Geometry and Topological Field Theory F09221011 莊秉勳

Week 1 Homework

<u>Exercise 1</u> $Z(\alpha, \varepsilon) := \int dx\, e^{-\left(\frac{\alpha}{2}x^2 + i\varepsilon x^3\right)}$ for ε small. Prove that

$$Z(\alpha, \varepsilon) = e^{\sum_{\Gamma} n_{\Gamma}}$$ Γ: connected graph, 3-regular, where

$$n_{\Gamma} = \frac{(-3!\, i\varepsilon)^{V}}{\alpha^{E}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \qquad \begin{aligned} V &= \#\text{ vertices of } \Gamma \\ E &= \#\text{ edges of } \Gamma \end{aligned}$$

proof: Using Taylor expansion, we get $Z(\alpha, \varepsilon) = \sum_{n=0}^{\infty} \int dx\, e^{-\frac{\alpha}{2}x^2} \frac{(-i\varepsilon x^3)^n}{n!}$.

Note that $\int (x^3)^r \cdot e^{-\frac{\alpha}{2}x^2} dx = \left(\frac{1}{\alpha}\right)^{3r/2} \cdot (\#\text{ of contractions of } r \text{ "}\curlyvee\text{"})$.
We want to compute the number of contractions of r $\curlyvee$. Note that the contraction of r $\curlyvee$ ends up with a 3-regular graph with r vertices. Then, for each 3-regular graph Γ (either connected or disconnected), we want to compute the number of contractions of $\curlyvee$ to Γ.
For each edge of Γ, we add a new vertex on this edge. Say the new graph Γ'. Consider the labeled r $\curlyvee$ (label the vertices and three nearby "edge"). To form a labeled Γ', we have $(r!)\cdot(3!)^r$ many possibilities. By the Burnside's lemma, the number of contractions of r $\curlyvee$ to Γ' is $\frac{(r!)(3!)^r}{|\mathrm{Aut}(\Gamma')|}$. Hence, we have

$$Z(\alpha, \varepsilon) = \sum_{n:\text{even}} \int dx\, e^{-\frac{\alpha}{2}x^2} \frac{(-i\varepsilon x^3)^n}{n!} = \sum_{n:\text{even}} \frac{(-i\varepsilon)^n}{n!} \int dx\, e^{-\frac{\alpha}{2}x^2} (x^3)^n.$$

$$= \sum_{n:\text{even}} \sum_{\substack{\Gamma:\text{3-regular} \\ \text{with } n \text{ vertices}}} \frac{(-i\varepsilon)^n}{n!} \cdot \left(\frac{1}{\alpha}\right)^{3n/2} \cdot \frac{(n!)(3!)^n}{|\mathrm{Aut}(\Gamma')|} \cdot \sqrt{\frac{2\pi}{\alpha}}$$

$$= \sqrt{\frac{2\pi}{\alpha}} \cdot \left(\sum_{\Gamma:\text{3-regular}} \frac{(-3!\, i\varepsilon)^{V}}{\alpha^{E}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma')|} \right)$$

(Note that a 3-regular graph has even vertices automatically by the formula $3V = 2E$.)

For a 3-regular graph Γ, write $\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + \cdots + a_k \Gamma_k$, where Γ_i are the connected components of Γ. Also, Γ_i are 3-regular. Note that $|\mathrm{Aut}(\Gamma)|$ is given by $\prod_{i=1}^{k} |\mathrm{Aut}(\Gamma_i)| \cdot (a_i!)$. Hence,

$$n_\Gamma = \frac{\prod_{i=1}^{k} n_{\Gamma_i}}{\prod_{i=1}^{k} (a_i)!}.$$

Therefore,
$$Z(\alpha, \varepsilon) = \sqrt{\frac{2\pi}{\alpha}} \cdot \left(\sum_{\Gamma: \text{3-regular}} \frac{(-3! \, i \, \varepsilon)^V}{\alpha^E} \cdot \frac{1}{|\mathrm{Aut}(\Gamma')|} \right)$$

$$= \sqrt{\frac{2\pi}{\alpha}} \cdot \underline{e^{\sum_\Gamma n_\Gamma}}.$$

This notation is defined by Taylor expansion.

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Week 1 Homework

<u>Exercise 3</u> Let $W(z_1 \dots z_N)$ be a quasi-homogeneous polynomial with $W(\lambda^{q_1} z_1 \dots \lambda^{q_N} z_N) = \lambda W(z)$, $\lambda \in \mathbb{C}$. Also, assume that $W(z_1, \dots, z_N) = 0$ defines an isolated hypersurface singularity. A commutative algebra fact states that a sequence $a_i \in A$ $(i = 1, 2 \dots \dim(A))$ in a regular local ring A is regular if and only if $\text{height}((a_i)) = \dim A$. Thus, $\{\partial_i W\}$ form a regular sequence in $\mathbb{C}[z_1 \dots z_N]$ if and only if $\bigcap_{i=1}^{N} \{\partial_i W = 0\}$ is zero dimensional. Hence, we know that $\{\partial_i W\}$ form a regular sequence of $\mathbb{C}[z_1 \dots z_N]$. Define $R^i = \mathbb{C}[z_1 \dots z_N] / \langle \partial_1 W, \dots, \partial_i W \rangle$. Since $\{\partial_i W\}$ is regular, we have short exact sequence:

$$0 \longrightarrow R^{i-1} \xrightarrow{\cdot \partial_i W} R^{i-1} \longrightarrow R^i \longrightarrow 0.$$

By the additivity of Poincaré polynomial and $wt(\partial_i W) = 1 - q_i$, we have

$$P(R^i) = P(R^{i-1}) - t^{1-q_i} P(R^{i-1}) = (1 - t^{1-q_i}) P(R^{i-1}).$$

By induction, we get $P(R) = P\left(\frac{\mathbb{C}[z_1 \dots z_N]}{\langle \partial_i W \rangle}\right) = \prod_{i=1}^{N} (1 - t^{1-q_i}) \cdot P(\mathbb{C}[z_1 \dots z_N])$

$$= \prod_{i=1}^{N} \frac{(1 - t^{1-q_i})}{(1 - t^{q_i})}.$$ Hence, let $t \to 1^-$, we get $\dim R = \prod_{i=1}^{N} \frac{1 - q_i}{q_i}$.

Also, $P(R)$ has some "symmetry". More precisely, the coefficient of t^{Q_α} and that of $t^{D - Q_\alpha}$, where $D = \sum_{i=1}^{N} (1 - 2q_i)$, are the same. Indeed, by replacing t by s^{-1}, we get

$$s^D \cdot P(s^{-1}) = s^D \cdot \prod_{i=1}^{N} \frac{1 - s^{q_i - 1}}{1 - s^{-q_i}} = s^D \cdot \prod_{i=1}^{N} \frac{s^{q_i} - s^{2q_i - 1}}{s^{q_i} - 1} = \prod_{i=1}^{N} \frac{s^{1-q_i} - 1}{s^{q_i} - 1} = P(s).$$

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Week 2 Homework

Exercise Show that "in physical sense" for $T = S^1_\beta$, let $\beta \to 0$

$$\chi(M) = \operatorname{tr}(-1)^F e^{-\beta H} = \frac{1}{(2\pi)^{n/2}} \int_M \operatorname{Pf}(-R).$$ Gauss - Bonnet - Chern.

"proof": The Lagrangian is given by:

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{\sqrt{-1}}{2} g_{ij} \left(\overline{\psi}^i D_t \psi^j - D_t \overline{\psi}^i \psi^j \right) - \frac{1}{2} R_{ijkl} \psi^i \overline{\psi}^j \psi^k \overline{\psi}^l.$$

To get an Euclidean action, we apply a Wick rotation:

$t \longmapsto -\sqrt{-1}\,\tau$, $dt \mapsto -\sqrt{-1}\, d\tau$, $\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \mapsto -\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$

$$\frac{\sqrt{-1}}{2} g_{ij} \left(\overline{\psi}^i \nabla_t \psi^j - \nabla_t \overline{\psi}^i \psi^j \right) \longmapsto -\frac{1}{2} g_{ij} \left(\overline{\psi}^i \nabla_t \psi^j - \nabla_t \overline{\psi}^i \psi^j \right)$$

(The last term remains unchange since it does not involve $\frac{\partial}{\partial t}$.)

Then, the Euclidean action become (obtained using integration by part.)

$$S_E = \int_{S^1_\beta} L\, dt = \int_0^\beta \underline{\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j} + \underline{g_{ij} \overline{\psi}^i (\nabla_t \psi^j)} + \underline{\frac{1}{2} R_{ijkl} \psi^i \overline{\psi}^j \psi^k \overline{\psi}^l \, d\tau}.$$

!! $S_{E_1} + S_{E_2} + S_{E_3}$.

Now, we can use path integral to compute

$$\operatorname{tr}(-1)^F e^{-\beta H} = \int DX\, D\psi\, D\overline{\psi}\, e^{-S_E(x, \psi, \overline{\psi})}$$

Since $T = S^1_\beta$, we have Fourier expansions, for $i = 1, 2 \ldots n$,

$$x^i = x_0^i + \sum_{n \neq 0} x_n^i e^{2\pi\sqrt{-1} n\tau/\beta}$$

$$\psi^i = \psi_0^i + \sum_{n \neq 0} \psi_n^i e^{2\pi\sqrt{-1} n\tau/\beta}, \quad \overline{\psi}^i = \overline{\psi}_0^i + \sum_{n \neq 0} \overline{\psi}_n^i e^{2\pi\sqrt{-1} n\tau/\beta}$$

Since the Witten index is independent of β, we may consider the limit $\beta \to 0$. Also, we choose Riemann normal coordinate at x_0 and we have $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$ at x_0. Then, we have

$$S_{E_1} = \int_0^\beta \frac{1}{2} \cdot \frac{(2\pi\sqrt{-1})^2\, m \cdot (-n)}{\beta^2} \sum_{m, n \neq 0} x_m^i x_{-n}^i e^{2\pi\sqrt{-1}(m-n)\tau/\beta} d\tau = \frac{2\pi^2}{\beta} \sum_{m \neq 0} m^2 x_m^i x_m^i.$$

since x^i are real, we have $x_m^i = x_{-m}^i$

$$S_{E_2} = \int_0^\beta \bar{\psi}^i \dot{\psi}^i d\tau = \int_0^\beta \left(\bar{\psi}_0^i + \sum_{n\neq 0} \bar{\psi}_n^i e^{2\pi\sqrt{-1}n\tau/\beta}\right)\left(\sum_{m\neq 0} \frac{2\pi\sqrt{-1}m}{\beta} \psi_m^i e^{2\pi\sqrt{-1}m\tau/\beta}\right) d\tau$$

$$= 2\pi\sqrt{-1} \sum_{m\neq 0} (-m) \bar{\psi}_m^i \psi_m^i .$$

Then, we obtain

$$S_E = \frac{2\pi^2}{\beta} \sum_{m\neq 0} m^2 x_m^i x_m^i + 2\pi\sqrt{-1} \sum_{m\neq 0} (-m) \bar{\psi}_m^i \psi_m^j + \underline{\frac{\beta}{2} R_{\bar{i}jk\bar{l}} \psi_0^i \bar{\psi}_0^j \psi_0^k \bar{\psi}_0^l} + O(\beta)$$

scaling $\psi_0^i, \bar{\psi}_0^i$ by $\beta^{-1/4}$.

