DIFFERENTIAL GEOMETRY II

FINAL REPORTS

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*We keep the same numbering as in the original "final report problem list".

A REPORT ON NEWLANDER-NIRENBERG THEOREM

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ABSTRACT. In 1956, Newlander and Nirenberg [1] proved that if the Nijenhuis tensor vanishes, then the almost complex structure is integrable. The case for the real analytic manifolds with real analytic almost complex structure was proved by Frölicher [2] in 1955, by the Frobenius Theorem. In [1], the condition is weakened to be the manifold is of class $2n+1$ and the almost complex structure is of class $C^{2n,\alpha}$.

Later in 1989 Webster [3] further proved for almost complex structure is only of $C^{r,\lambda}$, $r \geq 1$, then we can get a $C^{r+1,\lambda}$ complex structure. The proof is based on the Nash-Moser iteration. In 2010, Joachim Michel [4] gives the same result without the Nash-Moser iteration.

This report is based on the original proof [1]. Some notations are exchanged into our familier ones and adds some detail of calculations.

1. Introduction and Definitions

A manifold is called complex manifold if it can be covered by coordinate patches with complex coordinate in which the coordinates in overlapping patches are related by complex analytic transformations. On such a manifold, scalar multiplication by i in the tangent space has a invariant meaning.

Definition. *Let M be a $2n$-dimensional real manifold. M is called* **alomost complex** *if there exists a smooth linear transfomation $J : T_pM \to T_pM$ such that $J^2 = -id_{T_pM}$. Equivalently, there exists a real tensor field h^μ_λ satisfying*

$$h^\mu_\lambda h^\sigma_\mu = -\delta^\sigma_\lambda \tag{1.1}$$

On a even dimensional real manifold, in local coordinate $x^1, \cdots, x^{2n}$ one may introduce complex coordinates by setting, for example, $z^j = x^j + ix^{j+n}$, $j = 1, \cdots, n$.

Definition. *The almost complex structure given by J is called* **integrable** *if the manifold can be made into a complex manifold with local coordinates $z^1, \cdots, z^n$ so that operating with J is equivalent to transforming dz^j and $d\bar{z}^j$ into idz^j and $-id\bar{z}^j$ respectively.*

A natural question is to ask when a almost complex manifold is actually a complex manifold and we note that this problem is purely local since if z^j are local coordinates with dz^j and $d\bar{z}^j$ transformed into idz^j and $-id\bar{z}^j$ under J, then z^j will automatically be complex analytic functions of overlapping coordinates having the same transformation preperty with repect to J. (see [2]).

2. Main Theorem

In any chart, we may choose complex valued coordinate $z^1, \cdots, z^{2n}$ with $z^{j+n} = \bar{z}^j$ (In later discussing, we simply $z^1, \cdots, z^n$) such that at origin of the coordinate

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system, the values of h^μ_λ are:

$$h^\lambda_\mu = 0 \text{ if } \lambda \neq \mu, \quad h^j_j = i, \text{ and } \quad h^{j+n}_{j+n} = -i \tag{2.1}$$

Now suppose that we have a complex coordinate $\zeta^1, \cdots, \zeta^n$, then they must satisfy $d\zeta^j = \frac{\partial \zeta^j}{\partial z^\mu} d^\mu$ and $id\zeta^j = \frac{\partial \zeta^j}{\partial z^\mu} h^\mu_\lambda dz^\lambda$. After some calculation, the $\zeta^j = w$ satisty the system of equations:

$$\frac{\partial w}{\partial z^\mu}(h^\mu_\lambda - i\delta^\mu_\lambda) = 0, \quad \lambda = 1, \cdots, 2n \tag{2.2}$$

Definition. *A complex-valued function w satisfying* (2.2) *is called* **holomorphic** *with respect to the almost complex structure J.*

Also, we have $2id\zeta^j = \frac{\partial \zeta^j}{\partial z^\mu}(h^\mu_\lambda + i\delta^\mu_\lambda)dz^\lambda$, so the system of forms $d\zeta^j$ is equivalent to the system $(h^\mu_\lambda + i\delta^\mu_\lambda)dz^\lambda$ which follows from the following fact in [2]:

