A Note for Witten Complex

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6.Jan.2022

1 Clifford Actions and the Witten Deformation

Let M be a smooth compact manifold. For any $e \in TM$, and g^{TM} a Riemann metrics. We can define the dual e^* with g^{TM}. Then we can defined the Clifford operator:

$$c(e)\cdot = e^* \wedge \cdot - \iota_e \cdot, \quad \hat{c}(e)\cdot = e^* \wedge \cdot + \iota_e \cdot \tag{1}$$

Then we have the formula

$$\begin{aligned} c(e)c(e') + c(e')c(e) &= -2\langle e, e' \rangle \\ \hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) &= 2\langle e, e' \rangle \\ c(e)\hat{c}(e') + \hat{c}(e')c(e) &= 0 \end{aligned} \tag{2}$$

And we also have the formula from differential geometry

$$\begin{aligned} d &= \sum_{i=1}^{n} e^i \wedge \nabla_{e_i} \\ d^* &= -\sum_{i=1}^{n} \iota_{e^i} \nabla_{e_i} \end{aligned} \tag{3}$$

with ∇ the Levi-Civita connection. This give us

$$d + d^* = \sum_{i=1}^{n} c(e^i)\nabla_{e_i} \tag{4}$$

Now, for $V \in \Gamma(TM)$ and $T \in \mathbb{R}$, Witten define the deformation of the operator

$$D_T = d + d^* + T\hat{c}(V) \tag{5}$$

It is easy to see that D_T is self-adjoint and we have the Bochner type formula

Bochner type formula

For any $T \in \mathbb{R}$, we have the identity:

$$D_T^2 = D^2 + T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i} V) + T^2|V|^2 \tag{6}$$

proof:

Just using equation (2), (4) and (5).

□

2 Estimation Outside the Zero of V

Let $||\cdot||_0$ be the 0-th Sobolev norm on $\Omega^*(M)$ induced by the inner product. And $\mathbf{H}^0(M)$ be the corresponding Sobolev space. For each $p \in \text{zero}(V)$ (here we assume it is discrite points) let U_p be a neighborhood of it. Then we have the estimation:

Proposition 1.

There exist constants $C > 0$, $T_0 > 0$ such that for any section $s \in \Omega^*(M)$ with Supp(s)$\subset M \backslash U_{p\in zero(V)} U_p$ and $T \geq T_0$, one has

$$||D_T s||_0 \geq C\sqrt{T}||s||_0 \tag{7}$$

proof:

Since V is nowhere zero on$M \backslash U_{p\in \text{zero}(V)} U_p$, there is a constant $C_1 > 0$ such that on $M \backslash U_{p\in \text{zero}(V)} U_p$

$$|V|^2 \geq C_1$$

From equation(6), we then have a constant $C_2 > 0$ such that:

$$||D_T s||_0^2 = \langle D_T^2 s, s\rangle \geq (C_1 T^2 - C_2 T)||s||_0^2$$

for any $S \in \Omega^*(M)$ with support in $M \backslash U_{p\in \text{zero}(V)} U_p$. Then equation (7) is just a consequence of it.

□

3 Harmonic Oscillator on Euclidean Space

We first shrink the neighborhood U_p enough and redefine the metric g, such that on each neighborhood U_p, we have the metric is standard:

$$g = (dy^1)^2 + ... + (dy^n)^2$$

Hence U_p can be identify with an open neighborhood of the n-dimensional Euclidea space E_n. We assume that V can be locally written as $V = yA$ for some $A \in \mathrm{Gl}(n)$ Let $e_i = \frac{\partial}{\partial y_i}$ be an oriented orthonormal basis of E_n. Then we can refine the Bochner formula by:

$$\begin{aligned} D_T^2 &= -\sum_{i=1}^n (\frac{\partial}{\partial y^i})^2 + T\sum_{i=1}^n c(e_i)\hat{c}(e_iA) + T^2\langle yAA^*, y\rangle \\ &= -\sum_{i=1}^n (\frac{\partial}{\partial y^i})^2 - T\mathrm{Tr}(\sqrt{AA^*}) + T^2\langle yAA^*, y\rangle \\ &+ T\Big(\mathrm{Tr}(\sqrt{AA^*}) + \sum_{i=1}^n c(e_i)\hat{c}(e_iA)\Big) \end{aligned} \tag{8}$$

The operator

$$\begin{aligned} K_T &= -\sum_{i=1}^n (\frac{\partial}{\partial y^i})^2 - T\mathrm{Tr}(\sqrt{AA^*}) + T^2\langle yAA^*, y\rangle \\ &= -\sum_{i,j,k=1}^n (\frac{\partial}{\partial y^i} + Ty_k\sqrt{AA^*}_{ki})(\frac{\partial}{\partial y^i} - y_jT\sqrt{AA^*}_{ij}) \end{aligned} \tag{9}$$

is a rescaled harmonic oscillator. By the standard result of harmonic operator, we knows that when $T > 0$, K_T is a non negative elliptic operator with $\ker K_T$ being one-dimensional and generated by the Gaussian function:

$$\exp(\frac{-T|yA|^2}{2}) \tag{10}$$

Furthermore, the nonzero eigenvalues of K_T are all greater than CT for some fixed constant $C > 0$. For the remaining part, we have:

Lemma 2.

The linear operator

$$L = Tr(\sqrt{AA^*}) + \sum_{i=1}^n c(e_i)\hat{c}(e_iA) \tag{11}$$

acting on $\Lambda^*(E_n^*)$ *is nonegative. Moreover,* $\dim(\ker L) = 1$ *with* $\ker L \subset \Lambda^{even}(E_n^*)$ *if* $\det A > 0$, *and* $\ker L \subset \Lambda^{odd}(E_n^*)$ *if* $\det A < 0$.

proof:

We write

$$A = U\sqrt{A^*A}$$

with $U \in O(n)$(singular value decomposition). Also, let $W \in SO(n)$ be such that

$$\sqrt{A^*A} = W\mathrm{diag}s_1, ..., s_nW^* = WSW^*$$

then, one can easily deduces that

$$\mathrm{Tr}\sqrt{AA^*} = \sum_{i=1}^{n} s_i \tag{12}$$

and

$$\sum_{i=1}^{n} c(e_i)\hat{c}(e_i A) = \sum_{i=1}^{n} c(e_i)\hat{c}(e_i UWSW^*) \tag{13}$$

now, we define $\{UW\}_{ij} = w_{ij}$, we can get

$$\sum_{i=1}^{n} c(e_i)\hat{c}(e_i A) = \sum_{i=1}^{n} c(e_i)\hat{c}(e_j w_{ij} s_j W^*) = \sum_{j=1}^{n} s_j c(e_j W^* U^*)\hat{c}(e_j W^j) \tag{14}$$

Set $f_j = e_j W^*$. They are another oriented orthonormal basis of E_n, then by equation (12) and (13):

$$L = \sum_{i=1}^{n} s_i (1 + c(f_i U^*)\hat{c}(f_i)) \tag{15}$$

Now, we further define

$$\eta_j = c(f_j U^*)\hat{c}(f_j)$$

Then by equation (2) again, we have η_j is a self-adjoint operator and $\eta_j^2 = 1$. Thus the lowest eigenvalue of η_j is -1. This give us L is a nonnegative operator.
By applying equation (2), we actually can get relations:

$$\begin{aligned} \eta_i \eta_j &= \eta_j \eta_i \\ \hat{c}(f_j)\eta_j &= -\eta_j \hat{c}(f_j) \\ \hat{c}(f_i)\eta_j &= \eta_i \hat{c}(f_j) \qquad \text{if } i \neq j \end{aligned} \tag{16}$$

Then by induction, we have:

$$\dim\{x \in \Lambda^*(E_n^*) : (1 + \eta_j)x = 0 \text{ for } 1 \leq j \leq n\} = \frac{\dim \Lambda^*(E_n^*)}{2^n} = 1$$

Moreover, let $\rho \in \Lambda^*(E_n^*)$ denote one of the unit sections of $kerL$, then one has:

$$\rho = (-1)^n (\prod_{i=1}^{n} \eta_i)\rho = (-1)^n (\det U)(\prod_{i=1}^{n} c(f_i)\hat{c}(f_i))\rho$$

Now, it is easy to see that the chiral element:

$$(-1)^n (\prod_{i=1}^{n} c(f_i)\hat{c}(f_i)) = \pm \mathrm{Id}|_{\Lambda^{\mathrm{even/odd}}(E_n^*)}$$

Hence, we have $\rho \in \Lambda^{\mathrm{even/odd}}(E_n^*)$ if and only if $\det(U) = \pm 1$.