The path integral measure is given by

$$DX = \frac{dx_0^1 \cdots dx_0^n}{(2\pi)^{n/2}} \prod_{m\neq 0} \frac{dx_m^1 \cdots dx_m^n}{(2\pi)^{n/2}}$$

$$D\psi = d\psi_0^1 \cdots d\psi_0^n \prod_{m\neq 0} d\psi_m^1 \cdots d\psi_m^n, \quad D\bar{\psi} = d\bar{\psi}_0^1 \cdots d\bar{\psi}_0^n \prod_{m\neq 0} d\bar{\psi}_m^1 \cdots d\bar{\psi}_m^n$$

Then, as $\beta \to 0^+$, we get $\operatorname{tr} (-1)^F e^{-\beta H} = \int DX\, D\psi\, D\bar{\psi}\, e^{-S_E(X,\psi,\bar{\psi})}$

$$= \prod_{m\neq 0} \left(\int \frac{dx_m^1 \cdots dx_m^n}{(2\pi)^{n/2}} \exp\left(- \frac{2\pi^2}{\beta} m^2 x_m^i x_m^i \right)\right) \cdot$$

$$\prod_{m\neq 0} \int d\psi_m^1 \cdots d\psi_m^n d\bar{\psi}_m^1 \cdots d\bar{\psi}_m^n \exp\left(2\pi\sqrt{-1}\, m\, \bar{\psi}_m^i \psi_m^i \right) \cdot$$

$$\int \frac{dx_0^1 \cdots dx_0^n}{(2\pi)^{n/2}} \int d\psi_0^1 \cdots d\psi_0^n d\bar{\psi}_0^1 \cdots d\bar{\psi}_0^n \exp\left(- \frac{\beta}{2} R_{\bar{i}jk\bar{l}} \psi_0^i \bar{\psi}_0^j \psi_0^k \bar{\psi}_0^l \right)$$

$$= \prod_{m\neq 0} \left(\frac{1}{(2\pi)^{n/2}} \cdot \left(\frac{\sqrt{\pi\beta}}{\sqrt{2}\cdot\pi\cdot|m|} \right)^n \right) \cdot \prod_{m\neq 0} (2\pi\sqrt{-1}\, m)^n \int d\psi_m^1 \cdots d\psi_m^n d\bar{\psi}_m^1 \cdots d\bar{\psi}_m^n \left(\prod_{i=1}^n \bar{\psi}_m^i \psi_m^i \right)$$

$$\cdot \int \frac{dy_0^1 \cdots dy_0^n}{(2\pi)^{n/2}} \int d\eta_0^1 \cdots d\eta_0^n d\bar{\eta}_0^1 \cdots d\bar{\eta}_0^n \exp\left(- \frac{1}{2} R_{\bar{i}jk\bar{l}} \eta_0^i \bar{\eta}_0^j \eta_0^k \bar{\eta}_0^l \right) \cdot \beta^{n/2}$$

(Here, we apply change of variables by $y_0^i = \beta^{-1/2} x_0^i$, $\eta_0^i = \beta^{-1/4} \psi_0^i$, $\bar{\eta}_0^i = \beta^{-1/4} \bar{\psi}_0^i$. and the Berezinian is given by 1.)

$$= \underline{\beta^{-n/2}} \cdot \underline{1} \cdot \frac{1}{(2\pi)^{n/2}} \cdot \int_M \operatorname{Pf}(-R) \cdot \beta^{n/2} = \frac{1}{(2\pi)^{n/2}} \int_M \operatorname{Pf}(-R) .$$

$$\prod_{m=1}^\infty (2\pi\sqrt{-1}m) = e^{-\frac{\pi}{4}\sqrt{-1}} \text{ and } \prod_{m=1}^\infty (2\pi\sqrt{-1}\cdot(-m)) = e^{\frac{\pi}{4}\sqrt{-1}}.$$

#

$$\prod_{m=1}^\infty \frac{\sqrt{\beta}}{2\pi m} = \beta^{-1/2} \text{ by taking } \left.\frac{d}{ds}\right|_{s=0} \text{ on } \left(\frac{\sqrt{\beta}}{2\pi}\right)^s \cdot \zeta(s) = \sum_{m=1}^\infty \left(\frac{\sqrt{\beta}}{2\pi m}\right)^s.$$

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Week 3 Homework

$$L = \underbrace{\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j}_{①} + \underbrace{\frac{\sqrt{-1}}{2} g_{ij}\left(\bar{\psi}^i \underline{D_t} \psi^j - D_t \bar{\psi}^i \psi^j\right)}_{②} - \underbrace{\frac{1}{2} R_{ijkl} \psi^i \bar{\psi}^j \psi^k \bar{\psi}^l}_{③}$$

covariant derivative $D_t \psi^j = \dot{\psi}^j + \Gamma^j_{lm} \dot{x}^l \psi^m$

Super symmetry :

$$\delta x^i = \varepsilon \bar{\psi}^i - \bar{\varepsilon} \psi^i \qquad \delta \dot{x}^i = \varepsilon \dot{\bar{\psi}}^i - \bar{\varepsilon} \dot{\psi}^i$$

$$\delta \psi^i = \varepsilon\left(\sqrt{-1}\, \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k\right) \qquad \delta g_{ij} = g_{ij,k}\left(\varepsilon \bar{\psi}^k - \bar{\varepsilon} \psi^k\right)$$

$$\delta \bar{\psi}^i = \bar{\varepsilon}\left(-\sqrt{-1}\, \dot{x}^i - \Gamma^i_{jk} \bar{\psi}^j \psi^k\right) \qquad \delta \Gamma^i_{pq} = \Gamma^i_{pq,k}\left(\varepsilon \bar{\psi}^k - \bar{\varepsilon} \psi^k\right)$$

$$\delta \dot{\psi}^i = \varepsilon\left(\sqrt{-1}\, \ddot{x}^i - \Gamma^i_{pq,k} \dot{x}^k \bar{\psi}^p \psi^q - \Gamma^i_{pq} \dot{\bar{\psi}}^p \psi^q - \Gamma^i_{pq} \bar{\psi}^p \dot{\psi}^q\right)$$

$$\delta \dot{\bar{\psi}}^i = \bar{\varepsilon}\left(-\sqrt{-1}\, \ddot{x}^i - \Gamma^i_{pq,k} \dot{x}^k \bar{\psi}^p \psi^q - \Gamma^i_{pq} \dot{\bar{\psi}}^p \psi^q - \Gamma^i_{pq} \bar{\psi}^p \dot{\psi}^q\right)$$

Show that $\delta \int L\, dt = 0$.

proof:

$$\delta① = \frac{1}{2} g_{ij,k}\left(\underbrace{\varepsilon \bar{\psi}^k}_{⑪} - \underbrace{\bar{\varepsilon} \psi^k}_{⑫}\right) \dot{x}^i \dot{x}^j + g_{ij}\left(\underbrace{\varepsilon \dot{\bar{\psi}}^i}_{⑪} - \underbrace{\bar{\varepsilon} \dot{\psi}^i}_{⑬}\right) \dot{x}^j$$

$$\delta② = \sqrt{-1}\, g_{ij,k}\left(\underbrace{\varepsilon \bar{\psi}^k \bar{\psi}^i \dot{\psi}^j}_{⑮} + \varepsilon \bar{\psi}^k \bar{\psi}^i \Gamma^j_{lm} \dot{x}^l \psi^m - \underbrace{\bar{\varepsilon} \psi^k \bar{\psi}^i \dot{\psi}^j}_{⑯} - \bar{\varepsilon} \psi^k \bar{\psi}^i \Gamma^j_{lm} \dot{x}^l \psi^m\right)$$

$$+ \sqrt{-1}\, g_{ij}\Bigg(\bar{\varepsilon}\left(\underbrace{-\sqrt{-1}\, \dot{x}^i \dot{\psi}^j}_{⑬} - \underbrace{\Gamma^i_{pq} \bar{\psi}^p \psi^q \dot{\psi}^j}_{⑯} - \underbrace{\sqrt{-1}\, \dot{x}^i \Gamma^j_{lm} \dot{x}^l \psi^m}_{⑫} - \Gamma^i_{pq} \Gamma^j_{lm} \bar{\psi}^p \psi^q \dot{x}^l \psi^m\right)$$

$$+ \bar{\psi}^i \Big[\varepsilon\left(\underbrace{\sqrt{-1}\, \ddot{x}^j}_{⑪} - \Gamma^j_{pq,k} \dot{x}^k \bar{\psi}^p \psi^q - \underbrace{\Gamma^j_{pq} \dot{\bar{\psi}}^p \psi^q}_{⑭} - \underbrace{\Gamma^j_{pq} \bar{\psi}^p \dot{\psi}^q}_{⑮}\right)$$

$$+ \Gamma^j_{lm,k}\left(\varepsilon \bar{\psi}^k - \bar{\varepsilon} \psi^k\right) \dot{x}^l \psi^m + \Gamma^j_{lm}\left(\underbrace{\varepsilon \dot{\bar{\psi}}^l}_{⑭} - \underbrace{\bar{\varepsilon} \dot{\psi}^l}_{⑯}\right) \psi^m$$

$$+ \Gamma^j_{lm} \dot{x}^l \varepsilon\left(\underbrace{\sqrt{-1}\, \dot{x}^m}_{⑪} - \Gamma^m_{pq} \bar{\psi}^p \psi^q\right)\Big]\Bigg)$$

$$⑪ = \frac{1}{2} g_{ij,k} \varepsilon \bar{\psi}^k \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij} \varepsilon \dot{\bar{\psi}}^i \dot{x}^j - \underline{g_{ij} \bar{\psi}^i \varepsilon \ddot{x}^j} - g_{ij} \bar{\psi}^i \Gamma^j_{lm} \dot{x}^l \varepsilon \dot{x}^m$$

↓ integration by parts.

$$= \frac{1}{2} g_{ij,k} \varepsilon \bar{\psi}^k \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij} \varepsilon \dot{\bar{\psi}}^i \dot{x}^j + \underline{g_{ij,k} \dot{x}^k \bar{\psi}^i \varepsilon \dot{x}^j + g_{ij} \dot{\bar{\psi}}^i \varepsilon \dot{x}^j}$$

$$- \frac{1}{2}\left(g_{li,m} + g_{im,l} - g_{ml,i}\right) \bar{\psi}^i \dot{x}^l \varepsilon \dot{x}^m = 0.$$

$$⑫ = \frac{-1}{2} g_{ij,k} \bar{\varepsilon} \psi^k \dot{x}^i \dot{x}^j + g_{ij} \bar{\varepsilon} \dot{x}^i \Gamma^j_{lm} \dot{x}^l \psi^m = \frac{-1}{2} g_{ij,k} \bar{\varepsilon} \psi^k \dot{x}^i \dot{x}^j + \frac{\bar{\varepsilon}}{2}\left(g_{il,m} + g_{im,l} - g_{ml,i}\right) \dot{x}^i \dot{x}^l \psi^m = 0.$$

$⑬ = 0$, $⑭ = 0$ is clear. $⑮ = 0$ using $g_{ij} \Gamma^j_{pq} = \frac{1}{2}\left(g_{ip,q} + g_{iq,p} - g_{pq,i}\right)$.

$$⑯ = -\sqrt{-1}\, g_{ij,k} \bar{\varepsilon} \psi^k \bar{\psi}^i \dot{\psi}^j - \sqrt{-1}\, g_{ij} \bar{\varepsilon} \Gamma^i_{pq} \bar{\psi}^p \psi^q \dot{\psi}^j - \sqrt{-1}\, g_{ij} \bar{\psi}^i \Gamma^j_{lm} \bar{\varepsilon} \dot{\psi}^l \psi^m$$

$$= -\sqrt{-1}\left(g_{ij,k} \bar{\varepsilon} \psi^k \bar{\psi}^i \dot{\psi}^j + \frac{\bar{\varepsilon}}{2}\left(g_{jp,q} + \underline{g_{jq,p}} - g_{pq,j}\right) \bar{\psi}^p \psi^q \dot{\psi}^j + \frac{\bar{\varepsilon}}{2}\left(g_{il,m} + g_{im,l} - \underline{g_{ml,i}}\right) \bar{\psi}^i \psi^m \dot{\psi}^l\right) = 0.$$

$$\delta \textcircled{3} = -\frac{1}{2}\,\delta\left(R_{ipqr}\,\psi^i\,\bar\psi^p\,\psi^q\,\bar\psi^r\right)$$

$$\begin{aligned}
= -\frac{1}{2}\Big(& \underbrace{R_{ijkl,m}\left(\varepsilon\bar\psi^m - \bar\varepsilon\psi^m\right)\psi^i\bar\psi^j\psi^k\bar\psi^l}_{\textcircled{8}} \\
& + R_{ijkl}\,\varepsilon\left(\sqrt{-1}\,\dot x^i - \underbrace{\Gamma^i_{pq}\,\bar\psi^p\psi^q}_{\textcircled{4}}\right)\bar\psi^j\psi^k\bar\psi^l \\
& + R_{ijkl}\,\psi^i\,\bar\varepsilon\left(-\sqrt{-1}\,\dot x^j - \underbrace{\Gamma^j_{pq}\,\bar\psi^p\psi^q}_{\textcircled{5}}\right)\psi^k\bar\psi^l \\
& + R_{ijkl}\,\psi^i\bar\psi^j\,\varepsilon\left(\sqrt{-1}\,\dot x^k - \underbrace{\Gamma^k_{pq}\,\bar\psi^p\psi^q}_{\textcircled{6}}\right)\bar\psi^l \\
& + R_{ijkl}\,\psi^i\bar\psi^j\psi^k\,\bar\varepsilon\left(-\sqrt{-1}\,\dot x^l - \underbrace{\Gamma^l_{pq}\,\bar\psi^p\psi^q}_{\textcircled{7}}\right)\Big)
\end{aligned}$$

Note that $R_{ijkl;m} = R_{ijkl,m} - \Gamma^r_{im}R_{rjkl} - \Gamma^r_{jm}R_{irkl} - \Gamma^r_{km}R_{ijrl} - \Gamma^r_{lm}R_{ijkr}$.

$$\begin{aligned}
\textcircled{8} = {} & R_{ijkl,m}\left(\varepsilon\bar\psi^m - \bar\varepsilon\psi^m\right)\psi^i\bar\psi^j\psi^k\bar\psi^l \\
= {} & \underbrace{R_{ijkl;m}\left(\varepsilon\bar\psi^m - \bar\varepsilon\psi^m\right)\psi^i\bar\psi^j\psi^k\bar\psi^l}_{\textcircled{10}} + \Gamma^r_{im}R_{rjkl}\Big(\underbrace{\varepsilon\bar\psi^m}_{\textcircled{4}} - \underbrace{\bar\varepsilon\psi^m}_{\textcircled{9}}\Big)\psi^i\bar\psi^j\psi^k\bar\psi^l \\
& + \Gamma^r_{jm}R_{irkl}\Big(\underbrace{\varepsilon\bar\psi^m}_{\textcircled{9}'} - \underbrace{\bar\varepsilon\psi^m}_{\textcircled{5}}\Big)\psi^i\bar\psi^j\psi^k\bar\psi^l \\
& + \Gamma^r_{km}R_{ijrl}\Big(\underbrace{\varepsilon\bar\psi^m}_{\textcircled{6}} - \underbrace{\bar\varepsilon\psi^m}_{\textcircled{9}}\Big)\psi^i\bar\psi^j\psi^k\bar\psi^l \\
& + \Gamma^r_{lm}R_{ijkr}\Big(\underbrace{\varepsilon\bar\psi^m}_{\textcircled{9}'} - \underbrace{\bar\varepsilon\psi^m}_{\textcircled{7}}\Big)\psi^i\bar\psi^j\psi^k\bar\psi^l
\end{aligned}$$

$$\begin{aligned}
\textcircled{10} = {} & \left(R_{ijml;k} + R_{ijkm;l}\right)\left(\varepsilon\bar\psi^m - \bar\varepsilon\psi^m\right)\psi^i\bar\psi^j\psi^k\bar\psi^l \\
= {} & \underbrace{\varepsilon R_{ijml;k}\,\bar\psi^m\psi^i\bar\psi^j\psi^k\bar\psi^l}_{=0 \text{ by 1st Bianchi identity}} + \varepsilon R_{ijkm;l}\,\bar\psi^m\psi^i\bar\psi^j\psi^k\bar\psi^l \\
& + \bar\varepsilon R_{ijml;k}\,\psi^m\psi^i\bar\psi^j\psi^k\bar\psi^l + \underbrace{\bar\varepsilon R_{ijkm;l}\,\psi^m\psi^i\bar\psi^j\psi^k\bar\psi^l}_{=0 \text{ by 1st Bianchi identity}} = -\textcircled{10} \quad \text{implies} \quad \textcircled{10} = 0.
\end{aligned}$$

$\textcircled{9} = 0$ since we can change index m and i to get $\textcircled{9} = -\textcircled{9}$.