Necessary Condition. (The Complete Integrability Condition) *The exterior differential of any form $(h^\mu_\lambda + i\delta^\mu_\lambda)dz^\lambda$ of the system may be expressed as a sum of exterior products of the forms of the system with first order forms.*

By our choice of coordinate, the last n equations of the system are independent. For convience to fomulate these condition, we set

$$z^{j+n} = \bar{z^j} = \bar{z}^j; \quad \partial_j = \frac{\partial}{\partial z^j}; \quad \bar{\partial}_k = \frac{\partial}{\partial \bar{z}_j}, \quad \bar{\zeta^j} = \bar{\zeta}^j.$$

We may solve these for the derivatives $\bar{\partial}_j w$ and rewrite these equations in the form:

$$L_j w = \bar{\partial}_j w - a^k_j \partial_k w = 0, \tag{2.3}$$

where $a^k_j = 0$ at $z^1 = \cdots = z^n = 0$, $j = 1, \cdots, n$. By (2.3), the equivalent system of the forms $d\zeta^j$ is

$$\begin{aligned} d\zeta^j &= \partial_k \zeta^j dz^k + \bar{\partial}_k \zeta^j d\bar{z}^k \\ &= dz^j + a^j_k d\bar{z}^k \end{aligned}$$

and the integrability condition, in the sense of Frobenius Theorem (see Remark 1.61 in Warner [7]), becomes that the L_j operator commute:

$$\bar{\partial}_j a^k_m - a^p_j \partial_p a^k_m = \bar{\partial}_m a^k_j - a^p_m \partial_p a^k_j, \quad j, m, k = 1, \cdots, n \tag{2.4}$$

This shows that $Y_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{n} a^k_j \frac{\partial}{\partial \bar{z}^k}$ is a integrable complex struture. So our problem is to solve the converse.

Formulation. *Under the transformation of variables $(\zeta^1, \cdots, \zeta^n) \to (z^1, \cdots, z^n)$, the Cauchy-Riemann euqations $\frac{\partial w}{\partial \zeta^j} = 0$ are transfomed in to a system* (2.3) *satisfying the integrability relation* (2.4). *Show that conversely that a given system* (2.3) *satisfies* (2.4) *may be transformed to the Cauchy-Riemann equations by a nonsigular transformation of variables.*

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is **Hölder continuous** of order $0 < \alpha < 1$ if the quantity

$$\sup_{x,y\in[a,b],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is finite. We use $C^{k,\alpha}(B(0;r))$ to denote the space of all functions defined on $B(0;r) \subseteq \mathbb{R}^n$ for some $r > 0$ such that f is C^k and the k-th derivative of f is Hölder continuous with exponent α.

Now we can state the main Theorem we shall prove:

Theorem 2.1. *If the cofficients a_j^k in (2.3) are of class C^{2n} in a neighborhood of the origin and satisfy (2.4), then in some neighborhood of the origin, there exists n solutions $\zeta^1, \cdots, \zeta^n$ of (2.3) such that the Jacobian of $\zeta^1, \cdots, \zeta^n, \bar{\zeta}^1, \cdots, \bar{\zeta}^n$ with respect to $z^1, \cdots, \bar{z}^n$ is different from zero, so that the equations (2.3) reduce to $\frac{\partial w}{\partial \bar{\zeta}^j} = 0$. Also each ζ^j is of class $C^{2n+\beta}$ for all $0 < \beta < 1$. If, in addition, the coefficients a_j^k are of class $C^{m,\alpha}$, for some $m \geq 2n$, and $0 < \alpha < 1$, then each is of class $C^{m+1,\alpha}$.*

In our case, the manifold is of class C^{2n+1} and h_λ^μ is of class C^{2n}, then there exists a complex coordinate of class, however, C^{2n}. In order for theses to be of C^{2n+1}, we require that there exist suitable corrdinates in which h_λ^μ are of class $C^{2n,\alpha}$ for some $\alpha > 0$.

