquigarrow$ invariant length $f d\sigma = dl$. Reparameterize σ by arclength $\int dl$

$\rightsquigarrow$ $f = \frac{dl}{d\sigma}$ is independent of σ.

Next, use Weyl transformation to make (c). Thus, since f is Weyl-invariant, $\partial_\sigma f = 0$. Also, $\det(\gamma_{ab}) = -1$ gives $\partial_\sigma \gamma_{\sigma\sigma} = 0$ $\square$

The metric becomes
$$\begin{bmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{bmatrix} \underset{(c)}{=} \begin{bmatrix} -\gamma_{\sigma\sigma}(\tau)^{\,(b)} & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & \gamma_{\sigma\sigma}^{-1}(\tau)\left(1 - \gamma_{\tau\sigma}^2(\tau,\sigma)\right)_{(c)} \end{bmatrix}$$

Polyakov Lagrangian becomes

$$L = -\frac{1}{4\pi\alpha'}\int_0^l d\sigma \Big[\gamma_{\sigma\sigma}\left(2\partial_\tau x^- - \partial_\tau X^i \partial_\tau X^i\right) - 2\gamma_{\sigma\tau}\left(\partial_\sigma Y^- - \partial_\tau X^i \partial_\sigma X^i\right)$$
$$+ \gamma_{\sigma\sigma}^{-1}\left(1 - \gamma_{\tau\sigma}^2\right)\partial_\sigma X^i \partial_\sigma X^i\Big]$$

where $X^-(\tau,\sigma) = x^-(\tau) + Y^-(\tau,\sigma)$, $x^-(\tau) = \frac{1}{l}\int_0^l d\sigma\, X^-(\tau,\sigma)$.

open string condition $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\ell) = 0 \Rightarrow \gamma_{\tau\sigma}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0$ at $\sigma = 0, \ell$.

$\mu = "+" \Rightarrow \gamma_{\tau\sigma} = 0$ at $\sigma = 0, \ell$. Also, $\partial_\sigma^2 \gamma_{\tau\sigma} = 0 \Rightarrow \gamma_{\tau\sigma} \equiv 0$.

$\mu = i \Rightarrow \partial_\sigma X^i = 0$ at $\sigma = 0, \ell$.

$$L = -\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma \left(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\right)$$

only $x^-(\tau)$, $\gamma_{\sigma\sigma}(\tau)$, $X^i(\tau,\sigma)$.

- momentum conjugate $p_- = -p^+ = \frac{\partial L}{\partial(\partial_\tau x^-)} = -\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}$.
- momentum density conjugate $\Pi^i = \frac{\delta L}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p^+}{\ell}\partial_\tau X^i$.

$\rightsquigarrow$ Hamiltonian $H = p_-\partial_\tau x^- - L + \int_0^\ell d\sigma\, \Pi_i \partial_\tau X^i$

$$= \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma \left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma X^i\right)$$

$\rightsquigarrow$ eq. of motion $\partial_\tau x^- = \frac{\partial H}{\partial p_-} = \frac{H}{p^+}$, $\partial_\tau p^+ = \frac{\partial H}{\partial x^-} = 0$, $\quad c = \frac{\ell}{2\pi\alpha' p^+}$.

$$\partial_\tau X^i = \frac{\delta H}{\delta \Pi^i} = 2\pi\alpha' c\,\Pi^i, \quad \partial_\tau \Pi^i = -\frac{\delta H}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i$$

$\Rightarrow$ wave eq: $\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i$.

Solve it with boundary condition

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{n}\alpha_n^i \exp\left(-\frac{\pi i n c\tau}{\ell}\right)\cos\left(\frac{\pi n \sigma}{\ell}\right).$$

$\alpha_{-n}^i := (\alpha_n^i)^\dagger$.

where $x^i(\tau) = \frac{1}{\ell}\int_0^\ell d\sigma\, X^i(\tau,\sigma)$ and $p^i(\tau) = \int_0^\ell d\sigma\, \Pi^i(\tau,\sigma)$.