$\textcircled{9}'$ is similar to $\textcircled{9}$

The remaining terms are

$$\begin{aligned}
& \sqrt{-1}\,g_{ij,k}\left(\varepsilon\bar\psi^k\bar\psi^i\Gamma^j_{lm}\dot x^l\psi^m - \bar\varepsilon\psi^k\bar\psi^i\Gamma^j_{lm}\dot x^l\psi^m\right) \\
& + \sqrt{-1}\,g_{ij}\Big(-\bar\varepsilon\,\Gamma^i_{pq}\Gamma^j_{lm}\bar\psi^p\psi^q\dot x^l\psi^m - \bar\psi^i\varepsilon\Gamma^j_{pq,k}\dot x^k\bar\psi^p\psi^q + \bar\psi^i\Gamma^j_{lm,k}\left(\varepsilon\bar\psi^k - \bar\varepsilon\psi^k\right)\dot x^l\psi^m \\
& \qquad - \bar\psi^i\Gamma^j_{lm}\dot x^l\,\varepsilon\,\Gamma^m_{pq}\bar\psi^p\psi^q\Big) \\
& - \frac{1}{2}\Big(R_{ijkl}\,\varepsilon\sqrt{-1}\,\dot x^i\,\bar\psi^j\psi^k\bar\psi^l - R_{ijkl}\,\psi^i\,\bar\varepsilon\sqrt{-1}\,\dot x^j\,\psi^k\bar\psi^l \\
& \qquad + R_{ijkl}\,\psi^i\bar\psi^j\,\varepsilon\sqrt{-1}\,\dot x^k\,\bar\psi^l - R_{ijkl}\,\psi^i\bar\psi^j\psi^k\,\bar\varepsilon\sqrt{-1}\,\dot x^l\Big)
\end{aligned}$$

The terms involving ε are

$$\sqrt{-1}\, g_{ij,k}\left(\varepsilon \bar\psi^k \bar\psi^i \Gamma^j_{lm} \dot x^l \psi^m\right)$$

$$+\sqrt{-1}\, g_{ij}\left(-\bar\psi^i \varepsilon \Gamma^j_{pq,k} \dot x^k \bar\psi^p \psi^q + \bar\psi^i \Gamma^j_{lm,k} \varepsilon \bar\psi^k \dot x^l \psi^m - \bar\psi^i \Gamma^j_{lm} \dot x^l \varepsilon \Gamma^m_{pq} \bar\psi^p \psi^q\right)$$

$$-\frac{1}{2}\left(R_{ijkl}\, \varepsilon \sqrt{-1}\, \dot x^i \bar\psi^j \psi^k \bar\psi^l + R_{ijkl} \psi^i \bar\psi^j \varepsilon \sqrt{-1}\, \dot x^k \bar\psi^l\right)$$

$$= \sqrt{-1}\left(g_{is}\Gamma^s_{jk} + \underbrace{g_{sj}\Gamma^s_{ik}}_{(17)}\right)\varepsilon \bar\psi^k \bar\psi^i \Gamma^j_{lm} \dot x^l \psi^m$$

$$+\sqrt{-1}\, g_{ij}\left(-\bar\psi^i \varepsilon \Gamma^j_{pq,k} \dot x^k \bar\psi^p \psi^q + \bar\psi^i \Gamma^j_{lm,k} \varepsilon \bar\psi^k \dot x^l \psi^m\right) - \sqrt{-1}\, g_{ij} \bar\psi^i \Gamma^j_{lm} \dot x^l \varepsilon \Gamma^m_{pq} \bar\psi^p \psi^q$$

$$-\left(g_{lm}\left(\Gamma^m_{jk,i} - \Gamma^m_{ik,j} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}\right)\cdot \varepsilon \sqrt{-1}\, \dot x^i \bar\psi^j \psi^k \bar\psi^l\right)$$

(17) $= 0$ since we can interchange $i \leftrightarrow k$ to get (17) $= -$(17).

$$= -\varepsilon\sqrt{-1}\, g_{lm}\Gamma^m_{sj}\Gamma^s_{ik} \dot x^i \bar\psi^j \psi^k \bar\psi^l + \varepsilon\sqrt{-1}\, g_{lm}\Gamma^m_{jk,i} \dot x^i \bar\psi^j \psi^k \bar\psi^l - \varepsilon\sqrt{-1}\, g_{lm}\Gamma^m_{ik,j} \dot x^i \bar\psi^j \psi^k \bar\psi^l$$

$$+\varepsilon\sqrt{-1}\, g_{lm}\Gamma^m_{is}\Gamma^s_{jk} \dot x^i \bar\psi^j \psi^k \bar\psi^l$$

$$-\left(g_{lm}\left(\Gamma^m_{jk,i} - \Gamma^m_{ik,j} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}\right)\cdot \varepsilon \sqrt{-1}\, \dot x^i \bar\psi^j \psi^k \bar\psi^l\right) = 0.$$

Similarly, the $\bar\varepsilon$ term gives

$$-\sqrt{-1}\, g_{ij,k}\bar\varepsilon \psi^k \bar\psi^i \Gamma^j_{lm} \dot x^l \psi^m - \sqrt{-1}\, g_{ij}\bar\varepsilon \Gamma^i_{pq}\Gamma^j_{lm} \bar\psi^p \psi^q \dot x^l \psi^m - \sqrt{-1}\, \bar\psi^i \Gamma^j_{lm,k} \bar\varepsilon \psi^k \dot x^l \psi^m$$

$$+\frac{1}{2} R_{ijkl}\psi^i \bar\varepsilon \sqrt{-1}\, \dot x^j \psi^k \bar\psi^l + \frac{1}{2} R_{ijkl}\psi^i \bar\psi^j \psi^k \bar\varepsilon \sqrt{-1}\, \dot x^l = 0.$$

Hence, $\delta \int L\, dt = 0$.

To get the Noether charge, we assume $\varepsilon = \varepsilon(t)$, $\bar\varepsilon = \bar\varepsilon(t)$ is a function of t. Then,

$$\delta \int L\, dt = \int \quad g_{ij}\left(\dot\varepsilon \bar\psi^i - \dot{\bar\varepsilon} \psi^i\right)\dot x^j$$

$$+\sqrt{-1}\, g_{ij}\bar\psi^i\left(\dot\varepsilon\left(\sqrt{-1}\, \dot x^j - \Gamma^j_{pq}\bar\psi^p\psi^q\right) + \Gamma^j_{lm}\left(\varepsilon \bar\psi^l - \bar\varepsilon \psi^l\right)\psi^m\right)$$

$$+ g_{ij}\bar\psi^i \dot\varepsilon \dot x^j \qquad dt$$

$\overset{\shortparallel}{0}$ by changing $l \leftrightarrow m$.

$$= \int g_{ij}\dot\varepsilon \bar\psi^i \dot x^j - g_{ij}\dot{\bar\varepsilon}\psi^i \dot x^j \, dt = \int -\sqrt{-1}\,\dot\varepsilon\underbrace{\left(\sqrt{-1}\, g_{ij}\bar\psi^i \dot x^j\right)}_{Q} - \sqrt{-1}\,\dot{\bar\varepsilon}\underbrace{\left(-\sqrt{-1}\, g_{ij}\psi^i \dot x^j\right)}_{\bar Q}$$

Noether charge: Q $\quad \bar Q$

#

Geometry and Topological Field Theory F09221011 莊秉勳

Week 4 Homework

Computation of the two-point correlation function of vertex operator:

$$\langle e^{ik_1 X(t_1,s_1)} e^{ik_2 X(t_2,s_2)} \rangle$$

$$= \langle 0 | T[:e^{ik_1 X(t_1,s_1)}::e^{ik_2 X(t_2,s_2)}:] | 0 \rangle$$

First, we assume $t_1 > t_2$ and thus the time ordering product is $:e^{ik_1 X(t_1,s_1)}::e^{ik_2 X(t_2,s_2)}:$.

$$:e^{ik_1 X(t_1,s_1)}::e^{ik_2 X(t_2,s_2)}:|0\rangle \qquad z_j = e^{i(t_j - s_j)}, \quad \tilde{z}_j = e^{i(t_j + s_j)}$$

$$\exp\left(\frac{-k_1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{-1}{n}\left(\alpha_{-n} z_1^{n} + \tilde{\alpha}_{-n} \tilde{z}_1^{n}\right)\right) e^{ik_1 x_0} e^{ik_1 t p_0} \exp\left(\frac{-k_1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{n} z_1^{-n} + \tilde{\alpha}_{n} \tilde{z}_1^{-n}\right)\right)$$

$$\cdot \exp\left(\frac{-k_2}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{-1}{n}\left(\alpha_{-n} z_2^{n} + \tilde{\alpha}_{-n} \tilde{z}_2^{n}\right)\right) e^{ik_2 x_0} e^{ik_2 t p_0} \underline{\exp\left(\frac{-k_2}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{n} z_2^{-n} + \tilde{\alpha}_{n} \tilde{z}_2^{-n}\right)\right)|0\rangle}$$

$= |0\rangle$ since $\alpha_n, \tilde{\alpha}_n$ annilate $|0\rangle$ and $p_0 |k\rangle_0 = k|k\rangle_0$

$= |k_2\rangle_0 \otimes \bigotimes_{n\geq 1} |0\rangle_n$.

Now, we compute the operator

$$\exp\left(\frac{-k_1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{n} z_1^{-n} + \tilde{\alpha}_{n} \tilde{z}_1^{-n}\right)\right) \cdot \exp\left(\frac{-k_2}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{-1}{n}\left(\alpha_{-n} z_2^{n} + \tilde{\alpha}_{-n} \tilde{z}_2^{n}\right)\right) \text{ on } |k_2\rangle_0 \otimes \bigotimes_{n\geq 1} |0\rangle_n .$$

The only non-trivial commuting relation of $\alpha_{\pm n}, \tilde{\alpha}_{\pm n}$ are $[\alpha_n, \alpha_{-n}] = n = [\tilde{\alpha}_n, \tilde{\alpha}_{-n}]$.