□

Combining the result above, we finally get the result:

Propostion 3.

For $T > 0$, the operator

$$-\sum_{i=1}^{n}(\frac{\partial}{\partial y^i})^2 + T\sum_{i=1}^{n} c(e_i)\hat{c}(e_i A) + T^2\langle yAA^*, y\rangle$$

acting on $\Gamma(\Lambda^*(E_n^*))$ is nonnegative. Its kernel is of dimension one and is generated by

$$\exp(\frac{-T|yA|^2}{2}) \cdot \rho$$

Moreover, all the nonzero eigenvalues of this operator are greater than CT for some fixed constant $C > 0$ (independent of T).

4 Witten Deformation via Morse Function

Let $f \in C^\infty(M)$ be a Morse function on M. Then we have the Morse lemma:

Morse Lemma

For any critical points $x \in M$ of the Morse function f, there is an open neighborhood U_x of x ad an oriented coordinate system y such that on U_x, one has

$$f(y) = f(x) - \frac{1}{2}(y^1)^2 - ... - \frac{1}{2}(y^{n_f(x)})^2 + \frac{1}{2}(y^{n_f(x)+1})^2 + ... + \frac{1}{2}(y^n)^2 \quad (17)$$

We call the integer $n_f(x)$ the **Morse index** of f at x. Also, for later use, we assume that for any two different critical points $x, y \in M$ of f, $U_x \cap U_y = \emptyset$ and equip M with a metric such that the coordinate isometric to an open subset of the Euclidean space. Let m_i be the number of critical points such that $n_f = i$. Given a Morse function f(in this step, any function works). Witten suggested to deform the exterior differential operator d as follows:

$$d_{Tf} = e^{-Tf} d e^{Tf} \quad (18)$$

One obviously have $d_{Tf}^2 = 0$, and hence we can define a deform de Rham complex $(\Omega^*(M), d_{Tf})$

$0 \longrightarrow \Omega^0(M) \xrightarrow{d_{Tf}} \Omega^1(M) \xrightarrow{d_{Tf}} ... \xrightarrow{d_{Tf}} \Omega^{\dim(M)}(M) \xrightarrow{d_{Tf}} 0$ Then we have the cohomology group:

$$H^i_{Tf,dR}(M;\mathbb{R}) = \frac{\ker d_{Tf}|_{\Omega^i(M)}}{\mathrm{Im} d_{Tf}|_{\Omega^{i-1}(M)}}$$

The first simple conclusion is that

Propostion 4.

For any i, we have

$$\dim H^i_{Tf,dR}(M;\mathbb{R}) = \dim H^i_{dR}(M;\mathbb{R})$$

proof:

It is easy to see that $\alpha \to e^{-Tf}\alpha$ gives an well-defined isomorphism from $H^i_{dR}(M;\mathbb{R})$ to $H^i_{Tf,dR}(M;\mathbb{R})$.

□

Similarly to the undeformed one, we can develop the Hodge theory via the deformed exterior derivative d_{Tf}, for $\alpha, \beta \in \Omega^*(M)$,

$$\langle d_{Tf}\alpha, \beta\rangle = \langle e^{-Tf} d e^{Tf}\alpha, \beta\rangle = \langle \alpha, e^{Tf} d^* e^{-Tf}\beta\rangle$$

Thus we have

$$d^*_{Tf} = e^{Tf} d^* e^{-Tf} \tag{19}$$

is the formal adjoint of d_{Tf}. Then we also have the operator:

$$D_{Tf} = d_{Tf} + d^*_{Tf} \tag{20}$$

$$\square_{Tf} = D^2_{Tf} = d_{Tf}d^*_{Tf} + d^*_{Tf}d_{Tf} \tag{21}$$

and the Hodge theory tells us:

$$\dim(\ker \square_{Tf}|_{\Omega^i(M)}) = \dim H^i_{Tf,dR}(M;\mathbb{R}) = \dim H^i_{dR}(M;\mathbb{R}) \tag{22}$$

One can actually verifies that

$$d_{Tf} = d + Tdf\wedge, \quad d^*_{Tf} = d^* + T\iota_{df}$$

Hence we get

$$D_{Tf} = D + T\hat{c}(df) \tag{23}$$

is the special case of equation (5). For general D_T, the Laplacian D^2_T only preserve the Z_2 grading, but in this case $\square_{Tf}$ actually preserving the $\mathbb{Z}$ grading of $\Omega^*(M)$

By the Morse lemma, we have the special case of equation (8) on the local coordinate for $A = \mathrm{Id}_{n_f(x)\times n_f(x)} \oplus (\text{-Id})_{(n-n_f(x))\times(n-n_f(x))}$; hence, we have:

$$\begin{aligned}\square_{Tf} &= -\sum_{i=1}^{n}(\frac{\partial}{\partial y^i})^2 - nT + T^2|y|^2 + T\sum_{i=1}^{n_f(x)}(1 - c(e_i)\hat{c}(e_i)) + T\sum_{i=n_f(x)+1}^{n}(1 + c(e_i)\hat{c}(e_i)) \\ &= -\sum_{i=1}^{n}(\frac{\partial}{\partial y^i})^2 - nT + T^2|y|^2 + 2T\Big(\sum_{i=1}^{n_f(x)}\iota_{e_i}e_i^*\wedge + \sum_{i=n_f(x)+1}^{n} e_i^*\wedge\iota_{e_i}\Big)\end{aligned} \tag{24}$$

The kernel of the operator:

$$\sum_{i=1}^{n_f(x)} \iota_{e_i} e_i^* \wedge + \sum_{i=n_f(x)+1}^{n} e_i^* \wedge \iota_{e_i}$$

on the Euclidean space can be easily found to be

$$dy^1 \wedge \cdots \wedge dy^{n_f(x)}$$

Hence we can get a refinement of Proposition 3 by

Proposition 5.

For any $T > 0$, the operator

$$-\sum_{i=1}^{n}(\frac{\partial}{\partial y^i})^2 - nT + T^2|y|^2 + 2T\Big(\sum_{i=1}^{n_f(x)} \iota_{e_i} e_i^* \wedge + \sum_{i=n_f(x)+1}^{n} e_i^* \wedge \iota_{e_i}\Big)$$

acting on $\Gamma(\Lambda^*(E_n^*))$ is nonnegative. Its kernel is one-dimensional and is generated by

$$\exp(\frac{-T|y|^2}{2}) \cdot dy^1 \wedge \cdots \wedge dy^{n_f(x)}$$

Moreover, all the nonzero eigenvalues of this operator are greater than CT for some fixed constant $C > 0$.

5 Witten's instanton complex

The key result of the deformation is the proposition below:

Proposition 6.

For any $C > 0$, there exists $T_0 > 0$ such that when $T \geq T_0$, the number of eigenvalues in $[0, c]$ of $\Box_{Tf}|_{\Omega^i(M)}$, $0 \leq i \leq n$, equals to m_i.