To quantize, we impose $[x^-, p^+] = -i\eta^{-+} = -i$ i.e. $[x^i, p^j] = i\delta^{ij}$

$[X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma-\sigma')$. $\quad [\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n}$

$\rightsquigarrow$ eigenstate $|0;k\rangle$, $k = (k^+, k^i)$: $p^+|0;k\rangle = k^+|0;k\rangle$, $p^i|0;k\rangle = k^i|0;k\rangle$

$\alpha_m^i|0;k\rangle = 0$ for $m > 0$.

general state : $|N;k\rangle = \left(\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha_{-n}^i)^{N_{in}}}{(n^{N_{in}}N_{in}!)^{1/2}}\right)|0;k\rangle$,

Now, Hamiltonian becomes $H = \frac{p^ip^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \underline{A}\right)$, $A = \frac{D-2}{2}\sum_{n=1}^{\infty} n = \frac{2-D}{24}$.

commutators

The Guage $X^+(\tau) = \tau$ gives $p^- = H$. Then, the mass-shell eq. becomes

$m^2 = 2p^+H - p^ip^i = \frac{1}{\alpha'}\left(N + \frac{2-D}{24}\right)$, N is level $N = \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} n\cdot N_{in}$.

Now, we look at states.

- The lightest state is $|0;k\rangle$, $m^2 = \frac{2-D}{24}$.

  $m^2 < 0$ if $D > 2$. Such states is called "tachyon". Some physical reason

  $\longrightarrow$ we ignore it.

- The next lowest states are $n=1$ modes: $\underline{\alpha_{-1}^i |0;k\rangle}$, $m^2 = \frac{26-D}{24\alpha'}$

  there are $D-2$ many.

Now, we need to use Lorentz invariance:

(1) For a massive particle, one goes to the rest frame $p^\mu = (m.0\ldots0)$.

$\rightsquigarrow$ these states form a representation of $SO(D-1)$.

(2) For a massless particle, choose $p^\mu = (E.E.0\ldots0)$. $SO(D-2)$ action on them.

The massive particle has $D-1$ spin states;

the massless particle has $D-2$ spin states.

There are only $D-2$ many states at level 1: $\alpha_{-1}^i |0;k\rangle$.

$\Rightarrow$ they are all massless, i.e. $m=0$ $\Rightarrow$ $D=26$. $A=-1$

Closed strings

Very similar to open strings Impose Gauge $\begin{cases} X^+ = \tau & \cdots (a) \\ \partial_\sigma \gamma_{\sigma\sigma} = 0 & \cdots (b) \\ \det \gamma_{ab} = -1 & \cdots (c) \end{cases}$. But now, we

have an extra freedom: $\sigma' = \sigma + s(\tau)$ (mod ℓ). we can impose $\gamma_{\tau\sigma}(\tau, 0) = 0$.

Then, the remaining gauge freedom is $\sigma' = \sigma + s$ (mod ℓ).

The Lagrangian, canonical momenta, Hamiltonian, eq. of motion are the same as in the open string. Now, the solution to wave eq. becomes

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i\left(\frac{\alpha'}{2}\right)^{1/2} \cdot \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\underbrace{\frac{\alpha_n^i}{n} \exp\left(-\frac{2\pi i n(\sigma + c\tau)}{\ell}\right)}_{\text{left - move}} + \underbrace{\frac{\tilde{\alpha}_n^i}{n} \exp\left(\frac{2\pi i n(\sigma - c\tau)}{\ell}\right)}_{\text{right - move}} \right).$$

$\alpha_n^i, \tilde{\alpha}_n^i, x^i, p^i, x^-, p^+$ has commutation relation $[x^-, p^+] = -i$, $[x^i, p^j] = i\delta^{ij}$

$[\alpha_m^i, \alpha_n^i] = m\delta^{ij}\delta_{m,-n} = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^i]$

Ground state $|0,0;k\rangle$ annihilated by α_m^i and $\tilde{\alpha}_m^i$ for $m>0$.

Mass formula: $m^2 = 2p^+H - p^ip^i = \frac{2}{\alpha'}\left[\sum_{n=1}^{\infty}(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i) + A + \tilde{A}\right]$

$$= \frac{2}{\alpha'}\Big(N + \tilde{N} + \underbrace{A + \tilde{A}}_{\text{commutators} = \frac{2-D}{24}}\Big)$$

Note that we still have the Gauge freedom $\sigma' = \sigma + s \pmod{\ell}$.
The spectrum is obtained by restricting to Gauge-invariant states.
The operator generate σ-translation is

$$P = -\int_0^\ell d\sigma\, \Pi^i \partial_\sigma X^i = -\frac{2\pi}{\ell}\left[\sum_{n=1}^{\infty}\left(\alpha_{-n}^i \alpha_n^i - \tilde{\alpha}_{-n}^i \alpha_n^i\right) + A - \tilde{A}\right]$$

$$= -\frac{2\pi}{\ell}\left(N - \tilde{N}\right)$$

$\Rightarrow N = \tilde{N}$.

