Week 1 Homework

1. (a) Any complex submanifold M of $\mathbb{C}^n$ has to be non-compact unless it is finite points.

   (b) The complex projective space $\mathbb{C}P^n$ is compact.

2. (a) Verify the Fubini-Study metric ω_{FS} is positive definite.

   (b) Prove that $\int_{\mathbb{C}P^1} \omega_{FS} = 1$. (Bonus: $\int_{\mathbb{C}P^n} \omega_{FS}^n = 1$.)

Week 2 Homework

1. Given a Weierstrass data (ω, g) for the minimal immersion $M \to \mathbb{R}^3$. Show that $g = p \circ N$ where p is the stereographic projection and N is the Gauss map of M.

2. Show that a hermitian metric $g_{i\bar{j}}$ is K*ä*hler if and only if $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$. Under the complex coordinates, compute the Christoffel symbols and derive the formula for Riemannian curvature tensor. In particular, show that Ricci curvature is given by $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}})$.

Week 3 Homework

1. Determine the conformal transformation group of the Poincare disk.

2. Show the details about the Dirichlet principle (Lawson p.64-66). Namely solve the minimization problem as boundary value problem of (vector valued) harmonic functions, give the proof of Lemma 4 in p.64, and use it to complete the proof of Plateau problem.

Week 4 Homework

1. Assume that all base topological spaces are paracompact.
   (a) Show that a vector bundle over a contractible base is trivial.
   (b) Pull back bundles are isomorphic under homotopic maps.
2. Let $E \to X$ be an R (R = real or complex) vector bundle, prove the existence of bundle metrics (Riemannian or Hermitian respectively). If X is a manifold, prove the existence of R-linear connections. If moreover E has structure group G in GL(n, R) (i.e. transitions functions are in G) show that the connection can be chosen so that its curvature has values in the Lie algebra $\mathfrak{g} = Lie(G)$.

Week 5 Homework

1. Suppose that E, F are vector bundles over X and let ∇ be any connection on E.

   (a) (Naturality) Define the pullback connection $f^*\nabla$ on f^*E and show that $f^*c_k(E) = c_k(f^*E)$ for any $k \geq 0$.

   (b) (Whitney sum formula) $c(E \oplus F) = c(E)c(F)$.

   (c) (Tensor rule for line bundles) For line bundles L_i,

   $$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

   (d) (Normalization) Let S be the tautological line bundle over $\mathbb{C}P^n$ and let Q be the quotient bundle $(C^{n+1} \times \mathbb{C}P^n)/S$. Show that $S^* \otimes Q \simeq T_{\mathbb{C}P^n}$ and $[\omega_{FS}] = -c_1(S^*)$.

2. Show that the heat kernel $H(x, y, t)$ is smooth in x, y by showing that there are constants $C, \delta > 0$ such that $\lambda_n \geq Cn^\delta$ for $n \gg 0$.

Week 6 Homework

1. Show that the heat kernel for $H = -\sum_{i=1}^{n} \nabla_i^2 + F$ is given by

$$h(x,t) = \frac{1}{(4\pi t)^{n/2}} \det^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right) e^{-\frac{1}{4t}\left\langle x \middle| \frac{tR}{2}\coth\frac{tR}{2} \middle| x\right\rangle} e^{-tF}$$

   where $\nabla_i = \partial_i + \frac{1}{4}\sum_j R_{ij}x_j$.

2. (a) Show that $\mathrm{Str}[u,v] = 0$ for every $u, v \in C(V)$.

   (b) Let $(e)_{i=1}^{n}$ be an orthonormal basis of V and $I \subseteq \{1, ..., n\}$. If $|I| < n$, say j does not belong to I, prove that the Clifford product $e_I = -\frac{1}{2}[e_j, e_j e_I]$ where $e_I = \prod_{i \in I} e_i$.

   (c) Show that $S^{\pm} = \{v \mid \epsilon v = \pm v\}$ where $S = S^+ \oplus S^-$ is the spinor module.

Week 7 Homework

1. (a) $c(a)$ is skew-adjoint on E.

   (b) The Dirac operator D is self-adjoint.

2. Let (U, x) be a Riemann normal coordinate charts at $x_0 \in U$ and $E = S \otimes W$ be a complex Clifford module (bundle) over U. We trivialize $TM|_U \cong T_{x_0} \times U$ and $E \cong E_{x_0} \times U$ using parallel translations along radial lines in the chart (geodesics in U through x_0). Denote $c^i := c(dx^i) \in \mathrm{End}(E)$. Show that, on $C^\infty(U, E)$, the following asymptotic expansion for covariant derivatives near x_0 holds:

$$\nabla^E_{\partial_i} = \partial_i + \frac{1}{4} \sum_{j;k<l} R_{klij}(x_0) x^j c^k c^l + \sum_{k<l} f_{ikl}(x) c^k c^l + g_i(x),$$

   where $f_{ikl}(x) = O(|x|^2) \in C^\infty(U)$, $g_i(x) = O(|x|) \in C^\infty(U, \mathrm{End}(E))$.

   (**Hint**: Show that the Lie derivative of the connection matrix ω (1-form) satisfies $L_{\mathcal{R}}\omega = \iota_{\mathcal{R}}\Omega$, where Ω is the curvature matrix (2-form) and $\mathcal{R} := \sum x_i \partial_i$, and then apply the Taylor expansion to it.)

Homework 8

Let M^n be a compact oriented even dimensional manifold.

1. Define $\tilde{*} := i^{n/2+p(n-p)} *$. Show that

   (1) $\tilde{*}^2 = \mathrm{Id}$ on $\Lambda := \Lambda^*_{\mathbb{C}}(M)$ and in fact $\tilde{*} = \epsilon$,

   (2) Under $\Lambda = \Lambda^+ \oplus \Lambda^-$ induced by by ± 1 eigenspace of $\tilde{*}$, the index of $D^+ = (d + d^*)|_{\Lambda^+}$ is $\sigma(M)$ (If $n \notin 4\mathbb{N}$ both are zero).

   (3) Under $\Lambda \cong S \otimes W$, $W = W^+$ is of pure even grading.

2. Let $E \to M$ be a complex vector bundle. Consider the twisted signature operator $D^+ : \Gamma(\Lambda^+ \otimes E) \to \Gamma(\Lambda^- \otimes E)$. Show that

   (1) $\mathrm{ind}\, D^+ = (L(M).\mathrm{ch}(E))[M]$.

   (2) For $M = S^{2m}$, it is given by $\dfrac{1}{(m-1)!} c_m(E)[S^{2m}]$. (Hint: Show that all Pontryagin classes on S^{2m} vanishes.)

   (3) The only spheres which could admit almost complex structures are S^2 and S^6. (Bonus: Construct one on S^6.)

Homework 9

1. (Leray-Hirsch) Let $E \to M$ be a complex vector bundle of rank r with $f : \mathbb{P}(E) \to M$ be the induced projective bundle. Show that

$$H^*(\mathbb{P}(E), \mathbb{Z}) \simeq H^*(M, \mathbb{Z})[\zeta]/(f_E(\zeta)) \quad \text{as rings,}$$

   where $\zeta = c_1(S^*)$ and $f_E(\zeta) = \zeta^r + f^*c_1(E)\zeta^{r-1} + f^*c_2(E)\zeta^{r-2} + \cdots + f^*c_r(E)$ is the Chern polynomial.

2. Show that $e, p_1 : \pi_3(\mathbf{SO}(4)) \to H^4(S^4)$ are both group homomorphisms.