With he lemma, we can define $\mathrm{F}_{Tf,i}^{[0,c]} \subset \Omega^*(M)$ the m_i dimensional vector space generated by the eigenspaces of $\Box_{Tf}|_{\Omega^i(M)}$ associated with eigenvalues in $[0, c]$. Since we have

$$d_{Tf}\Box_{Tf} = \Box_{Tf} d_{Tf} = d_{Tf} d_{Tf}^* d_{Tf}$$

and

$$d_{Tf}^*\Box_{Tf} = \Box_{Tf} d_{Tf}^* = d_{Tf}^* d_{Tf} d_{Tf}^*$$

Hence we have d_{Tf}(resp. d_{Tf}^*) maps $\mathrm{F}_{Tf,i}^{[0,c]}$ to $\mathrm{F}_{Tf,i+1}^{[0,c]}$ (resp. $\mathrm{F}_{Tf,i-1}^{[0,c]}$). Thus one has the following finite dimensional subcomplex of $(\Omega^*(M), d_{Tf})$:

$$(\mathrm{F}_{Tf}^{[0,c]}, d_{Tf}): \; 0 \longrightarrow \mathrm{F}_{Tf,0}^{[0,c]} \xrightarrow{d_{Tf}} \mathrm{F}_{Tf,1}^{[0,c]} \xrightarrow{d_{Tf}} \ldots \xrightarrow{d_{Tf}} \mathrm{F}_{Tf,n}^{[0,c]} \xrightarrow{d_{Tf}} 0 \tag{25}$$

And the Hodge decomposition give us

$$\beta^{[0,c]}_{Tf,i} := \dim(\frac{\ker d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i}}}{\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i-1}}})$$

is equal to $\dim(\ker \Box_{Tf}|_{\Omega^i(M)})$, which is equal to $\beta_i = \dim(H^i(M;\mathbb{R}))$
One of the application of this complex is a proof of Morse inequality:

Morse Inequality

For any integer i such that $0 \leq i \leq n$,, one has

$$\beta_i \leq m_i$$

(weak Morse inequality) and

$$\beta_i - \beta_{i-1} + ... + (-1)^i \beta_0 \leq m_i - m_{i-1} + ... + (-1)^i m_0$$

(strong Morse inequality). Moreover

$$\beta_n - \beta_{n-1} + ... + (-1)^n \beta_0 = m_n - m_{n-1} + ... + (-1)^n m_0$$

proof:

The weak Morse inequality is a direct consequence of $\beta^{[0,c]}_{Tf,i} = \beta_i$. For strong one, we first find that

$$\begin{aligned} m_i = \dim(\mathrm{F}^{[0,c]}_{Tf,i}) &= \dim(\ker d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i}}) + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i}}) \\ &= \beta^{[0,c]}_{Tf,i} + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i-1}}) + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i}}) \end{aligned} \tag{26}$$

Hence we get

$$\begin{aligned} \sum_{j=0}^{i} (-1)^j m_{i-j} &= \sum_{j=0}^{i} (-1)^j (\beta_{i-j} + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i-j-1}}) + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i-j}})) \\ &= \sum_{j=0}^{i} (-1)^j \beta_{i-j} + \dim(\mathrm{Im} d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,i}}) \end{aligned}$$

Then the strong Morse inequalities follows.

□

For the case $c = 1$, the resulting complex is called **Witten's instanton complex**.

Now, the key point of the construction is to prove Propostion 6, which we start now. The first few step can be doing more general to the case of general deformation to vector field $V = Ay$. We first shrink all the neighborhood U_p isometric to open ball of radius $4a$. And let $\gamma : \mathbb{R} \to [0,1]$ being the bunp function with $\gamma(z) = 1$ if $|z| \leq a$ and $\gamma(z) = 0$ if $|z| \geq 2a$. Then we define

$$\begin{aligned} \alpha_{p,T} &= \int_{U_x} \gamma(|y|)^2 \exp(-T|yA_p|^2) dy^1 \wedge ... \wedge dy^n, \\ \rho_{p,T} &= \frac{\gamma(|y|)}{\sqrt{\alpha_{p,T}}} \exp(-\frac{T|yA_p|^2}{2}) \rho_p \end{aligned} \tag{27}$$

Then $\rho_{p,T} \in \Omega^*(M)$ is of unit length with compact support contained in U_p. Let E_T be the direct sum of the vector space generated by $\rho_{p,T}$'s, where p runs through the set of zero points of V. Since ρ_p is either even or odd, we have E_T admit the decompostion $E_T = E_{T,\text{even}} \oplus E_{T,\text{odd}}$. Let $E_T^{\perp}$ be the orthogonal complement of E_T in $\mathbf{H}^0(M)$. Then we have

$$\mathbf{H}^0(M) = E_T \oplus E_T^{\perp}$$

orthogonally. Let p_T and $p_T^{\perp}$ denote the orthonoal projection operators from $\mathbf{H}^0(M)$ to E_T and $E_T^{\perp}$ respectively. Then we defined

$$\begin{aligned} D_{T,1} = p_T D_T p_T, \quad D_{T,2} = p_T D_T p_T^{\perp}, \\ D_{T,3} = p_T^{\perp} D_T p_T, \quad D_{T,4} = p_T^{\perp} D_T p_T^{\perp} \end{aligned} \tag{28}$$

Let $\mathrm{H}^1(M)$ be the first Sobolev space with first Sobolev norm on $\Omega^*(M)$. Then we have

Proposition 7.

There exists a constant $T_0 > 0$ such that

1. *for any $T \geq T_0$ and $0 \leq u \leq 1$, the operator*

   $$D_T(u) = D_{T,1} + D_{T,4} + u(D_{T,2} + D_{T,3}) : H^1(M) \to H^0(M)$$

   is Fredholm;

2. *the operator $D_{T,4} : E_T^{\perp} \cap H^1(M) \to E_T^{\perp}$ is invertible.*

We first doing some estimate of $D_{T,i}$ for $i = 2, 3, 4$.

Lemma 8.

There exists constant $T_0 > 0$ such that for any $s \in E_T^{\perp} \cap H^1(M)$, $s' \in E_T$ and $T \geq T_0$, one has

$$||D_{T,2}s||_0 \leq \frac{||s||_0}{T},$$

$$||D_{T,3}s'||_0 \leq \frac{||s'||_0}{T},$$

proof:

It is easy to see that $D_{T,3}$ is the formal adjoint of $D_{T,2}$. Thus one needs only to prove the first estimate . Since each $\rho_{p,T}$, $p \in$zero(V), has support in U_p, by equation (27) and proposition 3, we have that for any $s \in E_T^{\perp} \cap \mathrm{H}^1(M)$,

$$\begin{aligned} D_{T,2}s &= \sum_{p\in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \langle \rho_{p,T}, D_T s\rangle dv_{U_p} \\ &= \sum_{p\in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \langle D_T \rho_{p,T}, s\rangle dv_{U_p} \\ &= \sum_{p\in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \langle D_T(\frac{\gamma(|y|)}{\sqrt{\alpha_{p,T}}} \exp(-\frac{T|yA_p|^2}{2})\rho_p), s\rangle dv_{U_p} \\ &= \sum_{p\in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \langle (\frac{c(d\gamma(|y|))}{\sqrt{\alpha_{p,T}}} \exp(-\frac{T|yA_p|^2}{2})\rho_p), s\rangle dv_{U_p} \end{aligned} \tag{29}$$

Since γ equals to one in an open neighborhood around zero(V). $d\gamma$ vanishes on this open neighborhood. Thus by equation (29), one can easily find that there exist constants $T_0 > 0$, $C_1 > 0$, $C_2 > 0$ such that when $T \geq T_0$, for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$,

$$||D_{T,2}s||_0 \leq C_1 T^{n/2} \exp(-C_2 T)||s||_0 \tag{30}$$

and the lemma is just the limited case.