Now, we look at states.

- The lightest state is $|0,0;k\rangle$, $m^2 = \frac{2-D}{6\alpha'}$. $\rightsquigarrow$ "tachyon".
- The next lowest states are $N = \tilde{N} = 1$ modes:

  $\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0,0;k\rangle$, $m^2 = \frac{26-D}{6\alpha'}$.

  $i, j = 2, \dots, D-1$, there are $(D-2)^2$ states. If $m > 0$, they fit into $SO(D-1)$-representation.

  But this is impossible: $(D-1) \nmid (D-2)^2$.

$\Rightarrow m = 0 \Rightarrow A = \tilde{A} = -1$, $D = 26$. (Wigner's classification.)

Critical Dimension of Bosonic String Theory is 26. 2021. 1. 3.

Reference : D. Freed, Determinants, Torsion, and Strings.

Determinant :

If $V \xrightarrow{D} W$: linear map. $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = n$.

$\rightsquigarrow \det(D) : \Lambda^n V \longrightarrow \Lambda^n W \rightsquigarrow$ regard $\det(D) \in (\det V^*) \otimes (\det W)$.

By the exact sequence $0 \longrightarrow \ker D \longrightarrow V \xrightarrow{D} W \longrightarrow \operatorname{coker} D \longrightarrow 0$, we have $(\det V)^* \otimes (\det W) \simeq (\det \ker D)^* \otimes (\det \operatorname{coker} D)$.

Use this, in general, we can define $(\det D)$ for $D : V \longrightarrow W$ having f.d. $\ker D$, $\operatorname{coker} D$

Now, consider a smooth family of operators $D_y : V_y \longrightarrow W_y$ with $\dim \ker D_y = \dim \operatorname{coker} D_y$.

Each $y \in Y$ gives a point on $(\det V_y)^* \otimes (\det W_y)$

$\xrightarrow[\text{together}]{\text{patch}}$ a section $\det(D)$ of a line bundle $\mathcal{L} \longrightarrow Y$.

We will mainly concern the Dirac operators on cpt. manifolds.

Def The <u>geometric anomaly</u> is the obstruction to trivializing the connection on $\mathcal{L}$.

i.e. to find a global flat non-zero section.

Examples

(1) (Gravitational anomalies)

X : cpt. spin manifold with $\dim X = n = 2r$.

$\mathrm{Met}(X)$: space of Riemannian metric on X. $GL(X)$: frame bundle.

α : spin structure on $X \rightsquigarrow \alpha$ gives a unique double cover $\widetilde{GL}(X) \longrightarrow GL(X)$.

$\mathcal{F} := \{ (g, f) \in \mathrm{Met}(X) \times \widetilde{GL}(X) \mid \text{frame } f \text{ is orthonormal under } g \}$.

$\rightsquigarrow$ $\begin{array}{c} \mathcal{F} \\ \downarrow \\ \mathrm{Met}(X) \times X \end{array}$: principal $\mathrm{Spin}(n)$-bundle. Restriction $\begin{array}{c} \mathcal{F}|_{\{g\} \times X} \\ \downarrow \\ \{g\} \times X \end{array}$ is just bundle of spin frames on X with metric g.

Half-spinor representation $\underbrace{\sigma_\pm}$ of $\mathrm{Spin}(n)$ applied to $\mathcal{F}$ $\rightsquigarrow$ $\begin{array}{c} E_\pm \\ \downarrow \\ \mathrm{Met}(X) \times X \end{array}$

Λ^{even}, Λ^{odd}

Restriction on $\{g\} \times X$ is just positive / negative spinor over X.

For each g, we can define chiral Dirac operator as usual

$\rightsquigarrow$ family of chiral Dirac operators (parameterized by $\mathrm{Met}(X)$).

$\mathrm{Diff}(X) \curvearrowright \underbrace{\mathrm{Met}(X)}_{\text{by pull back metric}} \times \underbrace{X}_{\text{natural action}}$. We restrict to $\mathrm{Diff}_\alpha(X) = \{ \text{diffeo. of } X \text{ preserving } \alpha \}$.

$\mathrm{Diff}_\alpha(X) \curvearrowright \mathrm{Met}(X) \times X$ lifts to action on $\mathcal{F}$.