Then, since $\alpha_n, \tilde{\alpha}_n$ annilate $|0\rangle_n$, the remaining term, for each n, is

$$\sum_{\ell \leq m} \frac{1}{\ell!}\left(\frac{-k_1}{\sqrt{2}n}\right)^{\ell} \cdot \left(\alpha_n z_1^{-n}\right)^{\ell} \cdot \frac{1}{m!}\left(\frac{k_2}{\sqrt{2}n}\right)^{m} \left(\alpha_{-n} z_2^{n}\right)^{m}$$

$$+ \sum_{\ell \leq m} \frac{1}{\ell!}\left(\frac{-k_1}{\sqrt{2}n}\right)^{\ell} \cdot \left(\tilde{\alpha}_n \tilde{z}_1^{-n}\right)^{\ell} \cdot \frac{1}{m!}\left(\frac{k_2}{\sqrt{2}n}\right)^{m} \left(\tilde{\alpha}_{-n} \tilde{z}_2^{n}\right)^{m}$$

$$= \sum_{\ell \leq m} \frac{1}{\ell!\, m!}\left(\frac{-k_1 z_1^{-n}}{\sqrt{2}n}\right)^{\ell}\left(\frac{k_2 z_2^{n}}{\sqrt{2}n}\right)^{m} \cdot m(m-1)\cdots(m-\ell+1)\, n^{\ell} \cdot \alpha_{-n}^{m-\ell}$$

$$+ \sum_{\ell \leq m} \frac{1}{\ell!\, m!}\left(\frac{-k_1 \tilde{z}_1^{-n}}{\sqrt{2}n}\right)^{\ell}\left(\frac{k_2 \tilde{z}_2^{n}}{\sqrt{2}n}\right)^{m} \cdot m(m-1)\cdots(m-\ell+1)\, n^{\ell} \cdot \tilde{\alpha}_{-n}^{m-\ell}$$

let $m-\ell=j_n$

$$= e^{-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n} \sum_{j_n=0}^{\infty} \frac{1}{(j_n)!}\left(\frac{k_2 z_2^n}{\sqrt{2}\,n}\right)^{j_n} \alpha_{-n}^{j_n} + e^{-\frac{k_1k_2}{2n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n} \sum_{j_n=0}^{\infty} \frac{1}{(j_n)!}\left(\frac{k_2 \tilde{z}_2^n}{\sqrt{2}\,n}\right)^{j_n} \tilde{\alpha}_{-n}^{j_n}$$

Then, we get

$$:e^{ik_1X(t_1,s_1)}::e^{ik_2X(t_2,s_2)}:|0\rangle = \exp\left(\frac{-k_1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{-1}{n}\left(\alpha_{-n} z_1^n + \tilde{\alpha}_{-n}\tilde{z}_1^n\right)\right) \underline{e^{ik_1x_0} e^{ik_1tp_0}} \cdot$$

$$\prod_{n\geq 1}\left(e^{-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n} \sum_{j_n=0}^{\infty} \frac{1}{(j_n)!}\left(\frac{k_2 z_2^n}{\sqrt{2}\,n}\right)^{j_n} \alpha_{-n}^{j_n} + e^{-\frac{k_1k_2}{2n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n} \sum_{j_n=0}^{\infty} \frac{1}{(j_n)!}\left(\frac{k_2 \tilde{z}_2^n}{\sqrt{2}\,n}\right)^{j_n} \tilde{\alpha}_{-n}^{j_n}\right)\underline{\left(|k_2\rangle_0 \otimes \bigotimes_{n\geq 1}|0\rangle_n\right)}$$

$$\Downarrow$$

$$e^{ik_1k_2t_1}\left(|k_1+k_2\rangle_0 \otimes \bigotimes_{n\geq 1}|0\rangle_n\right)$$

Thus, to find $\langle 0|:e^{ik_1X(t_1,s_1)}::e^{ik_2X(t_2,s_2)}:|0\rangle$, we need $j_n=0$ and $k_1+k_2=0$ since $\{|0\rangle, \alpha_{-n}^{j}|0\rangle, \tilde{\alpha}_{-n}^{j}|0\rangle\}$ are orthogonal. Hence, we get

$$\underline{2\pi\cdot\exp\left(\sum_{n\geq 1}-\frac{k_1k_2}{2n}\left(\frac{z_2}{z_1}\right)^n\right)\cdot\exp\left(\sum_{n\geq 1}-\frac{k_1k_2}{2n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n\right)e^{ik_1k_2t_1}}\cdot\delta(k_1+k_2).$$

This converges to $\exp\left(-\frac{k_1k_2}{2}\left(\sum_{n\geq 1}\frac{1}{n}\left(\frac{z_2}{z_1}\right)^n - \sum_{n\geq 1}\frac{1}{n}\left(\frac{\tilde{z}_2}{\tilde{z}_1}\right)^n + 2it_1\right)\right)$

$$= \exp\left(-\frac{k_1k_2}{2}\left(\log(z_1-z_2)+\log(\tilde{z}_1-\tilde{z}_2)\right)\right)$$

$$= \left[(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)\right]^{-\frac{k_1k_2}{2}}$$

$$= 2\pi\,\delta(k_1+k_2)\,\left[(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)\right]^{-\frac{k_1k_2}{2}}.$$

#

Geometry and Topological Field Theory F09221011 莊秉勳

Week5 Homework

Exercise Let $\tau=\tau_1+\sqrt{-1}\tau_2$. $(\zeta,\bar\zeta)=\left(\frac{s+it}{2\pi},\frac{s-it}{2\pi}\right)$: coordinate of torus with $\zeta\equiv\zeta+1\equiv\zeta+\tau$. Assume that

$$\begin{cases}\psi_-(t,s)=e^{-2\pi i a}\psi_-(t,s+2\pi)=e^{-2\pi i b}\psi_-(t+2\pi\tau_2,s+2\pi\tau_1)\\ \psi_+(t,s)=e^{2\pi i\tilde a}\psi_+(t,s+2\pi)=e^{2\pi i\tilde b}\psi_+(t+2\pi\tau_2,s+2\pi\tau_1)\end{cases}$$

Check that periodic Dirac Fermions coupled to flat connection

$$\underset{\text{on } S_-}{A^{0,1}=\frac{\pi\sqrt{-1}}{\tau_2}(b-\tau a)\,d\bar\zeta}\ ,\quad \underset{\text{on } S_+}{\tilde A^{1,0}=\frac{\pi\sqrt{-1}}{\tau_2}(\tilde b-\bar\tau\tilde a)\,d\zeta}$$

and solve the general solution.

proof: We want to find the twist $w(t,s)$ such that $\underset{\psi'_-(t,s)}{\underset{\parallel}{\psi_-(t,s)e^{w(t,s)}}}$ is periodic, i.e. $\psi'_-(t,s+2\pi)=\psi'_-(t,s)=\psi'_-(t+2\pi\tau_2,s+2\pi\tau_1)$.

$$\psi'_-(t,s+2\pi)=\psi_-(t,s+2\pi)e^{w(t,s+2\pi)}=\psi_-(t,s)e^{2\pi i a}e^{w(t,s+2\pi)}.$$

$$\begin{aligned}\psi'_-(t+2\pi\tau_2,s+2\pi\tau_1)&=\psi_-(t+2\pi\tau_2,s+2\pi\tau_1)e^{w(t+2\pi\tau_2,s+2\pi\tau_1)}\\&=\psi_-(t,s)e^{2\pi i b}e^{w(t+2\pi\tau_2,s+2\pi\tau_1)}\end{aligned}$$

Let $w(t,s)=ut+vs$. Then, we get $\begin{cases}v\cdot 2\pi+2\pi i a=0\\ u\cdot(2\pi\tau_2)+v\cdot(2\pi\tau_1)+2\pi i b=0.\end{cases}$

Solve and get $\begin{cases}u=\frac{1}{\tau_2}(-ib+i\tau_1 a)\\ v=-ia\end{cases}$

Then, $\psi'_-(t,s)=\psi_-(t,s)\cdot e^{\frac{i(\tau_1 a-b)}{\tau_2}t-ias}$ is periodic.

Now, changing variable $s=\pi(\zeta+\bar\zeta)$, $t=\frac{\pi(\zeta-\bar\zeta)}{i}$, we get

$$e^{\frac{i(\tau_1 a-b)}{\tau_2}t-ias}=e^{\pi\frac{a\bar\tau-b}{\tau_2}\zeta-\pi\frac{a\tau-b}{\tau_2}\bar\zeta}.$$

Then, the flat connection is given by

$$\nabla e^{-w(s,t)}=d+\Big(-\pi\frac{a\bar\tau-b}{\tau_2}d\zeta+\underbrace{\pi\frac{a\tau-b}{\tau_2}d\bar\zeta}_{=A^{0,1}}\Big).$$

Similar argument shows that $\nabla e^{-w(s,t)}=d+\Big(\underbrace{\pi\frac{\tilde a\bar\tau-\tilde b}{\tau_2}d\zeta}_{=\tilde A^{1,0}}-\pi\frac{\tilde a\tau-\tilde b}{\tau_2}d\bar\zeta\Big).$

Now, we solve the general solution to $\psi_\pm$.

Since $\psi'_\pm(t,s)$ is periodic, we have

$$\begin{cases} \psi'_-(t,s) = \sum_{n\in\mathbb{Z}} \psi_n(t)\, e^{ins} \\ \psi'_+(t,s) = \sum_{n\in\mathbb{Z}} \tilde{\psi}_n(t)\, e^{-ins} \end{cases} \qquad \begin{cases} \psi'_-(t,s) = \sum_{n\in\mathbb{Z}} \bar{\psi}_n(t)\, e^{ins} \\ \psi'_+(t,s) = \sum_{n\in\mathbb{Z}} \bar{\tilde{\psi}}_n(t)\, e^{-ins} \end{cases}$$

Then,

$$\psi_-(t,s) = \sum_{n\in\mathbb{Z}} \psi_n(t)\, e^{ins} \cdot e^{-\frac{i(\tau_1 a - b)}{\tau_2} t + ias}$$

$$= \sum_{r\in\mathbb{Z}+a} \psi_r(t)\cdot e^{-\frac{i(\tau_1 a - b)}{\tau_2} t} \cdot e^{irs}$$

$$\psi_+(t,s) = \sum_{n\in\mathbb{Z}} \tilde{\psi}_n(t)\cdot e^{-ins} \cdot e^{\frac{i(\tau_1 \tilde{a} - \tilde{b})}{\tau_2} t - i\tilde{a}s}$$

$$= \sum_{\tilde{r}\in\mathbb{Z}+\tilde{a}} \tilde{\psi}_{\tilde{r}}(t)\cdot e^{\frac{i(\tau_1 \tilde{a} - \tilde{b})}{\tau_2} t} \cdot e^{-i\tilde{r}s}$$

$$\bar{\psi}_-(t,s) = \sum_{n\in\mathbb{Z}} \bar{\psi}_n(t)\, e^{ins} \cdot e^{\frac{i(\tau_1 a - b)}{\tau_2} t - ias}$$

$$= \sum_{r\in\mathbb{Z}-a} \bar{\psi}_r(t)\cdot e^{\frac{i(\tau_1 a - b)}{\tau_2} t} \cdot e^{irs}$$

$$\bar{\psi}_+(t,s) = \sum_{n\in\mathbb{Z}} \bar{\tilde{\psi}}_n(t)\cdot e^{-ins} \cdot e^{-\frac{i(\tau_1 \tilde{a} - \tilde{b})}{\tau_2} t + i\tilde{a}s}$$

$$= \sum_{\tilde{r}\in\mathbb{Z}-\tilde{a}} \bar{\tilde{\psi}}_{\tilde{r}}(t)\cdot e^{-\frac{i(\tau_1 \tilde{a} - \tilde{b})}{\tau_2} t} \cdot e^{-i\tilde{r}s}$$

#

Geometry and Topological Field Theory F09221011 莊秉勳

Week 6 Homework

Homework 1 Chiral superfield Φ : $\bar{D}_\pm \Phi = 0$. Show that Φ is of the form

$$\Phi = \phi(y^\pm) + \theta^\alpha \underline{\psi_\alpha(y^\pm)} + \theta^+\theta^- F(y^\pm), \quad y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm .$$

Fermion function

proof: Suppose that (f are functions in x^+, x^-, $\theta^4 = \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+$)

$$\Phi = f + \theta^+ f_+ + \theta^- f_- + \theta^{\mp} f_{\mp} + \theta^{=} f_{=}$$
$$+ \theta^{+-} f_{+-} + \theta^{+=} f_{+=} + \theta^{+\mp} f_{+\mp} + \theta^{-=} f_{-=} + \theta^{-\mp} f_{-\mp} + \theta^{=\mp} f_{=\mp}$$
$$+ \theta^{+-=} f_{+-=} + \theta^{+-\mp} f_{+-\mp} + \theta^{+=\mp} f_{+=\mp} + \theta^{-=\mp} f_{-=\mp} + \theta^4 f_4$$

$$\bar{D}_+\Phi = -f_{\mp} + i\theta^+\partial_+ f + \theta^+ f_{+\mp} + \theta^- f_{-\mp} + \theta^{=} f_{=\mp}$$
$$+ i\theta^+(\theta^-\partial_+ f_- + \theta^{\mp}\partial_+ f_{\mp} + \theta^{=}\partial_+ f_{=}) - \theta^{+-} f_{+-\mp} - \theta^{+=} f_{+=\mp} - \theta^{-=} f_{-=\mp}$$
$$+ i\theta^+(\theta^{-=}\partial_+ f_{-=} + \theta^{-\mp}\partial_+ f_{-\mp} + \theta^{=\mp}\partial_+ f_{=\mp}) + \theta^{+-=} f_4$$
$$+ i\theta^4 \partial_+ f_{-=\mp}$$

$$\bar{D}_-\Phi = -f_{=} + i\theta^-\partial_- f + \theta^+ f_{+=} + \theta^- f_{-=} - \theta^{\mp} f_{=\mp}$$
$$+ i\theta^-(\theta^+\partial_- f_+ + \theta^{\mp}\partial_- f_{\mp} + \theta^{=}\partial_- f_{=}) - \theta^{+-} f_{+-=} + \theta^{+\mp} f_{+=\mp} + \theta^{-\mp} f_{-=\mp}$$
$$+ i\theta^-(\theta^{+=}\partial_- f_{+=} + \theta^{+\mp}\partial_- f_{+\mp} + \theta^{=\mp}\partial_- f_{=\mp}) - \theta^{+-\mp} f_4$$
$$- i\theta^4 \partial_- f_{+=\mp}$$

Then, we have $f_{\mp} = f_{-\mp} = f_{=\mp} = f_{-=\mp} = 0$, $f_{+\mp} + i\partial_+ f = 0$,
$f_{+-\mp} = i\partial_+ f_-$, $f_{+=\mp} = i\partial_+ f_{=}$, $f_4 + i\partial_+ f_{-=} = 0$ from $\bar{D}_+\Phi = 0$ and
$f_{=} = f_{+=} = f_{=\mp} = f_{+=\mp} = 0$, $f_{-=} + i\partial_- f = 0$, $f_{+-=} + i\partial_- f_+ = 0$,
$f_{-=\mp} + i\partial_- f_{\mp} = 0$, $f_4 + i\partial_- f_{+\mp} = 0$ from $\bar{D}_-\Phi = 0$.