□

By the lemma, $D_{T,2}$ and $D_{T,3}$ are compact operators (They are actually bounded of finite rank), and we already know D_T is Fredholm. Hence we have prove the first part of propostion 7.
For the second part, one just needs to show that there exist constants $T_0 > 0$, $C_3 > 0$ such that for any $T \geq T_0$ and $s \in E_T^{\perp} \cap \mathrm{H}^1(M)$,

$$||D_{T,4}||_0 \geq C_3||s||_0.$$

Notice since for $s \in E_T^{\perp} \cap \mathrm{H}^1(M)$ one has

$$D_T s = D_{T,2}s + D_{T,4}s,$$

Then by lemma 8., we just need to show that for some $C_4 \geq 0$.

$$||D_T s||_0 \geq C_4||s||_0$$

when $T > 0$ is large enough.

Lemma 9.

There exist constants $T_0 > 0$ and $C > 0$ such that for any $s \in E_T^{\perp} \cap H^1(M)$ and $T \geq T_0$,

$$||D_T s||_0 \geq C\sqrt{T}||s||_0 \tag{31}$$

proof:

We denote $U_p(b)$ for the open ball around p of radius b.
Step 1. Assume $\mathrm{Supp}(s) \in \cup_{p\in \mathrm{zero}(V)} U_p(4a)$, Then we can assume that we are in a union of Euclidean spaces E_p's containing $U'_p s$, $p \in \mathrm{zero}(V)$ and can thus applied the results of section 3. Thus, for any $T > 0$, $p \in \mathrm{zero}(V)$, set

$$\rho'_{p,T} = (\frac{T}{\pi})^{n/4}\sqrt{|\det(A_p)|}\exp(-\frac{T|yA_p|^2}{2}) \cdot \rho_p \tag{32}$$

And we can set p'_T for all s with $\mathrm{Supp}(s) \in \cup_{p\in \mathrm{zero}(V)} U_p(4a)$ by

$$p'_T s = \sum_{p\in \mathrm{zero}(V)} \rho'_{p,T} \int_{E_p} \langle \rho'_{p,T}, s\rangle dv_{E_p} \tag{33}$$

Since $p_T s = 0$, we can rewrite p'_T by

$$p'_T s = \sum_{p\in \mathrm{zero}(V)} \rho'_{p,T} \int_{E_p} \langle (1-\gamma(|y|))(\frac{T}{\pi})^{n/4}\sqrt{|\det(A_p)|}\exp(\frac{-2T|yA_p|^2}{2}) \cdot \rho_p, s\rangle dv_{E_p}$$

As γ equals to 1 near each p, hence there exists $C_5 > 0$ such that when $T \geq 1$,

$$||p'_T s||_0^2 \leq \frac{C_5}{\sqrt{T}}||s||_0^2 \tag{34}$$

By proposition 3, we have

$$D_T p'_T s = 0$$

By propostion 3 again, we actually have $C_6, C_7 \geq 0$ such that

$$||D_T s||_0^2 = ||D_T(s - p'_T s)||_0^2 \geq C_6 T||s - p'_T s||_0^2 \geq \frac{C_6 T}{2}||s_0||_0^2 - C_7\sqrt{T}||s||_0^2 \tag{35}$$

Hence there exist $T_1 > 0$ such that $T \geq T_1$ imply

$$||D_T s||_0 \geq \frac{\sqrt{C_6 T}}{2}||s||_0$$

Step 2. Suppose $\mathrm{Supp}(s) \subset M\backslash \cup_{p\in \mathrm{zero}(V)} U_p(2a)$. By proposition 1, we have $T_2 > 0$, $C_8 > 0$ such that for $T \geq T_2$, we have

$$||D_T s||_0 \geq C_8\sqrt{T}||s||_0 \tag{36}$$

Step 3. Let $\tilde{\gamma} \in C^\infty$ be such that on each U_p, $\tilde{\gamma}(y) = \gamma(|y|/2)$, and that $\tilde{\gamma}|_{M\backslash \cup_{p\in \mathrm{zero} U_p(4a)}} = 0$. Then for any $s \in E_T^\perp \cap \mathrm{H}^1(M)$, we have

$$\tilde{\gamma}s \in E_T^\perp \cap \mathrm{H}^1(M)$$

Then, we get, there exist $C_9 > 0$ such that

$$\begin{aligned}||D_T s||_0 &\geq \frac{1}{2}(||(1-\tilde{\gamma})D_T s||_0 + ||\tilde{\gamma} D_T s||_0) \\ &= \frac{1}{2}(||D_T((1-\tilde{\gamma})s) + [D,\tilde{\gamma}]s||_0 + ||D_T(\tilde{\gamma}s) + [D,\tilde{\gamma}]s||_0) \\ &\geq \frac{\sqrt{T}}{2}(C_8||(1-\tilde{\gamma})s||_0 + \sqrt{C_6}||\tilde{\gamma}s||_0) - C_9||s||_0 \\ & C_{10}\sqrt{T}||s||_0 - C_9||s||_0\end{aligned}$$

where $C_10 = \min\{\sqrt{C_6}/2, C_8/2\}$. Complete the proof.

□

Now we can come back to our concrete cases,for D_{Tf}^2 operator. In this case, for $x \in M$ a critical point of f, we have:

$$\begin{aligned}\alpha_{x,T} &= \int_{U_x} \gamma(|y|)^2 \exp(-T|y|^2) dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n \\ \rho_{x,T} &= \frac{\gamma(|y|)}{\sqrt{\alpha_{x,T}}} \exp(-T|y|^2) dy^1 \wedge dy^2 \wedge \cdots \wedge dy^{n_f}\end{aligned} \tag{37}$$

Apart from the estimate of lemma8 and lemma9, we have more information about this operator:

Propostion 10.

For any $T > 0$, we have

$$D_{T,1} = 0; \tag{38}$$

proof:

For any $s \in \mathbf{H}^0(M)$, we have:

$$p_T s = \sum_{x \in \mathrm{crit}(f)} \langle \rho_{x,T}, s \rangle_{\mathbf{H}^0(M)} \rho_{x,T} \tag{39}$$

and we also have:

$$D_{Tf} \langle \rho_{x,T}, s \rangle_{\mathbf{H}^0(M)} \rho_{x,T} \in \Omega^{n_f - 1}(M) \oplus \Omega^{n_f + 1}(M)$$

and has compact support in U_x. Thus $D_{T,1} = 0$.

□

Now for any positve constant $c > 0$, let $E_T(c)$ denote the direct sum of eigenspaces of D_{Tf} associated with the eigenvalues lying in $[-c, c]$. Clearly, $E_T(c)$ is a finite dimensional subspace of $\mathbf{H}^0(M)$. Let $P(c)$ denote the orthogonal projection to $E_T(c)$. Then

Lemma 11.