Take quotient $\rightsquigarrow$ $Z = \mathrm{Met}(X) \times X / \mathrm{Diff}_\alpha(X)$. Along the fiber, spin structure is consistent

$$\begin{array}{c} Z = \mathrm{Met}(X)\times X/\mathrm{Diff}_\alpha(X) \\ \downarrow X \\ Y = \mathrm{Met}(X)/\mathrm{Diff}_\alpha(X) \end{array}$$

$\rightsquigarrow$ spinors are defined.

Metric along fiber is preserved by $\mathrm{Diff}_\alpha(X)$ $\rightsquigarrow$ Z carries metric along fibers and there is a projection $P: TZ \longrightarrow T_{\mathrm{vert}} Z$.

Rmk: $\mathrm{Diff}_\alpha(X)$ acts on $\mathrm{Met}(X)$ may not free.
The isotropy group at $g \in \mathrm{Met}(X)$ is the isometry group of g.
If X is Riemann surface of genus ≥ 2, this group for metrics of curvature -1 is finite. Then, Y is an orbitfold.

(2) (σ-model)

(X, g) : cpt. spin manifold with $\dim X = 2r$. M : arbitrary manifold.

$E \to M$: Hermitian v.b. with unitary connection $\nabla^{(E)}$. $Y := \mathrm{Map}(X, M)$: mapping space.

For each $\varphi \in Y$, we have pull back bundle $\varphi^* E \to X$ with connection $\varphi^* \nabla^{(E)}$.

Dirac operator on X with $\varphi^* \nabla^{(E)}$ gives a Dirac operator on $\varphi^* E$-valued spinor field.
$\rightsquigarrow$ family parameterized by Y.

Set $Z = Y \times X$ $\rightsquigarrow$

$$\begin{tikzcd} (\mathbf{E}, \nabla^{(\mathbf{E})}) \arrow[r] \arrow[d] & (E, \nabla^{(E)}) \arrow[d] \\ Z \arrow[r, "e"] \arrow[d, "X"] & M \\ Y & \end{tikzcd}$$

evaluation map

(3) Polyakov formulation of string theory is the combination of these two examples : $X = \Sigma$: Riemann surface, bosonic fields are $\mathrm{Met}(\Sigma)$, $\mathrm{Map}(\Sigma, M)$.

(Geometric data)

(1) A smooth fibration $\pi : Z \xrightarrow{X} Y$ with $\dim X = n$ even s.t. the tangent bundle along fibers $T_{vert} Z \longrightarrow Z$ has fixed spin structure.

(In order to define family of Dirac operators. In family of $\bar{\partial}$, this is irrelevant.)

(2) A metric along fibers : $g^{(T_{vert} Z)}$ on $T_{vert} Z$.

(3) A projection $P : TZ \longrightarrow T_{vert} Z$ with kernel = horizontal complements to vertical tangent space

(4) A complex representation ρ of $Spin(n)$.

(To determine a bundle $V_\rho \longrightarrow Z$ with Dirac operator)

(5) A complex v.b. $E \longrightarrow Z$ with hermitian metric $g^{(E)}$ and connection $\nabla^{(E)}$.

Note : (2) & (3) determine a connection $\nabla^{(T_{vert} Z)}$ along fibers of $Z \longrightarrow Y$ by projection of Levi-Civita connection on Z via choosing arbitrary metric on Y. $\Omega^{(T_{vert} Z)}$: curvature.

<u>Thm</u> (Bismut - Freed) The geometric data determine functorially a smooth determinant line bundle $\mathcal{L} \longrightarrow Y$. It carries the Quillen metric $g^{(\mathcal{L})}$ and compatible connection $\nabla^{(\mathcal{L})}$. Moreover, if the Dirac operator have index 0, there is a section $\det D$ of $\mathcal{L}$.

sketch : Construct $\mathcal{L} \longrightarrow Y$ by patching. For jumping kernels, throw them in a finite dimensional space of "low eigenmode" for Laplacian D^*D, DD^*.

Then, use
$$\begin{cases} 0 \to \ker D \to V \xrightarrow{D} W \to \operatorname{coker} D \to 0 \\ (\det V)^* \otimes (\det W) \simeq (\det \ker D)^* \otimes (\det \operatorname{coker} D) \end{cases}$$
to patch these low eigenmode bundle.

For $y \in Y$ where D is invertible, $\det D$ trivialize $\mathcal{L}$ and $\| \det D \|^2_{(\mathcal{L})} = \det D^*D$.