Now, Φ becomes
$$f + \theta^+ f_+ + \theta^- f_- + \theta^{+-} f_{+-} + \theta^{+\mp} f_{+\mp} + \theta^{-=} f_{-=}$$
$$+ \theta^{+-=} f_{+-=} + \theta^{+-\mp} f_{+-\mp} + \theta^4 f_4 .$$

To find ϕ, ψ_α, F, we need the following lemma:

Lemma For polynomial function $g(z_1, z_2)$, we have
$$g(y^+, y^-) = g(x^+, x^-) - i\partial_+ g\, \theta^{+\mp} - i\partial_- g\, \theta^{-=} - \partial_-\partial_+ g\, \theta^4 .$$

subpf: Both sides are linear in g, so it suffices to prove the case when g is monomial, i.e. $g(z_1, z_2) = z_1^a z_2^b$.

$$g(y^+, y^-) = (x^+ - i\theta^{+\mp})^a (x^- - i\theta^{-=})^b = ((x^+)^a - ai(x^+)^{a-1}\theta^{+\mp})((x^-)^b - bi(x^-)^{b-1}\theta^{-=})$$
$$= (x^+)^a(x^-)^b - ai(x^+)^{a-1}(x^-)^b\theta^{+\mp} - bi(x^+)^a(x^-)^{b-1}\theta^{-=} - ab(x^+)^{a-1}(x^-)^{b-1}\theta^4$$
$$= g(x^+, x^-) - i\partial_+ g\, \theta^{+\mp} - i\partial_- g\, \theta^{-=} - \partial_-\partial_+ g\, \theta^4 . \quad \square$$

Now, we let $\phi = f(y^+, y^-)$, $\psi_+ = f_+(y^+, y^-)$, $\psi_- = f_-(y^+, y^-)$, $F = f_{+-}(y^+, y^-)$ and $\psi_\mp = \psi_= = 0$, where we view f, f_+, f_-, f_{+-} as function of $x^\pm$ and substitute $x^\pm = y^\pm$.

Then, we have

$$\phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)$$

lemma

$$\begin{aligned}
&= f - i\partial_+ f\,\theta^{+\mp} - i\partial_- f\,\theta^{-=} - \partial_-\partial_+ f\,\theta^4 \\
&\quad + \theta^+\left(f_+ - i\partial_+ f_+\theta^{+\mp} - i\partial_- f_+\theta^{-=} - \partial_-\partial_+ f_+\theta^4\right) \\
&\quad + \theta^-\left(f_- - i\partial_+ f_-\theta^{+\mp} - i\partial_- f_-\theta^{-=} - \partial_-\partial_+ f_-\theta^4\right) \\
&\quad + \theta^+\theta^-\left(f_{+-} - i\partial_+ f_{+-}\theta^{+\mp} - i\partial_- f_{+-}\theta^{-=} - \partial_-\partial_+ f_{+-}\theta^4\right) \\
&= f + f_{+\mp}\theta^{+\mp} + f_{-=}\theta^{-=} + f_4\theta^4 \\
&\quad + f_+\theta^+ + f_{+-=}\theta^{+-=} + f_-\theta^- + f_{+-\mp}\theta^{+-\mp} + f_{+-}\theta^{+-} = \Phi.
\end{aligned}$$

#

Homework 2 Find the conserved currents of $S = S_{kin} + S_W$ via Noether procedure.

proof: We first find the variation of Chiral superfield $\delta\Phi$.

Write Chiral superfield into $\Phi = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^- F(y^\pm)$.

$\delta = \varepsilon_+ Q_- - \varepsilon_- Q_+ - \bar\varepsilon_+\bar Q_- + \bar\varepsilon_-\bar Q_+$. For a function $g(y^\pm)$, we have

$$Q_\pm g = \left(\frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\frac{\partial}{\partial x^\pm}\right) g = \frac{\partial g}{\partial y^\pm}\cdot\left(-i\bar\theta^\pm\right) + i\bar\theta^\pm\left(\frac{\partial g}{\partial y^\pm}\right) = 0$$

$$\bar Q_\pm g = \left(-\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\frac{\partial}{\partial x^\pm}\right) g = \frac{\partial g}{\partial y^\pm}\left(-i\theta^\pm\right) - i\theta^\pm\left(\frac{\partial g}{\partial y^\pm}\right) = -2i\theta^\pm\partial_\pm g.$$

Note that $\delta\Phi$ is also Chiral since δ anti-commutes with $\bar D_\pm$, i.e. $\bar D_\pm\delta\Phi = -\delta\bar D_\pm\Phi = 0$.

Then, $\delta\Phi = \delta\phi + (\delta\theta^+)\psi_+ + \theta^+(\delta\psi_+) + (\delta\theta^-)\psi_- + \theta^-(\delta\psi_-) + \delta(\theta^+\theta^-)F + \theta^+\theta^-(\delta F)$.

$$\begin{aligned}
&= 2i\bar\varepsilon_+\theta^-\partial_-\phi - 2i\bar\varepsilon_-\theta^+\partial_+\phi - \varepsilon_-\psi_+ + \theta^+\left(2i\bar\varepsilon_+\theta^-\partial_-\psi_+ - 2i\bar\varepsilon_-\theta^+\partial_+\psi_+\right) \\
&\qquad + \varepsilon_+\psi_- + \theta^-\left(2i\bar\varepsilon_+\theta^-\partial_-\psi_- - 2i\bar\varepsilon_-\theta^+\partial_+\psi_-\right) \\
&\quad + \left(-\varepsilon_+\theta^+ - \varepsilon_-\theta^-\right)F + \underbrace{\theta^+\theta^-\left(2i\bar\varepsilon_+\theta^-\partial_- F - 2i\bar\varepsilon_-\theta^+\partial_+ F\right)}_{=0} \\
&= \left(\varepsilon_+\psi_- - \varepsilon_-\psi_+\right) + \theta^+\left(2i\bar\varepsilon_-\partial_+\phi + \varepsilon_+ F\right) + \theta^-\left(-2i\bar\varepsilon_+\partial_-\phi + \varepsilon_- F\right) \\
&\quad + \theta^+\theta^-\left(-2i\bar\varepsilon_+\partial_-\psi_+ - 2i\bar\varepsilon_-\partial_+\psi_-\right)
\end{aligned}$$

Hence, the corresponding variation on $\phi, \psi_\pm, F$ is given by

$$\begin{cases}
\delta\phi = \varepsilon_+\psi_- - \varepsilon_-\psi_+ \\
\delta\psi_\pm = \pm 2i\bar\varepsilon_\mp\partial_\pm\phi + \varepsilon_\pm F \\
\delta F = -2i\bar\varepsilon_+\partial_-\psi_+ - 2i\bar\varepsilon_-\partial_+\psi_-
\end{cases}$$

Also, by taking complex conjugate, we get:

$$\begin{cases}
\delta\bar\phi = -\bar\varepsilon_+\bar\psi_- + \bar\varepsilon_-\bar\psi_+ \\
\delta\bar\psi_\pm = \mp 2i\varepsilon_\mp\partial_\pm\bar\phi + \bar\varepsilon_\pm\bar F \\
\delta\bar F = -2i\varepsilon_+\partial_-\bar\psi_+ - 2i\varepsilon_-\partial_+\bar\psi_-.
\end{cases}$$

Now, we are ready to apply the Noether procedure to find the conserved charge. Our action is $S = S_{kin} + S_W = \int d^2x\, L$

$$= \int d^2x \left(\begin{array}{l} |\partial_0\phi|^2 - |\partial_1\phi|^2 - |W'(\phi)|^2 + i\bar{\psi}_-(\partial_0+\partial_1)\psi_- + i\bar{\psi}_+(\partial_0-\partial_1)\psi_+ \\ -W''(\phi)\psi_+\psi_- - \bar{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+ + |F+\bar{W}'(\bar{\phi})|^2 \end{array} \right).$$

The equation of motion is $F = -\bar{W}'(\bar{\phi})$.

For simplicity, we focus on the ε_+ terms in δL. The meaningful terms of $\delta\phi, \delta\psi_\pm, \delta\bar{\phi}, \delta\bar{\psi}_\pm$ are

$$\delta\phi = \varepsilon_+\psi_-, \quad \delta\psi_+ = \varepsilon_+ F, \quad \delta\psi_- = 0$$
$$\delta\bar{\phi} = 0, \quad \delta\bar{\psi}_+ = 0, \quad \delta\bar{\psi}_- = \varepsilon_+ \cdot 2i\partial_-\bar{\phi}.$$

$$\begin{aligned} \delta L = {} & \partial_0(\varepsilon_+\psi_-)\partial_0\bar{\phi} - \partial_1(\varepsilon_+\psi_-)\partial_1\bar{\phi} \underbrace{- W''(\phi)(\varepsilon_+\psi_-)\bar{W}'(\bar{\phi})}_{①} \\ & + i(\varepsilon_+\cdot 2i\partial_-\bar{\phi})(\partial_0+\partial_1)\psi_- + \underbrace{i\bar{\psi}_+(\partial_0-\partial_1)(\varepsilon_+F)}_{②} \\ & \underbrace{- W''(\phi)(\varepsilon_+F)\psi_-}_{①} \underbrace{- \bar{W}''(\bar{\phi})(\varepsilon_+\cdot 2i\partial_-\bar{\phi})\bar{\psi}_+}_{②} \\ = {} & (\partial_0\varepsilon_+)\psi_-(\partial_0\bar{\phi}) + \underbrace{\varepsilon_+(\partial_0\psi_-)(\partial_0\bar{\phi})}_{③} - (\partial_1\varepsilon_+)\psi_-(\partial_1\bar{\phi}) \underbrace{- \varepsilon_+(\partial_1\psi_-)(\partial_1\bar{\phi})}_{④} \\ & - \varepsilon_+[(\partial_0-\partial_1)\bar{\phi}][(\partial_0+\partial_1)\psi_-] - i\bar{\psi}_+[(\partial_0-\partial_1)\varepsilon_+]\bar{W}'(\bar{\phi}) \end{aligned}$$

①&② cancel out by the equation of motion $F = -\bar{W}'(\bar{\phi})$

$$\begin{aligned} = {} & (\partial_0\varepsilon_+)\psi_-(\partial_0\bar{\phi}) - (\partial_1\varepsilon_+)\psi_-(\partial_1\bar{\phi}) - \varepsilon_+(\partial_0\bar{\phi})(\partial_1\psi_-) + \varepsilon_+(\partial_1\bar{\phi})(\partial_0\psi_-) \\ & - i\bar{\psi}_+[(\partial_0-\partial_1)\varepsilon_+]\bar{W}'(\bar{\phi}) \end{aligned}$$

integration by parts!

Then,
$$\int d^2x\, L = \int d^2x \left[\begin{array}{l} (\partial_0\varepsilon_+)[(\partial_0-\partial_1)\bar{\phi}\psi_- + i\bar{\psi}_+\bar{W}'(\bar{\phi})] - \varepsilon_+(\partial_0\partial_1\bar{\phi})\psi_- \\ (\partial_1\varepsilon_+)[(\partial_0-\partial_1)\bar{\phi}\psi_- - i\bar{\psi}_+\bar{W}'(\bar{\phi})] + \varepsilon_+(\partial_0\partial_1\bar{\phi})\psi_- \end{array} \right].$$

cancel!!

Finally, since ε_+ is arbitrary, we know that the conservered currents are given by
$$G^0_- = [(\partial_0-\partial_1)\bar{\phi}\psi_- + i\bar{\psi}_+\bar{W}'(\bar{\phi})] = 2\partial_-\bar{\phi}\psi_- + i\bar{\psi}_+\bar{W}'(\bar{\phi})$$
$$G^1_- = [(\partial_0-\partial_1)\bar{\phi}\psi_- - i\bar{\psi}_+\bar{W}'(\bar{\phi})] = 2\partial_-\bar{\phi}\psi_- - i\bar{\psi}_+\bar{W}'(\bar{\phi})$$
and the corresponding conserved charges are $\int dx^1 G^0_-$ and $\int dx^0 G^1_-$.

Similar computations by focusing on the $\varepsilon_-, \bar{\varepsilon}_+, \bar{\varepsilon}_-$ terms in δL, we will get all the conservered currents $G^0_\pm, G^1_\pm, \bar{G}^0_\pm, \bar{G}^1_\pm$ and their corresponding supercharges.