There exist $C_1 > 0$, $T_3 > 0$ such that for any $T \geq T_3$ and any $\sigma \in E_T$,

$$||P_T(c)\sigma - \sigma||_0 \leq \frac{C_1}{T}||\sigma||_0 \tag{40}$$

proof:

Let $\delta = \{\lambda \in \mathbb{C} : |\lambda| = c\}$ be the counter-clockwise oriented circle. By our pprevios estimate, once can deduce that there is a T_0 such that for any $\lambda \in \delta$, $T > T_0$ and $s \in H^1(M)$,

$$\begin{aligned} ||(\lambda - D_{Tf}s)||_o &\geq \frac{1}{2}||\lambda_{p_T}s - D_{T,2}p_T^{\perp}s||_0 + \frac{1}{2}||\lambda p_T^{\perp}s - D_{T,3}p_Ts - D_{T,4}p_T^{\perp}s||_0 \\ &\geq \frac{1}{2}((c - \frac{1}{T})||p_Ts||_0 + (C\sqrt{T} - c - \frac{1}{T})||p_T^{\perp}s||_0) \end{aligned} \tag{41}$$

Hence, there exist $T_1 > 0$ and $C_2 > 0$ such that for any $T \geq T_1$ and $s \in \mathbf{H}^1(M)$,

$$||(\lambda - D_{Tf})s||_0 \geq C_2||s||_0 \tag{42}$$

Thus for any $T > T_1$ and $\lambda \in \delta$, we have

$$\lambda - D_{Tf} : \mathbf{H}^1(M) \to \mathbf{H}^0(M)$$

is invertible. Thus the resolvent $(\lambda - D_{Tf})^{-1}$ is well-defined. From spectral theorem, we have:

$$P_T(c)\sigma - \sigma = \frac{1}{2\pi\sqrt{-1}}\int_{\delta}((\lambda - D_{Tf})^{-1} - \lambda^{-}1)\sigma d\lambda. \tag{43}$$

and by Proposition 10, we have

$$((\lambda - D_{Tf})^{-1} - \lambda^{-1})\sigma = \lambda^{-1}(\lambda - D_{Tf})^{-1}D_{T,3}\sigma \tag{44}$$

Then, we get:

$$||(\lambda - D_{Tf})^{-1}D_{T,3}\sigma||_0 \leq C_2^{-1}||D_{T,3}\sigma||_0 \leq \frac{1}{C_2T}||\sigma||_0 \tag{45}$$

for $T > T_1$.

□

Now, we are ready to prove the most important Proposition 6.

proof of proposition 6:

By lemma 11, there is $T_5 > 0$ such that for $T > T_5$ $P_T(c)\rho_{x,T}$ are linear independent for $x\mathrm{crit}(f)$. Thus, for such T, we have:

$$\dim E_T(c) \geq E_T \tag{46}$$

Now, if $\dim E_T(c) > E_T$, then there should exist a nonzero element $s \in E_T(c)$ uch that s is perpendicular to $P_T(c)E_T$. That is

$$\langle s, P_T(c)\rho_{x,T}\rangle_0 = 0 \tag{47}$$

for all $x \in \mathrm{crit}(f)$. Then we deduced that

$$\begin{aligned} p_T s &= \sum_x \langle s, \rho_{x,T}\rangle \rho_{x,T} - \sum_x \langle s, P_T(c)\rho_{x,T}\rangle P_T(c)\rho_{x,T} \\ &= \sum_x \langle s, \rho_{x,T}\rangle(\rho_{x,T} - P_T(c)\rho_{x,T}) + \sum_x \langle s, \rho_{x,T} - P_T(c)\rho_{x,T}\rangle P_T(c)\rho_{x,T} \end{aligned} \tag{48}$$

By lemma 11, there is $C_3 > 0$ and $T \geq T_5$ such that:

$$||p_T s||_0 \leq \frac{C_3}{T}||s||_0 \tag{49}$$

Thus, there exists a constant $C_4 > 0$ such that when $T > 0$ is large enough,

$$||p_T^{\perp} s||_0 \geq C_4||s||_0 \tag{50}$$

Then we find that for Tlarge enough, we have

$$\begin{aligned} CC_4\sqrt{T}||s||_0 &\leq ||D_{Tf}p_T^{\perp}s||_0 \\ &= ||D_{Tf} - D_{Tf}p_T s|| \\ &= ||D_{Tf}s - D_{T,3}s||_0 \\ &\leq ||D_{Tf}s||_0 + ||D_{T,3}s||_0 \\ &\leq ||D_{Tf}s||_0 + \frac{1}{T}||s||_0 \end{aligned} \tag{51}$$

Hence we get

$$||D_{Tf}s||_0 \geq CC_4\sqrt{T}||s||_0 - \frac{1}{T}||s||_0$$

Hence if T large enough, the asumption $s \in E_T(c)$ nonzero is contradictive. Hence we get:

$$\dim E_T(c) = \dim E_T = \sum_{i=0}^{n} m_i$$

Moreoveer, $E_T(c)$ is generated by $P_T(c)\rho_{x,T}$ for all $x \in \mathrm{crit}(f)$. Now, let Q_i denote the orthogonal projection operator from $H^0(M)$ onto the L^2 space of

$\Omega^i(M)$. Since $\square_{Tf}$ preserves the $\mathbb{Z}-$grading structure, we have for any eigenvectors s of D_{Tf} associated with an eigenvalue $\mu \in [-c, c]$,

$$\square_{Tf} Q_i s = Q_i \square_{Tf} s = \mu^2 Q_i s$$

Hence $Q_i s$ is an eigenvector of $\square_{Tf}$ with eigenvalue μ^2. Moreover, by Lemma 11, we also have:

$$||Q_{n_f(x)} P_T(c) \rho_{x,T} - \rho_{x,T}|| \leq \frac{C_1}{T} \tag{52}$$

One can see that when $T > 0$ large enough, the forms $Q_{n_f(x)} P_T(c)$ is linear independent for all x, hence we have:

$$\dim Q_i E_T(c) \geq m_i \tag{53}$$

But we also have:

$$\sum_{i=0}^{i=n} \dim Q_i E_T(c) \leq \dim E_T(c) = \sum_{i=0}^{n} m_i \tag{54}$$

Conbining equation (53) and (54), we get the desired result:

$$\dim Q_i E_T(c) = m_i \tag{55}$$

□

6 Thom-Smale Complex

Let $f \in C^\infty(M)$ be a Morse function on an $n-$dimensional closed oriented manifold M. Let $g^T M$ be a metric on TM, and let

$$\nabla f = (df)^*$$

be the gradient vector field of f. Then we can defines a one parameter subgroup of the diffeomorphism group $(\psi_t)_{t\in\mathbb{R}}$ of M:

$$\frac{dy}{dt} = -\nabla f(y) \tag{56}$$

If $x \in \text{crit}(f)$, then we set:

$$\begin{aligned} W^u(x) &= \{y \in M : \lim_{t\to-\infty} \psi_t(y) = x\} \\ W^s(x) &= \{y \in M : \lim_{t\to+\infty} \psi_t(y) = x\} \end{aligned} \tag{57}$$

Be the unstable and stable cells at x respectively.
Assume that the vector field ∇f satisfies the **Smale transversality conditions**:

For any $x, y \in crit(f)$ and $x \neq y$, we have $W^u(x)$ and $W^s(y)$ intersect transversally

In particular, since we know that the dimension of $W^u(x)$ (resp. $W^s(x)$) should be $n_f(x)$(resp. $n - n_f(x)$). Hence if $n_f(y) = n_f(x) - 1$, then $W^u(x) \cap W^s(y)$ consists of finite set $\Gamma(x, y)$ of integral curves γ of the vector field $-\nabla f$, with $\gamma_{-\infty} = x$ and $\gamma_\infty = y$ along with $W^u(x)$ and $W^s(y)$ transversally.

By [S.Smale Theorem A], there always exists a metric g such that his transversality conidtions holds.

Now, we fixed an orientation on each $W^u(x)$. Let $x, y \in \text{crit}(f)$ with $n_f(y) = n_f(x) - 1$. Take $\gamma \in \Gamma(x, y)$, Then the tangent space $T_y W^u(y)$ is orthogonal to the tangent space $T_y W^s(y)$ and is oriented.