(To define $\det D^*D$, we use zeta-function regularization : $\zeta(s) = \sum_{\lambda : \text{eigenvalue}(D^*D)} \lambda^{-s} = \operatorname{Tr}((D^*D)^{-s})$.
ζ is analytic when $Re(s) \gg 0$ and has meromorphic continuation $\rightsquigarrow \det D^*D := e^{-\zeta'(0)}$.)

The connection $\nabla^{(\mathcal{L})}$ on section $\det D$ (D is invertible) is given by
$\nabla^{(\mathcal{L})}(\det D) = \underline{\operatorname{Tr}(\tilde{\nabla} D D^{-1})} \det D$, $\tilde{\nabla}$: connection $\nabla^{(T_{vert} Z)}$ operating pointwise with correction term making $\tilde{\nabla}$ to be unitary.

ζ-regularization :

($\omega(s) := \operatorname{Tr}((DD^*)^{-s} \tilde{\nabla} D D^{-1})$: analytic for $Re(s) \gg 0$, meromorphic continuation.
$\rightsquigarrow \omega(s)$ has a pole at $s=0$ in general, define $\operatorname{Tr}(\tilde{\nabla} D D^{-1}) = (s\omega(s)')(0)$.)

$\rightsquigarrow$ These construction extend to all $\mathcal{L}$ by patching.

If the Dirac operator D is s.t. $\ker D$, $\operatorname{coker} D$ have constant dimension,
we want to use sections of $\det D$ to trivialize $\mathcal{L}$.
Choose smoothing varying base $\{\varphi_i\}$ for $\ker D$, $\{\psi_\alpha\}$ for $\ker D^*$.
Pass to determinant, get nonzero section of $\mathcal{L}$.

Then, $$\|s\|^2_{(\mathcal{L})} = \underbrace{\frac{\det(\psi_\alpha, \psi_\beta)}{\det(\varphi_i, \varphi_j)}}_{L^2\text{-inner product of harmonic spinors}} \underbrace{\det{}' D^* D}_{\zeta\text{-function regularization but omit zero eigenvalues.}}$$

The connection is given by
$$\nabla^{(\mathcal{L})}(s) = \Big[\sum_\alpha (\tilde{\nabla}\psi_\alpha, \psi_\alpha) + \sum_i (\tilde{\nabla}\varphi_i, \varphi_i) + \underbrace{\mathrm{Tr}'}_{\mathrm{Tr}\text{ restrict to }(\ker D^*)^\perp}(\tilde{\nabla} D D^{-1}) \Big] \cdot s$$

■

<u>Thm</u> (Bismut - Freed) The curvature of the determinant line bundle $\mathcal{L} \longrightarrow Y$ is the 2-form
$$\Omega^{(\mathcal{L})} = \Big[2\pi i \int_X \hat{A}(\Omega^{(T_{vert} Z)})\, ch(\Omega^{(T_{vert} Z)})\, ch(\Omega^{(E)}) \Big]_{(2)}.$$
$\uparrow$ integrate over fiber X in the fibration $Z \longrightarrow Y$.

$$\hat{A}(\Omega) = \det\left(\frac{\Omega/4\pi}{\sinh(\Omega/4\pi)} \right)^{1/2}, \quad ch(\Omega) = \mathrm{Tr}\, e^{i\Omega/2\pi}.$$

The formula is obtained by Bismut's heat equation approach to the index theorem for families.

Next time: Specialize to Polyakov formulation for bosonic string in flat $\mathbb{R}^d$.

Critical Dimension of Bosonic String Theory is 26. 2021.1.10.

Reference : D. Freed, Determinants, Torsion, and Strings.

Let Σ be a Riemann surface of genus ≥ 2.

Partition function : $Z = \int_{g \in Met(\Sigma)} \underline{[dg]} \int_{\varphi \in Map(\Sigma, \mathbb{R}^d)} \underline{[d\varphi]} \underbrace{\exp\left(-\frac{1}{2}\int_\Sigma (d\varphi, d\varphi)_g\right)}_{\text{invariant under translation of } \mathbb{R}^d,\ diff(\Sigma),\ C^\infty_+(\Sigma)}$.

conformal rescaling of metric.

$[dg], [d\varphi]$: formal measure divided by vol. of the orbits of the symmetric group.