3. Integral Equations

We first treat the case of one complex dimension. We want to find out the solution of

$$\bar{\partial}_z w = a(z)\partial_z w \tag{3.1}$$

in $|z| < r$. According to the Cauchy integral formula, we define a operator T by

$$Tf = \frac{1}{2\pi i}\iint_{|\tau|<r} \frac{f(\tau)}{z-\tau} d\bar{\tau} d\tau$$

Then if f is Hölder continuous in $|z| < r$, then $\bar{\partial}_z Tf(z) = f(z)$. So w is a solution of (3.1) if and only if w satisfies the integral equation:

$$w(z) = T(a\partial_z w) + z$$

Use the Picard's interation $w_{n+1} = T(a\partial_z w_n) + z$ and $w_0 = 0$. To exam the convergency, note that

$$\|Tf\|^*_{1,\alpha;B(0;r)} \leq Cr\left(\|f\|^*_{0;B(0;r)} + [f]^*_{0,\alpha;B(0;r)}\right) = Cr\|f\|^*_{0,\alpha;B(0;r)}, \tag{3.2}$$

where the norms and seminorms equipped on C^k and $C^{k,\alpha}$ here is:

$$[f]^*_{k;B(0;r)} = \sup_{y\neq x,|\beta|=k} r^k|D^\beta f(x)|$$

$$\|f\|^*_{k;B(0;r)} = \sum_{j=0}^{k}[f]_{j;B(0;r)}$$

$$[f]^*_{k,\alpha;B(0;r)} = \sup_{y\neq x,|\beta|=k} r^{k+\alpha}\frac{|D^\beta f(x) - D^\beta(y)|}{|x-y|}$$

$$\|f\|^*_{k,\alpha;B(0;r)} = \|f\|_{k;B(0;r)} + [f]_{k,\alpha;B(0;r)}$$

and D is both ∂_z and $\bar{\partial}_z$

So the convergence in $C^{1,\alpha}$ for r suffiently small such that $\partial_z w \neq 0$.

With the experience above, we define

$$T^j f = \frac{1}{2\pi i}\iint_{|\tau|<r} f(z^1, \cdots, z^{j-1}, \tau, z^{j+1}, \cdots, z^n)\frac{d\tau d\bar{\tau}}{z^j - \tau}. \tag{3.3}$$

Clearly T^j commute each other, and for $j \neq k$, T^j also commute with ∂_k. If $\boldsymbol{F} = \{f_1, \cdots, f_n\}$ satisfies them compatibility condition: for all j, k

$$\bar{\partial}_j f_k = \bar{\partial}_k f_j.$$

then the inhomogeneous Cauchy-Riemann equation for each j

$$\bar{\partial}_j w = f_j \tag{3.4}$$

has the solution

$$w = \sum_{s=0}^{n=1} \frac{(-1)^s}{(s+1)!}\sum{}' T^{j_1}\bar{\partial}_{j_1}\cdots T^{j_s}\bar{\partial}_{j_s}\cdot T^k f_k \equiv \boldsymbol{TF}, \tag{3.5}$$

where $\sum'$ denotes summation over all $(s+1)$-tuples with $j_1, \cdots, j_s, k$ distinct. Directly differentiate, we get

$$\begin{aligned}\bar{\partial}_j w - f_j &= \bar{\partial}_j w - T^k\bar{\partial}_k f_j \\ &= \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!}\sum{}'^{j} T^{j_1}\bar{\partial}_{j_1}\cdots T^{j_s}\bar{\partial}_{j_s}T^k(\bar{\partial}_j f_k - \bar{\partial}_k f_j) = 0\end{aligned} \tag{3.6}$$

where $\sum^j$ denotes summation over all $(s+1)-tuples$ with $j_1, \cdots, j_s, k$ distinct and different from j.

By unfortunately, since right hand side of (2.3) invovles derivatives with respect to all z^k, so the standard Picard iteration will not converges as one can imagine. So we need a new viewpoint of the relation of ξ^j and z^k.

In the begining, we pick a arbitrary coordinates z^k and try to find out the ξ^j variable to be the solution of (2.3). We now pick arbitrary ξ^j variable and try to find out which kind of z^k coordinates we should choose. If we find out such z^k, they shall give as a system like (2.3) and the ξ^j will autometically satisfy the system! For this purpose, let's see our $z^1, \cdots, z^n$ as functions of $\zeta^1, \cdots, \zeta^n$. Set

$$d_j = \frac{\partial}{\partial\zeta^j}, \quad \bar{d}_j = \frac{\partial}{\partial\bar{\zeta}_j}$$