Next, we compute the conserved charge of axial rotation: $\phi \mapsto \phi$, $\psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm$

The variation of axial rotation is given by $\delta_A\phi = 0$, $\delta_A\bar{\phi} = 0$.

$$\delta_A\psi_\pm = \left.\frac{\partial e^{\mp i\varepsilon c}\psi_\pm}{\partial\varepsilon}\right|_{\varepsilon=0} = \mp ic\psi_\pm, \quad \delta_A\bar{\psi}_\pm = \pm ic\bar{\psi}_\pm.$$

Then, we have

$$
\begin{aligned}
\delta_A S = \int d^2x\ & c\cdot\bar\psi_-(\partial_0+\partial_1)\psi_- - \bar\psi_-(\partial_0+\partial_1)(c\cdot\psi_-) - c\cdot\bar\psi_+(\partial_0-\partial_1)\psi_+ + \bar\psi_+(\partial_0-\partial_1)(c\cdot\psi_+) \\
& - W''(\phi)(-ic\psi_+)\psi_- - W''(\phi)\psi_+(ic\psi_-) - \bar W''(\bar\phi)(-ic\bar\psi_-)\bar\psi_+ - \bar W''(\bar\phi)\bar\psi_-(ic\bar\psi_+) \\
= \int d^2x\ & (\partial_0 c)\underbrace{\left[-\bar\psi_-\psi_- + \bar\psi_+\psi_+\right]}_{=J_A^0} + (\partial_1 c)\underbrace{\left[-\bar\psi_-\psi_- - \bar\psi_+\psi_+\right]}_{=J_A^1}
\end{aligned}
$$

corresponding currents : J_A^0 J_A^1

supercharges: $F_A = \int J_A^0\, dx^1$.

Finally, we compute the conserved charge of vector rotation : $\phi \mapsto e^{(2/k)i\alpha}\phi$, $\psi_\pm \mapsto e^{((2/k)-1)i\alpha}\psi_\pm$ under the condition $W(\Phi) = c\cdot\Phi^k$.

The variation of axial rotation is given by

$$
\delta_V\phi = \left.\frac{\partial e^{(2/k)i\varepsilon a}\phi}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{2}{k}ia\phi,\quad \delta_V\psi_\pm = \left.\frac{\partial e^{((2/k)-1)i\varepsilon a}\psi_\pm}{\partial\varepsilon}\right|_{\varepsilon=0} = \left(\frac{2}{k}-1\right)ia\psi_\pm
$$

$$
\delta_V\bar\phi = -\frac{2}{k}ia\bar\phi,\quad \delta_V\bar\psi_\pm = -\left(\frac{2}{k}-1\right)ia\bar\psi_\pm
$$

Then, we have

$$
\begin{aligned}
\delta_V S = \int d^2x\ & \partial_0\left(\frac{2}{k}ia\phi\right)\cdot\partial_0\bar\phi + (\partial_0\phi)\partial_0\left(-\frac{2}{k}ia\bar\phi\right) - \partial_1\left(\frac{2}{k}ia\phi\right)\cdot\partial_1\bar\phi - (\partial_1\phi)\partial_1\left(-\frac{2}{k}ia\bar\phi\right) \\
& + ck(k-1)\left(\frac{2}{k}ia\phi^{k-1}\right)\cdot ck\bar\phi^{k-1} + ck\phi^{k-1}\cdot ck(k-1)\left(-\frac{2}{k}ia\bar\phi^{k-1}\right) \\
& + \left(\frac{2}{k}-1\right)a\bar\psi_-(\partial_0+\partial_1)\psi_- - \bar\psi_-\left(\frac{2}{k}-1\right)(\partial_0+\partial_1)(a\psi_-) \\
& + \left(\frac{2}{k}-1\right)a\bar\psi_+(\partial_0-\partial_1)\psi_+ - \bar\psi_+\left(\frac{2}{k}-1\right)(\partial_0-\partial_1)(a\psi_+) \\
& 0 = \left(\begin{aligned}
& - ck(k-1)(k-2)\left(\frac{2}{k}ia\right)\phi^{k-2}\psi_+\psi_- + ck(k-1)(k-2)\left(\frac{2}{k}ia\right)\bar\phi^{k-2}\bar\psi_-\bar\psi_+ \\
& - ck(k-1)\phi^{k-2}\left(\frac{2}{k}-1\right)ia\psi_+\psi_- + ck(k-1)\bar\phi^{k-2}\left(\frac{2}{k}-1\right)ia\bar\psi_-\bar\psi_+ \\
& - ck(k-1)\phi^{k-2}\psi_+\left(\frac{2}{k}-1\right)ia\psi_- + ck(k-1)\bar\phi^{k-2}\bar\psi_-\left(\frac{2}{k}-1\right)ia\bar\psi_+
\end{aligned}\right) = 0 \\
= \int d^2x\ & (\partial_0 a)\underbrace{\left[\frac{2i}{k}\left(\phi\cdot\partial_0\bar\phi - \bar\phi\cdot\partial_0\phi\right) - \left(\frac{2}{k}-1\right)\left(\bar\psi_-\psi_- + \bar\psi_+\psi_+\right)\right]}_{=J_V^0} \\
& + (\partial_1 a)\underbrace{\left[\frac{2i}{k}\left(-\phi\cdot\partial_1\bar\phi + \bar\phi\cdot\partial_1\phi\right) + \left(\frac{2}{k}-1\right)\left(-\bar\psi_-\psi_- + \bar\psi_+\psi_+\right)\right]}_{=J_V^1}
\end{aligned}
$$

conserved currents.

and the supercharge is $F_V = \int J_V^0\, dx^1$.

#

Geometry and Topological Field Theory F09221011 莊秉勳

Week 7 Homework

Homework $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi, \bar{\Phi})$ and $(g_{i\bar{j}}) > 0$.

$$L_{kin} = \int d^4\theta\, K(\Phi, \bar{\Phi}) = -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + i\, g_{i\bar{j}} \bar{\psi}^{\bar{j}}_{\mp} D_{\pm} \psi^i_{\mp} + R_{i\bar{j}k\bar{\ell}} \psi_+^i \psi_-^k \bar{\psi}_-^{\bar{j}} \bar{\psi}_+^{\bar{\ell}}$$

$$+ g_{i\bar{j}} \left(F^i - \Gamma^i_{\ell k} \psi_+^\ell \psi_-^k \right) \left(\bar{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{\ell}\bar{k}} \bar{\psi}_-^{\bar{\ell}} \bar{\psi}_+^{\bar{k}} \right).$$

Show above equality !!

proof: First, we have the expression

$$\Phi^i = \phi^i(x^\pm) - i\theta^{+\bar{+}} \partial_+ \phi^i - i\theta^{-\bar{-}} \partial_- \phi^i - \theta^4 \partial_- \partial_+ \phi^i + \theta^+ \psi_+^i - i\theta^{+-\bar{-}} \partial_- \psi_+^i + \theta^- \psi_-^i + i\theta^{+-\bar{+}} \partial_+ \psi_-^i + \theta^{+-} F^i.$$

$$\bar{\Phi}^{\bar{j}} = \bar{\phi}^{\bar{j}}(x^\pm) + i\theta^{+\bar{+}} \partial_+ \bar{\phi}^{\bar{j}} + i\theta^{-\bar{-}} \partial_- \bar{\phi}^{\bar{j}} - \theta^4 \cdot \partial_- \partial_+ \bar{\phi}^{\bar{j}} - \theta^{\bar{+}} \bar{\psi}_+^{\bar{j}} - i\theta^{-\bar{-}\bar{+}} \partial_- \bar{\psi}_+^{\bar{j}} - \theta^{\bar{-}} \bar{\psi}_-^{\bar{j}} + i\theta^{+\bar{-}\bar{+}} \partial_+ \bar{\psi}_-^{\bar{j}} + \theta^{\bar{-}\bar{+}} \bar{F}^{i}.$$

Use Taylor expansion on $K(\Phi^i, \bar{\Phi}^{\bar{i}})$, we get

$$K(\Phi, \bar{\Phi}) = K(\phi, \bar{\phi}) + \partial_i K(\phi, \bar{\phi}) \cdot \begin{pmatrix} -i\theta^{+\bar{+}} \partial_+ \phi^i - i\theta^{-\bar{-}} \partial_- \phi^i - \theta^4 \partial_- \partial_+ \phi^i + \theta^{+-} F^i \\ + \theta^+ \psi_+^i - i\theta^{+-\bar{-}} \partial_- \psi_+^i + \theta^- \psi_-^i + i\theta^{+-\bar{+}} \partial_+ \psi_-^i \end{pmatrix}$$

$$+ \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot \begin{pmatrix} + i\theta^{+\bar{+}} \partial_+ \bar{\phi}^{\bar{j}} + i\theta^{-\bar{-}} \partial_- \bar{\phi}^{\bar{j}} - \theta^4 \partial_- \partial_+ \bar{\phi}^{\bar{j}} + \theta^{\bar{-}\bar{+}} \bar{F}^{\bar{j}} \\ - \theta^{\bar{+}} \bar{\psi}_+^{\bar{j}} - i\theta^{-\bar{-}\bar{+}} \partial_- \bar{\psi}_+^{\bar{j}} - \theta^{\bar{-}} \bar{\psi}_-^{\bar{j}} + i\theta^{+\bar{-}\bar{+}} \partial_+ \bar{\psi}_-^{\bar{j}} \end{pmatrix}$$

$$+ \tfrac{1}{2} \partial_k \partial_i K(\cdots)(\cdots) + \partial_i \partial_{\bar{j}} K(\cdots)(\cdots) + \tfrac{1}{2} \partial_{\bar{k}} \partial_{\bar{j}} (\cdots)(\cdots) + \cdots.$$

Since we are integral over the measure $d^4\theta$, we only focus on the coefficient of θ^4 in $K(\Phi, \bar{\Phi})$:

$$K(\Phi, \bar{\Phi})\Big|_{\theta^4} = \partial_i K(\phi, \bar{\phi}) \cdot (-\partial_- \partial_+ \phi^i) + \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot (-\partial_- \partial_+ \bar{\phi}^{\bar{j}})$$

$$+ \partial_i \partial_j K(\phi, \bar{\phi}) (-\partial_+ \phi^i \cdot \partial_- \phi^j) + \partial_{\bar{i}} \partial_{\bar{j}} K(\phi, \bar{\phi}) (-\partial_+ \bar{\phi}^{\bar{i}} \cdot \partial_- \bar{\phi}^{\bar{j}})$$

$$+ \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot \begin{pmatrix} \partial_+ \phi^i \cdot \partial_- \bar{\phi}^{\bar{j}} + \partial_- \phi^i \partial_+ \bar{\phi}^{\bar{j}} + F^i \bar{F}^{\bar{j}} \\ + i \psi_+^i \partial_- \bar{\psi}_+^{\bar{j}} - i \partial_- \psi_+^i \cdot \bar{\psi}_+^{\bar{j}} + i \psi_-^i \cdot \partial_+ \bar{\psi}_-^{\bar{j}} - i \partial_+ \psi_-^i \cdot \bar{\psi}_-^{\bar{j}} \end{pmatrix}$$

$$+ \partial_k \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot \left(-i \partial_+ \phi^k \cdot \psi_-^i \bar{\psi}_-^{\bar{j}} - i \partial_- \phi^k \cdot \psi_+^i \bar{\psi}_+^{\bar{j}} - \psi_+^k \psi_-^i \bar{F}^{\bar{j}} \right)$$

$$+ \partial_{\bar{k}} \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot \left(i \partial_+ \bar{\phi}^{\bar{k}} \cdot \psi_-^i \bar{\psi}_-^{\bar{j}} + i \partial_- \bar{\phi}^{\bar{k}} \psi_+^i \bar{\psi}_+^{\bar{j}} + \bar{\psi}_+^{\bar{k}} \bar{\psi}_-^{\bar{j}} F^i \right)$$

$$+ \partial_\ell \partial_{\bar{k}} \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}) \cdot \psi_+^\ell \bar{\psi}_+^{\bar{k}} \psi_-^i \bar{\psi}_-^{\bar{j}}.$$

Note that under the Kähler metric $ds^2 = g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$, we have $\partial_k g_{i\bar{j}} = g_{s\bar{j}} \Gamma^s_{ki}$, $\partial_{\bar{k}} g_{i\bar{j}} = g_{i\bar{s}} \Gamma^{\bar{s}}_{\bar{k}\bar{j}}$, and $\partial_\ell \partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{k}} (g_{s\bar{j}} \Gamma^s_{\ell i}) = g_{s\bar{p}} \Gamma^{\bar{p}}_{\bar{k}\bar{j}} \Gamma^s_{\ell i} + R_{i\bar{j}\ell\bar{k}}$.