Gaussian integral $\rightarrow$ $= \int_{g \in Met(\Sigma)} [dg] \left(\frac{\det' \frac{1}{2}\Delta_g}{(1,1)_g}\right)^{-d/2}$

change of variable $\rightarrow$ $= \underline{\int [dg]' \left(\frac{\det' \frac{1}{2}\Delta_g}{(1,1)_g}\right)^{-d/2} \left(\frac{\det' \bar\partial_L^* \bar\partial_L}{\det(\phi_i, \phi_j)}\right)}$. (*)

$Met(\Sigma) = Conf(\Sigma) \times C^\infty_+(\Sigma)$

$\bar\partial_L$: $\bar\partial$ on holomorphic tangent bundle $L \longrightarrow \Sigma$.

$\{\phi_i\}$: basis of holomorphic quadratic differential, invariant under $C^\infty_+(\Sigma)$.

$\underline{Q}$: Whether the product of $\underline{\text{determinants}}$ pass to $Conf(\Sigma) = Met(\Sigma)/C^\infty_+(\Sigma)$ i.e. invariant under $C^\infty_+(\Sigma)$.

This is called conformal anomaly and it vanishes exactly at $d=26$.

Geometric data :

$Y = Met(\Sigma)$, $Z = Met(\Sigma) \times \Sigma$ with natural projection $Z \xrightarrow{\Sigma} Y$ and vertical metric.

Choose representation ρ of of $Spin(n)$ s.t. $V_\rho = \underbrace{-\frac{d}{2}}_{\text{negative trivial } \frac{d}{2}\text{-dim bundle}} + \underbrace{L}$. (bundle of Z)

holomorphic tangent bundle of Σ

$L \longrightarrow Met(X) \times \Sigma$.

$\rightsquigarrow$ These data gives a family of operators $\bar\partial_{-d/2+L}$ on Σ, parameterized by $Met(\Sigma)$.

$\rightsquigarrow$ $\mathcal{L} \longrightarrow Met(\Sigma)$: the determinant line bundle for $\bar\partial_{-d/2+L}$.

Now, $\ker \bar\partial_{-d/2+L}$, $\operatorname{coker} \bar\partial_{-d/2+L}$ has constant dimension (not index zero !)

Kodaira vanishing thm $\Rightarrow$ $\ker \bar\partial_L = 0$. $\ker \bar\partial = \{$ holo. fun. on $\Sigma \} = \mathbb{C}$.

Serre duality $\Rightarrow$ $\ker \bar\partial^* = H^0(\Sigma, \Omega^1)^\vee$.

Let $\{\omega_\alpha\}$: locally a basis of holo. one-form on $Met(X)$. $\rightsquigarrow$ give a section s of $\begin{matrix}\mathcal{L}\\ \downarrow \\ Met(X)\end{matrix}$

with $\|s\|^2_{(\mathcal{L})} = \left(\frac{\det' \bar\partial^* \bar\partial}{(1,1)_g \cdot \det(\omega_\alpha, \omega_\beta)}\right)^{-d/2} \cdot \frac{\det' \bar\partial_L^* \bar\partial_L}{\det(\phi_i, \phi_j)}$.

$\uparrow$
Last time

$\frac{1}{2}\Delta_g = \bar\partial^*\bar\partial \rightsquigarrow \|s\|^2_{(\mathcal{L})}$ differs from (*) only in $\det(\omega_\alpha, \omega_\beta)^{-d/2}$.

May pick ω_α inv. under $C^\infty_+(\Sigma)$.

Now, since $\bar\partial$ only depend on $\mathrm{Conf}(\Sigma)$, L^2-inner product $(\cdot,\cdot)$: inv. under $\mathrm{Conf}(\Sigma)$

$\rightsquigarrow$ irrelevant to computation of conformal anomaly :

conformal anomaly $=0 \iff \|s\|^2$ on $\mathrm{Met}(\Sigma)$ is inv. under $C^\infty_+(\Sigma)$.

Lift action $C^\infty_+(\Sigma) \curvearrowright \mathrm{Met}(\Sigma)$ to $C^\infty_+(\Sigma) \curvearrowright \mathrm{Met}(\Sigma) \times \underbrace{\Sigma}_{\text{trivial action}}$

$\rightsquigarrow C^\infty_+(\Sigma)$ action on $\mathcal{L}$.

<u>Claim:</u> s is inv. under $C^\infty_+(\Sigma)$.

$$\|s\|^2_{(\mathcal{L})} = \left(\frac{\det' \bar\partial^*\bar\partial}{(1,1)_g \cdot \det(\omega_\alpha, \omega_\beta)}\right)^{-d/2} \cdot \frac{\det' \bar\partial_L^*\bar\partial_L}{\det(\phi_i, \phi_j)}$$

$\bar\partial$ operator : complex structure unchange $\Rightarrow$ inv. under rescaling $C^\infty_+(\Sigma)$.