If w satifies (2.3), then the Cauchy-Riemann Equation with respect to ζ^j becomes

$$\begin{aligned}0 = \bar{d}_j w &= \frac{\partial w}{\partial z^k}\frac{\partial z^k}{\partial\bar{\zeta}^j} + \frac{\partial w}{\partial\bar{z}^k}\frac{\partial\bar{z}^k}{\partial\bar{\zeta}^j} \\ &= \partial_k w\bar{d}_j z^k + a_k^m\partial_m w\bar{d}_j\bar{z}^k \\ &= \partial_k w(\bar{d}_j z^k + a_m^k\bar{d}_j\bar{z}^m)\end{aligned}$$

provided

$$\bar{d}_j z^k + a_m^k\bar{d}_j\bar{z}^m = 0 \tag{3.7}$$

Converserly, if z^k satisfies 3.7, then

$$\begin{aligned} 0 = \bar{d}_j w &= \bar{d}_j z^k \partial_k w + \bar{d}_j \bar{z}^k \bar{\partial}_k w \\ &= -a^k_m \bar{d}_j \bar{z}^m \partial_k w + \bar{d}_j \bar{z}_k \bar{\partial}_k w \\ &= \bar{d}_j \bar{z}^k (\bar{\partial}_k w - a^i_k \partial_i w) \end{aligned}$$

Thus the Cauchy-Riemann euqations $\bar{d}_j w = 0$ are equaivalent to the system (2.3) if z^j satisfies (3.7) and the matrix $[\bar{d}_j \bar{z}^k]$ is nonsingular.

The system (3.7) is convenient because the differentiation occurs with respect to only one of the indepentdent variable in each equation.

Also we set the similar operator T^j and $\boldsymbol{T}$ by replacing all $\bar{\partial}_j$ with $\bar{d}_j$. Set $\boldsymbol{Z} = \{z^1, \cdots, z^n\}$ and denote a^i_m by $a^i_m(\boldsymbol{Z})$ as it is a function of $z^1, \cdots, z^n$. Furthermore, setting

$$f^i_j[\boldsymbol{Z}] = -a^i_m \bar{d}_j \bar{z}^m \quad \text{and define} \quad \boldsymbol{F}^i = \{f^i_1, \cdots, f^i_n\} \tag{3.8}$$

The corresponding integral system of (3.7) is

$$z^i = \zeta^i + \boldsymbol{T}(\boldsymbol{F}^i[\boldsymbol{Z}]) - z^i_0[\boldsymbol{Z}], \quad \text{where} \quad z^i_0 = \boldsymbol{T}\boldsymbol{F}^i[\boldsymbol{Z}]\big|_{\zeta^1 = \cdots = \zeta^n = 0} \tag{3.9}$$

4. A Special Normed Linear Space

Throughout this section, we use D_k to denote both d_k and $\bar{d}_k$ and $D^k = D_{i_1} \cdots D_{i_k}$ be the mixed derivative. Also, similarly, $D^{k,j}$ denotes all $i_1, \cdots, i_k$ are distinct to j. Further, the seminorms and norms use in the following section are still denoted with the super-script $*$.

Consider complex valued functions of $\zeta^1, \cdots, \zeta^n$ in the polycylinder $|\zeta^j| < r \leq \frac{1}{4}$ for all j and $r > 0$ to be determine later.

Definition. *Fixed $0 < \alpha < 1$, the* **difference quotient operators of Hölder type** *δ_i is defined by*

$$\delta_i f = \frac{f(\zeta^1, \cdots, \zeta^j + \delta\zeta^j, \cdots, \zeta^n) - f}{|\delta\zeta^j|^\alpha}, \quad |\zeta^j + \delta\zeta^j| < r,$$

and denote $\delta^m = \delta_{j_1} \cdots \delta_{j_m}$, $0 \leq m \leq n$, $j_1, \cdots, j_m$ are dinstinct.

Clearly, δ_j commute with D_k for $j \neq k$.