For any $t \in (-\infty, \infty)$, the orthogonal space $T^\perp_{\gamma_t} W^s(y)$ to $T_{\gamma_t} W^s(y)$ in $T_{\gamma_t}(M)$ carries a natural orientation, whith is induced form the orientaion on $T_y W^u(y)$. On the other hand, the orthognal space $T'_{\gamma_t} W^u(x)$ to $-\nabla f$ in $T_{\gamma_t} W^u(x)$ can be oriented in such a way that s is an oriented basis of $T'_{\gamma_t} W^u(x)$ if $(-\nabla f(\gamma_t), s)$ is an orented basis of $T_{\gamma_t} W^u(y)$.

Since $W^u(x)$ and $W^s(y)$ intersect transversally along γ, for any $t \in (-\infty, \infty)$, $T^\perp_{\gamma_t} W^s(y)$ and $T'_{\gamma_t} W^u(x)$ can be identified, and hence can compare the induced orentations on them. Then we defined

$$n_\gamma(x, y) = \begin{cases} 1 & \text{if the orientation are the same.} \\ -1 & \text{if the orientation are different.} \end{cases} \tag{58}$$

Then we can defined our complex:

$$C_i(W^u) = \bigoplus_{n_f(x) = i} \mathbb{R}[W^u(x)] \tag{59}$$

and the boundary map

$$\partial W^u(x) = \sum_{n_f(y) = n_f(x) - 1} \sum_{\gamma \Gamma(x, y)} n_\gamma(x, y) W^y(y). \tag{60}$$

The basic result,is

Theorem 12.

$(C_*(W^u), \partial)$ is a chain complex. Moreover, we have a canonical identification between its homology group $H_*(C_*(W^u), \partial)$ to singular homology group $H_*(M)$

We now consider its dual complex $(C^*(W^u), \partial)$ and we are going to construct an isomorphism from this dual complex to the singular cohomology group.

The de Rham map of the Thom-Smale Complex

We first state the result of Lauudenbach.

Proposition 13.

1. *If $x \in \mathrm{crit}(f)$, the nthe clousure $\bar{W}^u(x)$ is an $n_f(x)$ dimensional submanifold of M with conical singularities.*
2. *$\bar{W}^u(x) \backslash W^u(x)$ is stratified by unstable manifolds of critical points of index strictly less than $n_f(x)$.*

By this proposition, we have a well defined integration:

$$\int_{\bar{W}^u(x)} \alpha$$

for $\alpha \in \Omega^*(M)$. Moreover, if $\alpha \in \Omega^i(M)$, this integral is not zero only if $n_f(x) = i$. Hence we get a $\mathbb{Z}-$graded map from $\Omega^*(M)$ to $H_*(C_*(W^u), \partial)$:

$$P_\infty : \alpha \longmapsto \sum_{x \in \mathrm{crit}(f)} [W^u(x)]^* \int_{\bar{W}^u(x)} \alpha \tag{61}$$

where $[W^u(x)]^*$ is the dual basis of $[W^u(x)]$.
We are going to prove the theorem:

Theorem 14

P_∞ is an quasi-isomorphism
By Stokes theorem and proposition 13, it is easy to see that P_∞ is a chain map. And we are going to prove this theorem via Witten's instanton complex.

7 Proof of the Isomorphism via Witten's Instanton Complex

In the following, we always assume T is sufficient large such that Proposition 6 is valid.
We first endow $C^*(W^u)$ with an inner product such that $[W^u]^*$ become an orthonormal basis. And we now define a linear map $J_T : C^*(W^u) \to \Omega^*(M)$

$$J_T W^u(x)^* = \rho_x, T \tag{62}$$

Clearly, J_T is an isometry preserving the $\mathbb{Z}$-gradings.

Noew, we let P^T denote the orthogonal projection from $\Omega^*(M)$ on $F_{Tf}^{[0,1]}$.(Actually, it is the $P_T(1)$ we defined before.)Futhermore, we definede$e_T : C^*(W^u) \to F_{Tf}^{[0,1]}$:

$$e_T = P_T J_T. \tag{63}$$

Then we have an estimation:

Theorem 15

There exists $c > 0$ such that as $T \to \infty$, for any $s \in C^*(W^u)$,

$$(e_T - J_T)s = O(e^{-cT})||s||_0 \tag{64}$$

uniformly on M. In particular, e_T is an isomorphism

proof:

Let $\delta = U(1) \in \mathbb{C}$ be the counter-clockwise oriented circle. By equation (43), we have for any $x \in \mathrm{crit}(f)$ and $T > 0$ large enough,

$$\begin{aligned} (e_T - J_T)W^u(x)^* &= P_T\rho_{x,T} - \rho_{x,T} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_\delta((\lambda - D_{Tf})^{-1} - \lambda^{-1})\rho_{x,T}d\lambda \\ &= \frac{1}{2\pi\sqrt{-1}}\int_\delta(\lambda - D_{Tf})^{-1}\frac{D_{Tf}\rho_{x,T}}{\lambda}d\lambda \end{aligned} \tag{65}$$

For any $p \geq 0$, let $||\cdot||_p$ denote the p-th Soolev norm on $\Omega^*(M)$.
By the construction of $\rho_{x,T}$, for small neighborhood of x, we have:

$$D_{Tf}\rho_{x,T} = 0; \tag{66}$$

Hence by definition, for any positive p, there is $v_p > 0$ such that as $T \to \infty$,

$$||D_{Tf}\rho_{x,T}||_p = O(e^{-c_pT}) \tag{67}$$

Take $p > 1$, Since D is a first order elliptic operator, by Gårding's inequality, we have $C, C_1, C_2 > 0$ such that for $s \in \Omega^* M)$:

$$\begin{aligned} ||s||_q &\leq C_1(||Ds||_{q-1} + ||s||_0) \\ &\leq C_1(||(\lambda - D_{Tf})s||_{q-1} + C_2T||s||_{q-1} + ||s||_0) \\ &\leq CT^q(||(\lambda - D_{Tf})s||_{q-1} + ||s||_0) \end{aligned} \tag{68}$$

and by equation (42), there also exist $C' > 0$ such that for $\lambda \in \delta$, $s \in \Omega^*(M)$ and T large enough

$$||(\lambda - D_{Tf})^{-1}s||_0 \leq C'||s||_0 \tag{69}$$

Combine equation (68) and (69), we get:

$$||(\lambda - D_{Tf})^{-1}s||_q \leq CT^q(||s||_{q-1} + C'||s||_0) \leq C''||s||_{q-1} \tag{70}$$

Hence, there exist $c_q > 0$ such that for T large enough, we have:

$$||(\lambda - D_{Tf})^{-1}D_{Tf}\rho_{x,T}||_q = O(e^{-cT}) \tag{71}$$

uniformmly on $\lambda \in \delta$. Then by Sobolev inequality, we get there exist $c > 0$ such that:

$$|(\lambda - D_{Tf})^{-1}D_{Tf}\rho_{x,T}| = O(e^{-cT}) \tag{72}$$

uniformlly on λ. Hence prove the first assertion. Since J_T is an isometry, in particular, we have e_T is an isomorphism for T large enough.

□

Now, we define the deform P_∞ by $P_{\infty,T} : F_{T,f}^{[0,1]} \to C^*(W^u)$. By:

$$P_{\infty,T} : \alpha \mapsto P_\infty e^{Tf}\alpha \tag{73}$$

Being composition of chain map, $P_{\infty,T}$ is again a chain map. We also define two operator $\mathcal{F}, N$ on $C^*(W^u)$ by

$$\begin{aligned} \mathcal{F}[W^u(x)]^* &= f(x)[W^u(x)]^* \\ N[W^u(x)]^* &= n_f[W^u(x)]^* \end{aligned} \tag{74}$$

Then we have an estimate:

Theorem 16.