ω_α, ϕ_i : chosen to be $C^\infty_+(\Sigma)$ inv.

Now, s is inv. under $C^\infty_+(\Sigma) \iff g^{(\mathcal{L})}$ is inv. under $C^\infty_+(\Sigma)$.

Suffice to show $\nabla^{(\mathcal{L})}$ is inv. : $\quad G = C^\infty_+(\Sigma)$

<u>Lemma</u> Let $\mathcal{L} \longrightarrow Y$ be a line bundle with $\nabla^{(\mathcal{L})}$ and G: connected Lie group acting on $\mathcal{L}$ and Y freely. $\omega^{(\mathcal{L})}$: connection one form, $\Omega^{(\mathcal{L})}$: curvature two form.

Suppose for each $X \in \mathrm{Lie}(G)$, (1) $L_X \omega^{(\mathcal{L})} = 0$, (2) $L_X \Omega^{(\mathcal{L})} = 0$.

Then, $\nabla^{(\mathcal{L})}$ pass to $\mathcal{L}/G \longrightarrow Y/G$.

proof: For section s and vector field Y, need to prove $g(\nabla_Y s) = \nabla_{dg(Y)} s + \nabla_Y g(s)$ for all $g \in G$. Since G: connected, suffices to check $X(\nabla_Y s) = \nabla_{L_X Y} s + \nabla_Y X(s)$ for all $X \in \mathrm{Lie}(G)$. Write $s = s^i \cdot \sigma_i$, where $\{\sigma_i\}$: basis of sections.

$\nabla_{L_X Y} s + \nabla_Y X(s) = \iota_{L_X Y}(ds^i \cdot \sigma_i + \omega^i_j s^j \sigma_i) + \nabla_Y (X(s^i)\cdot\sigma_i + s^i \cdot X(\sigma_i))$

$= (ds^i([X,Y]) + \omega^i_j([X,Y]) s^j)\sigma_i + \iota_Y (dX(s^i)\sigma_i + X(s^j)\,\omega^i_j \sigma_i) + \nabla_Y (s^i \cdot X(\sigma_i))$

$= ([X,Y]s^i + (X(\omega^i_j(Y)) - Y(\omega^i_j(X)) - d\omega^i_j(X,Y))s^j)\sigma_i + YX(s^i)\sigma_i + \omega^i_j(Y)X(s^j)\sigma_i$

$\quad + \omega^i_j(Y) s^j X(\sigma_i) + Y(s^i) X(\sigma_i) \quad \overset{\shortparallel}{0} \quad (\Omega^i_j + \omega^k_j \wedge \omega^i_k)(X,Y) = 0$.

$= X(\omega^i_j(Y) s^j \sigma_i + Y(s^i)\sigma_i) = X(\nabla_Y s)$.

$\square$

<u>Proposition</u> The curvature $\Omega^{(\mathcal{L})}$ of the determinant line bundle for the $\bar{\partial}_{-d/2+L}$ family vanishes if $d=26$.

proof: Let $x = \frac{i}{2\pi}\Omega^{(L)} \in \Omega^2(\mathrm{Met}(\Sigma)\times\Sigma)$ be curvature of holo. tangent bundle L.

$$\rightsquigarrow ch(\Omega^{(-d/2+L)}) = (1-\frac{d}{2}) + x + \frac{x^2}{2} + \cdots$$

Apply <u>Thm</u> $\Omega^{(\mathcal{L})} = \left[2\pi i \int_X \hat{A}(\Omega^{(T_{vert}Z)})\, ch(\Omega^{(T_{vert}Z)})\, ch(\Omega^{(E)})\right]_{(2)}$ last time. When consider $\bar{\partial}$ operator, $\hat{A}$ is replaced by Todd polynomial.

$$Todd(\Omega^{(L)}) = 1 + \frac{x}{2} + \frac{x^2}{12} + \cdots$$

$$\left[(1+\frac{x}{2}+\frac{x^2}{12}+\cdots)((1-\frac{d}{2})+x+\frac{x}{2}+\cdots)\right]_{(2)} = \frac{26-d}{24}x^2 = \frac{26-D}{24}\cdot\frac{-1}{4\pi^2}(\Omega^{(L)})^2.$$

$$\rightsquigarrow \Omega^{(\mathcal{L})} = i\cdot\left(\frac{d-26}{48\pi}\right)\int_\Sigma (\Omega^{(L)})^2 = 0 \text{ if } d=26.$$

□

Now, condition (2) holds if $d=26$ and condition (1) is the direct computation of connection one form using

$$\nabla^{(\mathcal{L})}(s) = \left[\sum_\alpha(\tilde{\nabla}\psi_\alpha, \psi_\alpha) + \sum_i(\tilde{\nabla}\varphi_i, \varphi_i) + Tr'(\tilde{\nabla}DD^{-1})\right]\cdot s.$$

$\Rightarrow$ the conformal anomaly vanishes at $d=26$.