integration by parts

$$
\begin{aligned}
\text{Then, } L_{kin} &= \partial_i K(\phi,\bar\phi)\cdot(-\partial_-\partial_+\phi^i) + \partial_{\bar j} K(\phi,\bar\phi)\cdot(-\partial_-\partial_+\bar\phi^{\bar j}) \\
&\quad + \partial_i\partial_j K(\phi,\bar\phi)\left(-\partial_+\phi^i\cdot\partial_-\phi^j\right) + \partial_{\bar i}\partial_{\bar j}K(\phi,\bar\phi)\left(-\partial_+\bar\phi^{\bar i}\cdot\partial_-\bar\phi^{\bar j}\right) \\
&\quad + g_{i\bar j}\cdot\left(\begin{aligned}&\partial_+\phi^i\cdot\partial_-\bar\phi^{\bar j} + \partial_-\phi^i\,\partial_+\bar\phi^{\bar j} + F^i\bar F^{\bar j} \\ &+ i\,\psi_+^i\,\partial_-\bar\psi_+^{\bar j} - i\,\partial_-\psi_+^i\cdot\bar\psi_+^{\bar j} + i\,\psi_-^i\cdot\partial_+\bar\psi_-^{\bar j} - i\,\partial_+\psi_-^i\cdot\bar\psi_-^{\bar j}\end{aligned}\right) \\
&\quad + g_{s\bar j}\,\Gamma^s_{ki}\cdot\left(-i\,\partial_+\phi^k\cdot\psi_-^i\bar\psi_-^{\bar j} - i\,\partial_-\phi^k\cdot\psi_+^i\bar\psi_+^{\bar j} - \psi_+^k\psi_-^i\bar F^{\bar j}\right) \\
&\quad + g_{i\bar s}\,\Gamma^{\bar s}_{\bar k\bar j}\cdot\left(i\,\partial_+\bar\phi^{\bar k}\cdot\psi_-^i\bar\psi_-^{\bar j} + i\,\partial_-\bar\phi^{\bar k}\,\psi_+^i\bar\psi_+^{\bar j} + \bar\psi_+^{\bar k}\bar\psi_-^{\bar j}F^i\right) \\
&\quad + \left(g_{s\bar p}\,\Gamma^{\bar p}_{\bar k\bar j}\Gamma^s_{\ell i} - R_{i\bar j\ell\bar k}\right)\cdot\psi_+^\ell\bar\psi_+^{\bar k}\psi_-^i\bar\psi_-^{\bar j} \\
&= \underline{\partial_{\bar j}\partial_i K(\phi,\bar\phi)\cdot\partial_-\phi^j\cdot\partial_+\phi^i}_{\,①} + g_{i\bar j}\cdot\partial_-\bar\phi^{\bar j}\cdot\partial_+\phi^i + \underline{\partial_{\bar i}\partial_{\bar j}K(\phi,\bar\phi)\cdot\partial_-\bar\phi^{\bar i}\cdot\partial_+\bar\phi^{\bar j}}_{\,②} + g_{i\bar j}\cdot\partial_-\phi^i\cdot\partial_+\bar\phi^{\bar j} \\
&\quad + \underline{\partial_i\partial_j K(\phi,\bar\phi)\left(-\partial_+\phi^i\cdot\partial_-\phi^j\right)}_{\,①} + \underline{\partial_{\bar i}\partial_{\bar j}K(\phi,\bar\phi)\left(-\partial_+\bar\phi^{\bar i}\cdot\partial_-\bar\phi^{\bar j}\right)}_{\,②} \\
&\quad + g_{i\bar j}\left(\partial_+\phi^i\cdot\partial_-\bar\phi^{\bar j} + \partial_-\phi^i\,\partial_+\bar\phi^{\bar j}\right) \\
&\quad + i\cdot g_{i\bar j}\left(\begin{aligned}&-\left(\partial_+\bar\psi_-^{\bar j} + \partial_+\bar\phi^{\bar k}\cdot\Gamma^{\bar j}_{\bar k\bar p}\bar\psi_-^{\bar p}\right)\cdot\psi_-^i + \bar\psi_-^{\bar j}\left(\partial_+\psi_-^i + \partial_+\phi^k\cdot\Gamma^i_{kp}\psi_-^p\right) \\ &-\left(\partial_-\bar\psi_+^{\bar j} + \partial_-\bar\phi^{\bar k}\,\Gamma^{\bar j}_{\bar k\bar p}\bar\psi_+^{\bar p}\right)\cdot\psi_+^i + \bar\psi_+^{\bar j}\left(\partial_-\psi_+^i + \partial_-\phi^k\cdot\Gamma^i_{kp}\psi_+^p\right)\end{aligned}\right) \\
&\quad + R_{i\bar j k\bar\ell}\,\psi_+^i\psi_-^k\bar\psi_-^{\bar j}\bar\psi_+^{\bar\ell} \\
&\quad + g_{i\bar j}\left(F^i\bar F^{\bar j} - \Gamma^i_{pq}\psi_+^p\psi_-^q\bar F^{\bar j} - \Gamma^{\bar j}_{\bar p\bar q}\bar\psi_-^{\bar p}\bar\psi_+^{\bar q}F^i + \Gamma^i_{\ell m}\Gamma^{\bar j}_{\bar k\bar n}\cdot\psi_+^\ell\psi_-^m\bar\psi_-^{\bar n}\bar\psi_+^{\bar k}\right) \\
&= \frac{1}{2}g_{i\bar j}\left((\partial_0+\partial_1)\phi^i\cdot(\partial_0-\partial_1)\bar\phi^{\bar j} + (\partial_0-\partial_1)\phi^i\cdot(\partial_0+\partial_1)\bar\phi^{\bar j}\right) \\
&\quad + \frac{i}{2}g_{i\bar j}\left(-(D_0+D_1)\bar\psi_-^{\bar j}\cdot\psi_-^i + \bar\psi_-^{\bar j}(D_0+D_1)\psi_-^i - (D_0-D_1)\bar\psi_+^{\bar j}\cdot\psi_+^i + \bar\psi_+^{\bar j}(D_0-D_1)\psi_+^i\right) \\
&\quad + R_{i\bar j k\bar\ell}\,\psi_+^i\psi_-^k\bar\psi_-^{\bar j}\bar\psi_+^{\bar\ell} \\
&\quad + g_{i\bar j}\left(F^i - \Gamma^i_{pq}\psi_+^p\psi_-^q\right)\left(\bar F^{\bar j} - \Gamma^{\bar j}_{\bar p\bar q}\bar\psi_-^{\bar p}\bar\psi_+^{\bar q}\right) \\
&= \underline{g_{i\bar j}\left(\partial_0\phi^i\cdot\partial_0\bar\phi^{\bar j} - \partial_1\phi^i\cdot\partial_1\bar\phi^{\bar j}\right)} = -g_{i\bar j}\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar j} \\
&\quad + i\,g_{i\bar j}\,\bar\psi_-^{\bar j}(D_0+D_1)\psi_-^i + i\,g_{i\bar j}\,\bar\psi_+^{\bar j}(D_0-D_1)\psi_+^i + R_{i\bar j k\bar\ell}\,\psi_+^i\psi_-^k\bar\psi_-^{\bar j}\bar\psi_+^{\bar\ell} \\
&\quad + g_{i\bar j}\left(F^i - \Gamma^i_{pq}\psi_+^p\psi_-^q\right)\left(\bar F^{\bar j} - \Gamma^{\bar j}_{\bar p\bar q}\bar\psi_-^{\bar p}\bar\psi_+^{\bar q}\right).
\end{aligned}
$$

Integration by parts

#

Geometry and Topological Field Theory F09221011 莊秉勳

Week 8 Homework

<u>Homework</u> Compute 2 point, 4 point functions up to 1-loop.

$$\langle \mathcal{O} \rangle = \frac{1}{Z(M.C)} \int d^n x \, e^{-\frac{1}{2} x_i M_{ij} x_j + C_{ijkl} x_i x_j x_k x_l} \, \mathcal{O}(x_1, \dots x_n)$$

proof: The two point function is

$$\langle x_\alpha x_\beta \rangle = \frac{1}{Z(M.C)} \int d^n x \, e^{-\frac{1}{2} x_i M_{ij} x_j} \sum_{r=0}^{\infty} \frac{1}{r!} \left(C_{ijkl} x_i x_j x_k x_l \right)^r \cdot x_\alpha x_\beta$$

To compute the single term $\int d^n x \, e^{-\frac{1}{2} x_i M_{ij} x_j} \frac{1}{r!} \left(C_{ijkl} x_i x_j x_k x_l \right)^r x_\alpha x_\beta$,

we need to introduce new variables $J = (J_1 \dots J_n)^T$:

$$f(J) = \int d^n x \, e^{-\frac{1}{2} x_i M_{ij} x_j + J^T \cdot x} \underset{\uparrow}{=\!=} \int d^n y \, e^{-\frac{1}{2} y^T M y + J^T (M^{-1})^T J - \frac{1}{2} v^T M v}$$

change of variable $y = x - v$, where $v = (M^{-1})^T J$.

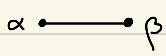
$$= \frac{(\sqrt{2\pi})^n}{\sqrt{\det(M)}} \cdot e^{\frac{1}{2} J^T M^{-1} J}$$

Then, to compute $\int d^n x \, e^{-\frac{1}{2} x_i M_{ij} x_j} x_i x_j x_k x_l x_\alpha x_\beta$, we take the differentation $\frac{\partial}{\partial J_i} \cdots \frac{\partial}{\partial J_l} \frac{\partial}{\partial J_\alpha} \frac{\partial}{\partial J_\beta}$ on the above equality at $J = 0$.

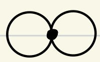
In the differentiation, we need to get three pairs in $\{i.j.k.l.\alpha.\beta\}$.

Then, we have two types of contractions:

(1) $\alpha \leftrightarrow \beta$



i,j,k,l : two pair

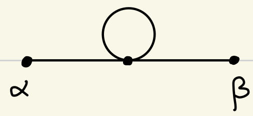


3 terms

(2) $\alpha \leftrightarrow$ one of $\{i.j.k.l\}$

$\beta \leftrightarrow$ another of $\{i.j.k.l\}$

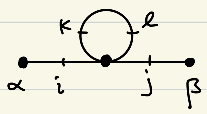
the rest two of $\{i.j.k.l\}$



12 terms

$M^{\alpha i} M^{\beta j} M^{kl}$

i.e. $\alpha \leftrightarrow i$, $\beta \leftrightarrow j$, $k \leftrightarrow l$



The first type is counted as zero loop since the term $(M^{ij} M^{kl} + M^{ik} M^{jl} + M^{il} M^{jk})$ will appear in the denominator $Z(M.C)$.

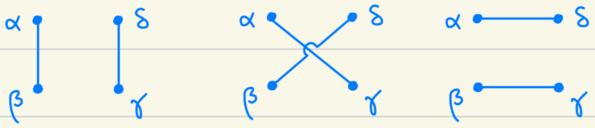
Thus, the two-point function at 1-loop level is given by

$$\langle x_\alpha x_\beta \rangle_{(1)} = \sum_{i,j,k,l} C_{ijkl} \left(M^{\alpha i} \cdot M^{\beta j} \cdot M^{kl} + 11\text{- terms of the form } M^{\alpha \cdot} M^{\beta \cdot} M^{\cdot\cdot} \right)$$

The four point function is

$$\langle X_\alpha X_\beta X_\gamma X_\delta \rangle = \frac{1}{Z(M,C)} \int d^n x \; e^{-\frac{1}{2} X_i M_{ij} X_j} \sum_{r=0}^{\infty} \frac{1}{r!} \left(C_{ijkl} X_i X_j X_k X_l \right)^r \cdot X_\alpha X_\beta X_\gamma X_\delta$$

Using the same method, we need to take 4-time derivatives to get the terms when $r=0$ in above summation. Then, the first three terms of $\langle X_\alpha X_\beta X_\gamma X_\delta \rangle$ is given by $M^{\alpha\beta} M^{\gamma\delta} + M^{\alpha\gamma} M^{\beta\delta} + M^{\alpha\delta} M^{\beta\gamma}$. These three terms are just



the product of two-point functions. $\langle X_\alpha X_\beta \rangle_{(0)} \langle X_\gamma X_\delta \rangle_{(0)}$, $\langle X_\alpha X_\gamma \rangle_{(0)} \langle X_\beta X_\delta \rangle_{(0)}$, $\langle X_\alpha X_\delta \rangle_{(0)} \langle X_\beta X_\gamma \rangle_{(0)}$.

Next, we consider $r=1$. We need to take 8-time derivatives and get 3 types of contraction:

(1) $\{i.j.k.l\}$ form two pairs
$\{\alpha.\beta.\gamma.\delta\}$ form two pairs

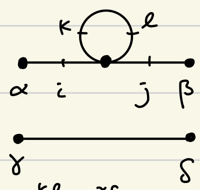
9 terms

e.g. product of four two-point functions

(2) two of $\{i.j.k.l\}$ form a pair
two of $\{\alpha.\beta.\gamma.\delta\}$ form a pair
one of $\{i.j.k.l\} \longleftrightarrow$ one of $\{\alpha.\beta.\gamma.\delta\}$ $\underline{\times 2}$

72 terms

e.g. $\alpha \leftrightarrow i$
$\beta \leftrightarrow j$
$k \leftrightarrow l$
$\gamma \leftrightarrow \delta$

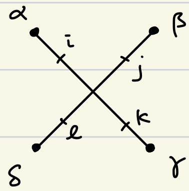


$\rightsquigarrow M^{\alpha i} M^{\beta j} M^{kl} M^{\gamma\delta}$.

(3) each one of $\{i.j.k.l\}$ pair each one of $\{\alpha.\beta.\gamma.\delta\}$

24 terms

e.g. $\alpha \leftrightarrow i$
$\beta \leftrightarrow j$
$\gamma \leftrightarrow k$
$\delta \leftrightarrow l$



$\rightsquigarrow M^{\alpha i} M^{\beta j} M^{\gamma k} M^{\delta l}$.