There exists $c > 0$ such that as $T \to \infty$,

$$P_{\infty,T}e_T = e^{T\mathcal{F}}(\frac{\pi}{T})^{N/2-n/4}(1 + O(e^{-cT})) \tag{75}$$

In particular, $P_{\infty,T}$ is an isomorphism for $T > 0$ large enough.

proof:

Take $x \in \mathrm{crit}(f)$, $s = W^u(x)^*$. By definition, we have:

$$P_{\infty,T}e_T s = \sum_{y, n_f(y) = n_f(x)} e^{Tf(y)}W^u(y)^* \int_{\bar{W}(y)} e^{T(f-f(y))}e_T s \tag{76}$$

By the definition of unstable manifold, we must have:

$$f - f(y) \leq 0 \tag{77}$$

on $\bar{W}^u(y)$. Apply Theorem 15, we have:

$$\int_{\bar{W}(y)} e^{T(f-f(y))} e_T s = \int_{\bar{W}(y)} e^{T(f-f(y))} J_T s + O(e^{-cT}) \tag{78}$$

for some $c > 0$. Since $\mathrm{supp}(J_T s) \in U_x$, we can using the definition of $\rho_{x,T}$ to give us:

$$\int_{\bar{W}(x)} e^{T(f-f(x))} e_T s = (\frac{\pi}{T})^{n_f(x)/2-n/4}(1 - O(e^{-cT})) \tag{79}$$

By Proposition 13, we have $\bar{W}(y)\backslash W^u(y)$ is a union of certain $\bar{W^u}(y')$, with $n_f(y') < n_f(y)$. Thus we find that for $y \in \mathrm{crit}(f)$m with $y \neq x$ and $n_f(y) = n_f(x)$, we then have

$$x \notin \bar{W}^u(y) \tag{80}$$

Hence, the by the definition of $\rho_{x,T}$ again, we have:

$$J_T s = O(e^{-c'T}) \tag{81}$$

on $\bar{W}^u(y)$ for some $c' > 0$. Hence we get:

$$\int_{\bar{W}(y)} e^{T(f-f(y))} e_T s = O(e^{-cT}) \tag{82}$$

Combine all the result, we are done.

□

Finally, we get our proof for theorem 14:

proof of theorem 14:

Since we have already seen that $e^{Tf} : F^{[0,1]}_{Tf} \to \Omega^*$ is an quasi-isomorphism by Proposition 4. And now $P_{\infty,T} = P_\infty \circ e^{Tf}$ is an quasi-isomorphism (actually isomorphism) for T large enough, too. Hence P_∞ is an quasi-isomorphism.

□

8 The Product Structure of The Thom-Smale Complex (Notation)

As we all know, the cohomology is a graded ring with cup product as its product structure. In this section, we will follows the discussion of C. Viterbo to encode the product structure on the Thom-Smale Complex. For this, we first clear the notation on his paper.
Let f being a Morse function on a smooth compact manifold M (In his paper, the result can be generalized to non-compact cases in certain ways, but we assume the compactness for simplicity.) And also assume all good property as

before. For all critical points $x, y \in M$ of f. We define $P(x,y)$ being all gradient flow with start points x and end points y.

$$P(x,y) = \{\gamma : \mathbb{R} \to M | \dot{\gamma}(s) = df(s)^*, \lim_{s\to-\infty} \gamma(s) = x, \lim_{s\to\infty} \gamma(s) = y\} \tag{83}$$

There is a natural $\mathbb{R}$ action on $P(x,y)$ defined by $(t.\gamma)(s) = \gamma(s+t)$; Hence, we can defined:

$$\hat{P}(x,y) = P(x,y)/\mathbb{R} \tag{84}$$

Clearly, this is the set of all gradient flow from y to x regardless of the midpoint. Finally, we can define the set:

$$M(x,y) = \{\gamma(0) \in M | \gamma \in P(x,y)\} \tag{85}$$

Be the set of points on $P(x,y)$, this is the same as the intersection $W^u(x) \cap W^s(y)$ we defined last time up to the points x, y. By the uniqueness theorem, we know that $P(x,y)$ is diffeomophic to $M(x,y)$.
Let $i(x)$ denote the Morse index of the critical point x, we then have:

$$\begin{aligned} \dim P(x,y) &= i(y) - i(x) \\ \dim \hat{P}(x,y) &= i(y) - i(x) - 1 \end{aligned} \tag{86}$$

Then the Thom-Smale complex $W^*(f)$ is defined to be the free $\mathbb{R}$-module with one generator for each critical points and graded by its Morse index (when M is not compact, it is useful to define $W^*(f; a, b)$ to restrict the discussion to the region $f^{-1}[a,b]$), and we can define the coboundary map by:

$$\begin{aligned} \delta :& W^k(f) \to W^{k+1}(f) \\ \delta(x) =& \sum_{y \in \mathrm{crit}(f), i(y) = i(x)+1} n(x,y) \cdot y \end{aligned} \tag{87}$$

Where the coefficient $n(x,y)$ is the as before (the intersection number of $W^u(x)$ and $W^s(y)$). We fix the orientation of $W^u(x)$ arbitrarily and define the orientation of $W^s(x)$ by requiring that $W^s(x) \cap W^u(x) = (+1) \cdot x$. We have known that this cohomology is isomorphic the the de Rham cohomology. Now, we are going to define the "cap" product structure on the cohomology:

$$H^*(M) \otimes H^*_{TS}(M,f) \to H^*_{TS}(M,f)$$

9 An $H^*(M)$-module Structure on the Cohomology of (W, δ)

Theorem 17.

Let omega be a closed d-form on M, and let $\pi(\omega)$ be the map:

$$\begin{aligned} \pi(\omega) :W^k(f) &\to W^{k+d}(f) \\ x \quad &\to \sum_{y \in crit(f), i(y) = i(x)+d} (\int_{M(x,y)} \omega) \cdot y \end{aligned} \tag{88}$$

Then we have $\pi(\omega)$ commutes with δ, inducing a map in cohomology:

$$P(\omega) : H^k_{TS}(M, f) \to H^{k+d}_{TS}(M, f) \tag{89}$$

Moreover, this map depends only on the cohomology class of ω in $H^*(M)$, and we have the associativity;

$$P(\omega)P(\omega') = P(\omega \cup \omega') \tag{90}$$

As a result, P defines an $H^*(M)$ module structure on $H^*_{TS}(M, f)$
We will need some lemma to prove this theorem. First is to explore the structure of the set $M(x, y)$:

Lemma 18.

The closure of $M(x, z)$ may be described as

$$\bar{M}(x, z) = \bigcup M(x, y_1) \cup M(y_1, y_2) \cup ... \cup M(y_q, z) \tag{91}$$

The union being over all sequences $y_1, ..., y_q$ of critical points such that $M(x, y_1), M(y_1, y_2), ..., M(y_q, z)$ are all non empty. Moreover, for any such sequence $(y_1, ..., y_q)$, there is a map

$$G : \hat{P}(x, y_1) \times ... \times \hat{P}(y_q, z) \times \Delta^{q+1} \to \bar{M}(x, z) \tag{92}$$

where

$$\Delta^{q+1} = \{(\lambda_0, ..., \lambda_q) \in [-\infty, +\infty]^[q + 1]|1 + \lambda_j \leq \lambda_{j+1}\} \tag{93}$$

and

1. *The image of G is a neighborhood of $M(x, y_1) \cup ... \cup M(y_q, z)$ in $\bar{M}(x, z)$*
2. *The restriction of G to $\hat{P}(x, y_1) \times ... \times \hat{P}(y_q, z) \times \Delta^{q+1\circ}$ is a diffeomorphism onto its image.*
3.

$$G(a_0, ..., a_q, -\infty, ..., \infty, \mu_j, ..., \mu_{j+p}, +\infty, ..., +\infty) = G(a_j, ..., a_{j+p}, \mu_j, ..., \mu_{j+p}) \tag{94}$$

proof:

Since the paper itself does not contains the complete proof, either, and the theorem is very intuitive, we omit the proof.