In $d=26$, the line bundle $\mathcal{L} \longrightarrow \mathrm{Met}(\Sigma)$ pass to $\mathcal{L}/C_+^\infty(M) \longrightarrow \mathrm{Met}(\Sigma)/C_+^\infty(M)$, $\mathcal{L}/C_+^\infty(M) = \underline{\mathcal{L}}$, $\mathrm{Met}(\Sigma)/C_+^\infty(M) = \mathrm{Conf}(\Sigma)$.

with metric, connection, non-vanishing section s.

<u>Next Step</u>: In critical dim $d=26$, interpert the partition function Z to be norm square of a holo. function F on $\mathrm{Teich}(\Sigma) = \mathrm{Conf}(\Sigma)/\underline{\mathrm{Diff}_0(\Sigma)}$.

Teichmüller space (diffeo. of connected comp.)

$\mathrm{Conf}(\Sigma) = \{\text{cpx. structure on } \Sigma\}$: a cpx. mfd. $\rightsquigarrow \mathrm{Conf}(\Sigma)\times\Sigma$: cpx. mfd.

$\rightsquigarrow \mathrm{Conf}(\Sigma)\times\Sigma$ is holo. with $\mathrm{Diff}_0(\Sigma)$ action freely, preserving cpx. structure

$$\begin{array}{c}\mathrm{Conf}(\Sigma)\times\Sigma \\ \big\downarrow \text{projection} \\ \mathrm{Conf}(\Sigma)\end{array}$$

$$\begin{array}{c} \rightsquigarrow \quad Z := \mathrm{Conf}(\Sigma)\times\Sigma / \mathrm{Diff}_0(\Sigma) \quad : \text{holo. fiberation.} \\ \downarrow \Sigma \\ Y := \mathrm{Conf}(\Sigma)/\mathrm{Diff}_0(\Sigma) = \mathrm{Teich}(\Sigma) \end{array}$$

Also, $T(\mathrm{Conf}(\Sigma)\times\Sigma)$ pass to quotient TZ.

$$\begin{array}{ccc} T(\mathrm{Conf}(\Sigma)\times\Sigma) & & TZ \\ \downarrow & & \downarrow P \\ T(\mathrm{Conf}(\Sigma)) & & T_{vert} Z \end{array}$$

Back to bosonic string:

We have line bundle $\mathcal{L} \longrightarrow \mathrm{Conf}(\Sigma)$ with $g^{(\mathcal{L})}$, $\nabla^{(\mathcal{L})}$, non-vanishing section s.

$\mathrm{Diff}_0(\Sigma)$ preserve all these data $\rightsquigarrow$ pass to quotient and get $\mathcal{L} \longrightarrow \mathrm{Teich}(\Sigma)$.

<u>Theorem</u> The bundle $\mathcal{L} \longrightarrow \mathrm{Teich}(\Sigma)$ is holo. and $\nabla^{(\mathcal{L})}$ is the unique holo. conn. compatible with $g^{(\mathcal{L})}$. Also, s is a holo. section.

(Group action on <u>geometric data</u> pass to quotient.
including . metric, connection, holomorphic structure ...)

<u>Theorem</u> In $d=26$, the partition function Z is the norm square of a holo. function on $\mathrm{Teich}(\Sigma)$.

proof: Had seen Z is realized as $\|s\|^2$ for a section $\mathcal{L} \longrightarrow \mathrm{Teich}(\Sigma)$.

Suffice to find a holo. section of $\mathcal{L} \longrightarrow \mathrm{Teich}(\Sigma)$ with unit norm.

curvature $\Omega^{(\mathcal{L})}$ vanishes, $\mathrm{Teich}(\Sigma)$: simply connected

$\Rightarrow \exists$ global flat section s_0 of unit norm.

s_0 is holo. since $\nabla^{(\mathcal{L})}$ is holo.

Finally, pick $F = s/s_0$.

$\square$