The four point function of 0-loop level is given by type 3:

$$\langle X_\alpha X_\beta X_\gamma X_\delta \rangle_{(0)} = \sum_{i.j.k.l} C_{ijkl} \left(M^{\alpha i} M^{\beta j} M^{\gamma k} M^{\delta l} + 23 \text{ terms} \right).$$

Here, we see the four-point function are related to the two-point functions by $\langle X_\alpha X_\beta X_\gamma X_\delta \rangle = \sum_{cyc} \langle X_\alpha X_\beta \rangle \langle X_\gamma X_\delta \rangle + \sum_{i=0}^{\infty} \langle X_\alpha X_\beta X_\gamma X_\delta \rangle_{(i)}$, where the type (1) & (2) appear in the first summand.

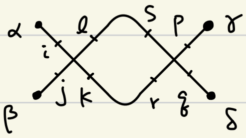
For $r=2$, we compute $\int d^n x \; e^{-\frac{1}{2} M_{ij} X_i X_j} C_{ijkl} C_{pqrs} \, x^{ijklpqrs\alpha\beta\gamma\delta}$. We have several types of six contractions in $\{i.j.k.l.p.q.r.s.\alpha.\beta.\gamma.\delta\}$.

(1) $\{\alpha,\beta,\gamma,\delta\}$: self form two pairs
(2) two of $\{\alpha.\beta.\gamma.\delta\}$ form a pair, other two pair with $\{i,j \ldots r.s\}$

$\Rightarrow$ These two types will appear in the first summand, i.e. $\sum_{cyc} \langle X_\alpha X_\beta \rangle \langle X_\gamma X_\delta \rangle$.

(3) all of $\{\alpha.\beta.\gamma.\delta\}$ pair with $\{i.j.k.l.p.q.r.s\}$

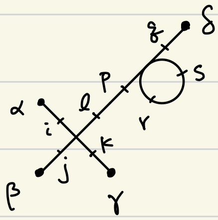
(3-I) e.g.



$\rightsquigarrow M^{\alpha i} M^{\beta j} M^{ls} M^{kr} M^{p\gamma} M^{q\delta}$.

There are 576 terms.

(3-II) e.g.



$\rightsquigarrow M^{\alpha i} M^{\beta j} M^{\gamma k} M^{lp} M^{rs} M^{q\delta}$.

There are 576 terms.

Then, the four point function at 1-loop level is given by

$$\langle X_\alpha X_\beta X_\gamma X_\delta \rangle_{(1)} = \frac{1}{2!} \sum_{\substack{i,j,k,l \\ p,q,r,s}} C_{ijkl} C_{pqrs} \left(M^{\alpha i} M^{\beta j} M^{ls} M^{kr} M^{p\gamma} M^{q\delta} + 575 \text{ terms} \right)$$

$$+ \frac{1}{2!} \sum_{\substack{i,j,k,l \\ p,q,r,s}} C_{ijkl} C_{pqrs} \left(M^{\alpha i} M^{\beta j} M^{\gamma k} M^{lp} M^{rs} M^{q\delta} + 575 \text{ terms} \right)$$

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Geometry and Topological Field Theory F09221011 莊秉勳

Week 9 Homework

Homework 1 Show that $(\phi_1 \dots \phi_N) \mapsto (e^{i\gamma}\phi_1 \dots e^{i\gamma}\phi_N)$ induces the transformation in $v_\mu \mapsto v_\mu - \partial_\mu \gamma$ via $v_\mu = \frac{i}{2} \frac{\sum_{i=1}^N (\bar{\phi}_i \partial_\mu \phi_i - \partial_\mu \bar{\phi}_i \phi_i)}{\sum_{i=1}^N |\phi_i|^2}$.

proof: Using direct computation, v_μ is transformed in

$$v_\mu \longmapsto \frac{i}{2} \frac{1}{\sum_{i=1}^N |e^{i\gamma}\phi_i|^2} \sum_{i=1}^N \left(e^{-i\gamma}\bar{\phi}_i \partial_\mu (e^{i\gamma}\phi_i) - \partial_\mu (e^{-i\gamma}\bar{\phi}_i) e^{i\gamma}\phi_i \right)$$

$$= \frac{i}{2} \frac{1}{\sum_{i=1}^N |\phi_i|^2} \sum_{i=1}^N \left(\bar{\phi}_i \cdot i \cdot \partial_\mu \gamma \cdot \phi_i + \bar{\phi}_i \partial_\mu \phi_i + i \cdot \partial_\mu \gamma \cdot \bar{\phi}_i \phi_i - \partial_\mu \bar{\phi}_i \cdot \phi_i \right)$$

$$= \frac{i}{2} \frac{1}{\sum_{i=1}^N |\phi_i|^2} \sum_{i=1}^N \left(\bar{\phi}_i \cdot \partial_\mu \phi_i - \partial_\mu \bar{\phi}_i \cdot \phi_i \right) - \partial_\mu \gamma = v_\mu - \partial_\mu \gamma.$$

Next, we show how to get such v_μ satisfies such transformation.

Now, our Lagrangian is given by $L = -\sum_{i=1}^N |D_\mu \phi_i|^2 - U(\phi)$, where $U(\phi) = \frac{e^2}{2} \left(\sum_{i=1}^N |\phi_i|^2 - r \right)^2$.

When $r > 0$, our vacuum manifold M_{vac} is the sphere S^{2N-1} of radius $\sqrt{r}$.

We need to derive the equation of motion of L so that we can solve the Gauge field v_μ.

Our L is invariant under the transformation $(\phi_1 \dots \phi_N) \mapsto (e^{i\gamma}\phi_1 \dots e^{i\gamma}\phi_N)$, then we can find the variations on M_{vac} at $(\phi_1, \dots, \phi_N)$:

$$\delta\phi_i = \frac{\partial(e^{i\gamma\varepsilon}\phi_i)}{\partial\varepsilon} = i \cdot \gamma \cdot \phi_i, \quad \delta\bar{\phi}_i = -i \cdot \gamma \cdot \bar{\phi}_i.$$

Also, $\delta U(\phi) = e^2 \underbrace{\left(\sum_{i=1}^N |\phi_i|^2 - r \right)}_{= 0 \text{ on } M_{vac}} \cdot \delta \left(\sum_{i=1}^N |\phi_i|^2 - r \right) = 0$.

Then, $\delta L = -\sum_{i=1}^N (\delta D_\mu \phi_i) D_\mu \bar{\phi}_i + D_\mu \phi_i (\delta D_\mu \bar{\phi}_i) + \underbrace{\delta U(\phi)}_{= 0}$

$$= -\sum_{i=1}^N \left[i(\partial_\mu \gamma \cdot \phi_i + \gamma \cdot \partial_\mu \phi_i) - i(\delta v_\mu) \cdot \phi_i - v_\mu \cdot \gamma \cdot \phi_i \right] \cdot D_\mu \bar{\phi}_i$$
$$+ D_\mu \phi_i \cdot \left[-i(\partial_\mu \gamma \cdot \bar{\phi}_i + \gamma \cdot \partial_\mu \bar{\phi}_i) + i \cdot (\delta v_\mu) \cdot \bar{\phi}_i - v_\mu \cdot \gamma \cdot \bar{\phi}_i \right]$$

$$= i(\partial_\mu \gamma + \delta v_\mu) \cdot \left(\sum_{i=1}^N \bar{\phi}_i \cdot D_\mu \phi_i - D_\mu \bar{\phi}_i \cdot \phi_i \right)$$

Since γ is arbitrary, we get the equation of motion $\sum_{i=1}^{N} \bar{\phi}_i \cdot D_\mu \phi_i - D_\mu \bar{\phi}_i \cdot \phi_i = 0$.

Hence, we can solve v_μ via the equation of motion.

$$\begin{cases} D_\mu \phi_i = (\partial_\mu + i v_\mu)\phi_i \\ D_\mu \bar{\phi}_i = (\partial_\mu - i v_\mu)\bar{\phi}_i \end{cases} \Rightarrow \sum_{i=1}^{N} \bar{\phi}_i \left(\partial_\mu \phi_i + i v_\mu \phi_i\right) - \left(\partial_\mu \bar{\phi}_i - i v_\mu \bar{\phi}_i\right)\phi_i = 0$$

$$\Rightarrow v_\mu = \frac{i}{2} \frac{\sum_{i=1}^{N} (\bar{\phi}_i \partial_\mu \phi_i - \partial_\mu \bar{\phi}_i \phi_i)}{\sum_{i=1}^{N} |\phi_i|^2}.$$

#

Homework 2 Show that under a suitable Gauge transformation $V \longmapsto V + i(\bar{A} - A)$, the real superfield V can be expressed as

$$V = \theta^\mp \bar{\theta}^\mp (v_0 \mp v_1) - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma}$$

real one form complex scalar field

$$+ i\theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_- + \bar{\theta}^+ \bar{\lambda}_+) + i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D$$

complex Dirac fermion real

proof: We need to find a Chiral superfield A so that $V + i(\bar{A} - A)$ does not have the constant and degree 1 terms in θ.

A Chiral superfield has expression:

$A = \phi + \theta^+ \psi_+ + \theta^- \psi_- + (\text{higher order terms})$

$\bar{A} = \bar{\phi} - \bar{\theta}^+ \bar{\psi}_+ - \bar{\theta}^- \bar{\psi}_- + (\text{higher order terms})$.

Suppose that $V = V_0 + \theta^+ V_+ + \theta^- V_- + \bar{\theta}^+ V_{\bar{+}} + \bar{\theta}^- \theta_{\bar{-}} + (\text{higher order terms})$.

Since V is real, we know that V_0 is real, $V_+ = -\bar{V}_{\bar{+}}$, and $V_- = -\bar{V}_{\bar{-}}$.

Note that $i(\bar{A} - A) = 2\,\mathrm{Im}\,A$

$= 2\,\mathrm{Im}\,\phi - i(\theta^+ \psi_+ + \theta^- \psi_- + \bar{\theta}^+ \bar{\psi}_+ + \bar{\theta}^- \bar{\psi}_-) + (\text{higher order terms})$

Now, choose ϕ, ψ_+, ψ_- so that $2\,\mathrm{Im}\,\phi = -V_0$, $i\psi_+ = V_+$, and $i\psi_- = V_-$.

Then, $V + i(\bar{A} - A)$ has no constant and first order terms.

After this Gauge transformation, we get

$$V = \theta^- \bar{\theta}^- (v_0 - v_1) + \theta^+ \bar{\theta}^+ (v_0 + v_1) - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma}$$
$$+ i\theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_- + \bar{\theta}^+ \bar{\lambda}_+) + i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D.$$

Since $\overline{\theta^- \bar{\theta}^-} = \theta^- \bar{\theta}^-$, $\overline{\theta^+ \bar{\theta}^+} = \theta^+ \bar{\theta}^+$, v_0 and v_1 are real. Also, $\overline{\theta^- \bar{\theta}^+} = \theta^+ \bar{\theta}^-$, the coefficients of $\theta^- \bar{\theta}^+$ and $\theta^+ \bar{\theta}^-$ are conjugate to each other. Similarly for $\lambda_\pm$.

For the last term, since $\overline{\theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^-} = \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^-$, we have D is real.

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Geometry and Topological Field Theory F09221011 莊秉勳

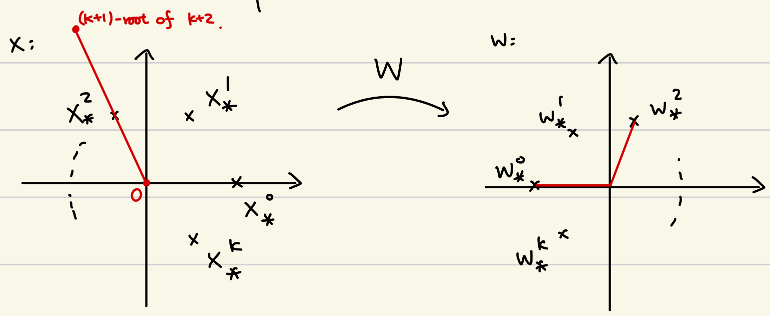
Week 10 Homework

<u>Homework</u> $W(x) = \frac{1}{k+2} x^{k+2} - x$

$W'(x) = x^{k+1} - 1 \rightsquigarrow$ critical points are $(k+1)$-root of unity.

Show that there exists a unique soliton connecting each pair of critical points.

proof: The critical points are $x_*^n = e^{\frac{2\pi i n}{k+1}}$, $w_*^n = -\frac{k+1}{k+2} e^{\frac{2\pi i n}{k+1}}$ $(n=0,1,\dots,k)$.



We want to compute the number of solitons connecting x_*^m and x_*^n. We pick a particular path γ connecting w_*^m, 0, w_*^n and find its primage. Note that the primage of 0 is $\{0, (k+1)\text{-roots of } k+2\}$. To find the primage of $0w_*^n$ passing through x_*^n, it must lie on the same argument. That is, the only possible two segments are $x_*^n 0$ and $x_*^n P$, where P is the $(k+1)$-roots of $(k+2)$ with same argument as x_*^n. Hence, using our particular path γ, the soliton connecting x_*^m and x_*^n is given by $x_*^m 0\, x_*^n$ which is unique.

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