□

The next lemma is to develop a new Stokes' formula for our application.

Lemma 19.

Let x, z with $i(x) - i(z) = k + 1$. Define

$$\partial M(x,z) = \sum_{i(y)=i(x)+1} n(x,y)M(y,z) + \sum_{i(y)=i(z)-1} n(y,z)M(x,y) \tag{95}$$

Then we have for any k-form ϕ:

$$\int_{\partial M} \phi = \int_{M(x,z)} d\phi \tag{96}$$

proof:

We first using a partition of unity restrict our attention to a neighborhood of $M(x,y_1) \cup ... \cup M(y_q,z)$. Since G^{-1} is a diffeomorphism of such a neigborhood into $\hat{P}(x,y_1) \times ... \times \hat{P}(y_q,z) \times \Delta^\circ_{q+1}$, we may pull back the form $G^*d\phi$ to doing the integral. By the Stoke's formula for manifolds with corners, we see that $G^*\phi$ must be integrated only on:

$$\hat{P}(x,y_1) \times ... \times \hat{P}(y_q,z) \times \{-\infty\}\Delta^\circ_q \bigcup \hat{P}(x,y_1) \times ... \times \hat{P}(y_q,z)\Delta^\circ_q \times \{\infty\} \tag{97}$$

By the third property of G, this integration is on:

$$\hat{P}(y_1,y_2) \times ... \times \hat{P}(y_q,z) \times \Delta^\circ_q \tag{98}$$

or

$$\hat{P}(x,y_1) \times ... \times \hat{P}(y_{q-1},y_q) \times \Delta^\circ_q \tag{99}$$

which have dimension $< k$ unless $i(x) - i(y_1) = 1$ (resp. $i(y_q) - i(z) = 1$). Thus the only integrals of ϕ that appear are these on $M(y,z)$ (resp. $M(x,y)$) with $i(x) - i(y) = 1$ (resp. $i(y) - i(z) = 1$) and the integral appears once for each elemet $P(x,y)$ counted with the proper sign. This concludes the proof.

□

. With the help of this Stokes formula, we can now prove the key lemma:

Lemma 20.

$$\pi(d\omega) = \delta\pi(\omega) + \pi(\omega)\delta \tag{100}$$

proof:

$$\begin{aligned}
\pi(d\omega)x &= \sum_t (\int_{M(x,t)} d\omega)t \\
&= \sum_t (\int_{\partial M(x,t)} \omega)t \\
&= \sum_t (\sum_{i(y)=i(x)+1} n(x,y)\int_{M(y,t)} \omega + \sum_{i(z)=i(t)-1} n(z,t)\int_{M(x,z)} \omega)t \\
&= \sum_t \sum_{i(y)=i(x)+1} n(x,y)(\int_{M(y,t)} \omega)t + \sum_t \sum_{i(z)=i(t)-1} n(z,t)(\int_{M(x,z)} \omega)t \\
&= \pi(\omega)\delta x + \delta\pi(\omega)x \\
&= (\delta\pi(\omega) + \pi(\omega)\delta)x
\end{aligned}$$

Corollary.

If ω is closed, $\pi(\omega)$ induces a map $P(\omega) : H^k_{TS}(M,f) \to H^{k+d}_{TS}(M,f)$ which depends only on the cohomology class of ω.

proof:

If $d\omega = 0$, $\delta\pi(\omega) = -\pi(\omega)\delta$, hence $(-1)^{deg} \cdot \pi(\omega)$ is a chain map, hence define a map in cohomology.
If $\omega = d\phi$, $\pi(d\phi) = \delta\pi(\phi) + \pi(\phi)\delta$, hence $\phi(d\phi)$ sends cocycles to coboundaries: it induces the zero map in cohomology.

□

Now, we prove the final part of the theorem

Lemma 21.

Let ω_1, ω_2 be closed forms, Then we have $P(\omega_1 \wedge \omega_2) = P(\omega_1)P(\omega_2)$

proof:

We have

$$\begin{aligned}
P(\omega_1 \wedge \omega_2)x &= \sum_z (\int_{M(x,z)} \omega_1 \wedge \omega_2)z \\
P(\omega_1)P(\omega_2)x &= \sum_z (\sum_y (\int_{M(x,y)} \omega_1 \int_{M(y,z)} \omega_2))z
\end{aligned} \tag{101}$$

Hence, what we need to prove is the equality:

$$\int_{M(x,z)} \omega_1 \wedge \omega_2 = \sum_y \int_{M(x,y)} \omega_1 \int_{M(y,z)} \omega_2 \tag{102}$$

We only have term with $i(y) = i(x) + k_1 = i(z) - k_2$ in the left hand side.
By the technique of partition of unity, we may assume that ω_1 and ω_2 vanish outside neighborhood of $\bar{M}(x,y)$ and $\bar{M}(y,z)$. Then by lemma 18, we can pullback the problem on the image of $\hat{P}(x,y) \times \hat{P}(y,z) \times \Delta_2^\circ$. Now, considering the cone sapce $C\hat{P}(x,y) \times C\hat{P}(y,z)$, where
$C\hat{P}(x,y) = \hat{P}(x,y) \times [-\infty,\infty]/\hat{P}(x,y) \times \{+\infty\}$ And we have a map:

$$C\hat{P}(x,y) \times C\hat{P}(y,z) \to \quad \hat{P}(x,y) \times \hat{P}(y,z) \times \Delta_2(a_1,t_1,a_2,t_2) \to \quad (a_1,a_2,t_1,t_1+t_2) \tag{103}$$

and maps $(a_1,-\infty,a_2,\infty)$ to $\{y\}$. Now, we can pullback ω_1 and ω_2 on $C\hat{P}(x,y) \times C\hat{P}(y,z)$, and we get two forms ϕ_1, ϕ_2.
By the fact that ω_1 vanishes away from $M(x,y)$, we get that in fact

$$\begin{aligned} \phi_1 &\in H(C\hat{P}(x,y) \times C\hat{P}(y,z), D_1) \\ \phi_2 &\in H(C\hat{P}(x,y) \times C\hat{P}(y,z), D_1) \end{aligned} \tag{104}$$

where $D_i = \{(a_1,t_1,a_2,t_2) | t_i \geq C\}$ for some C. If we let $A = \hat{P}(x,y)$ and $B = \hat{P}(y,z)$, then we have:

$$\begin{aligned} \phi_1 &\in H(CA \times CB, A \times CB) \\ \phi_2 &\in H(CA \times CB, CA \times B) \end{aligned} \tag{105}$$

Then by the fact of algebraic topology, we have the formula:

$$\int_{CA \times CB} \phi_1 \wedge \phi_2 = \int_{CA} \phi_1 \int_{CB} \phi_2 \tag{106}$$

And since $C\hat{P}(x,y) \times \{(a_1,+\infty)\}$ goes to $M(x,y)$, we thus have:

$$\int_{C\hat{P}(x,y)} \phi_1 = \int_{M(x,y)} \omega_1 \tag{107}$$

The lemma follows.

□

Finally, we are going to prove that his product structure coincide with the original cup product one.

Proposition 22.

With the above assumptions. Under the identification of the Thom-Smale cohomology and the de Rham cohomology. The product structure above is just the usual cup product.

proof:

In our case, this proposition is not hard at all. Recall that our identification is given by:

$$\begin{aligned} \mu : H^i(M) &\to W^i(X, f) \\ \omega &\to \sum_x (\int_{W^s}) x \end{aligned} \tag{108}$$

But this is exactly the definition of $P(\omega) \cdot 1$. Hence the identification of the cup product and the product we just define is the direct result of the associativity.

□

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