Definition. *Given a function z in polycylinder, we define a semi-norm*

$$[z]^*_\alpha = \sum_{m=0}^{n} \frac{r^{m\alpha}}{m!} \sup_{\delta^m} |\delta^m z|$$

If $z \in C^n$, instead of the standard norm on C^n, we equip the following norm

$$\|z\|^*_n = \sum_{k=0}^{n} \frac{r^k}{k!} \sup |D^k z|$$

Similarly, for $z \in C^{n,\alpha}$, we use the norm

$$\|z\|^*_{n,\alpha} = \sum_{k=0}^{n} \frac{r^k}{k!} \sup [D^k z]^*_\alpha \leq \sum_{k,m=0}^{n} \frac{r^{k+m\alpha}}{k!m!} \sup |\delta^m D^k z|$$

and denote $\widetilde{C}^{n,\alpha} = \{z \,|\, \|z\|^*_{n,\alpha} < \infty\}$.

Finally, since the requirement of f_j, we introduce the norm

$$\|f\|^{*,j}_{n-1,\alpha} = \sum_{k=1}^{n} \frac{r^k}{k!} \sup[D^{k,j} f]^*_\alpha$$

It is not difficult to see that $(\widetilde{C}^{n,\alpha}, \|\cdot\|^*_{n,\alpha})$ is a Banach algebra under the usual addition and multiplication.

Now for n-tuples of functions $\boldsymbol{Z} = \{z^1, \cdots, z^n\}$ and $\boldsymbol{F} = \{f_1, \cdots, f_n\}$, we define the norm

$$\|\boldsymbol{Z}\|^*_{n,\alpha} = \max_j \|z^j\|^*_{n,\alpha},$$
$$\|\boldsymbol{F}\|^*_{n-1,\alpha} = \max_j \|f_j\|^{*,j}_{n-1,\alpha}.$$

We denote $\boldsymbol{B} = \{\boldsymbol{Z} | z^j \in \widetilde{C}^{n,\alpha}, \forall j\}$ with the norm $\|\cdot\|^*_{n,\alpha}$.

Our first lemma is to estimate $a(\boldsymbol{Z})$ if $\boldsymbol{Z} \in \boldsymbol{B}$, especially for $a(\boldsymbol{0}) = 0$.

Lemma 4.1. *Suppose that $\|\boldsymbol{Z}\|^*_{n,\alpha} \leq 1$ and that $a(\boldsymbol{Z})$ has continuous derivative up to $2n-1$ order and bounded, say by K. Then there exists a constant $c = c(n, \alpha)$ such that for all $1 \leq m \leq n$*

$$\|a(\boldsymbol{Z})\|^{*,m}_{n-1,\alpha} \leq cK. \tag{4.1}$$

If moreover, $a(\boldsymbol{0}) = 0$, then

$$\|a(\boldsymbol{Z})\|^{*,m}_{n-1,\alpha} \leq cK\|\boldsymbol{Z}\|^*_{n,\alpha}. \tag{4.2}$$

Proof. We use D^k_z to denote the derivative of a. Observe that by mean-valued Theorem for the case $a(\boldsymbol{0}) = 0$

$$|a(\boldsymbol{Z})| \leq \begin{cases} K, \\ c'K\|\boldsymbol{Z}\|^*_{n,\alpha} \text{ if } a(\boldsymbol{0}) = 0. \end{cases} \tag{4.3}$$

For $j > 0$, by chain rule $D^j a$ may be express as a linear combination of terms t of the from

$$t = (D^k_z a)(D^{j_1} z^{i_1}) \cdots (D^{j_s} z^{i_s})(D^{j_{s+1}} \bar{z}^{i_{s+1}}) \cdots D^{j_k} \bar{z}^{i_k}, \quad 1 \leq k \leq j.$$

with $D^{j_1} \cdots D^{j_k} = D^j$. So $r^j|t| \leq K\|\boldsymbol{Z}\|^*_{n,\alpha}$ and hence $r^j|D^j a| \leq c'K\|\boldsymbol{Z}\|^*_{n,\alpha}$. So combining with (4.3), we get

$$\|a(\boldsymbol{Z})\|^*_n \leq \begin{cases} cK, \\ c''K\|\boldsymbol{Z}\|^*_{n,\alpha} \text{ if } a(\boldsymbol{0}) = 0. \end{cases} \tag{4.4}$$

Now we turn to the estimation of $r^j[D^j a]^*_\alpha$ for $j < n$. First by mean-value Theorem, we get

$$[z]^*_\alpha \leq c\|z\|^*_n \tag{4.5}$$

Indeed,

$$[z]^*_\alpha = \sum_{m=0}^{n} \frac{r^{m\alpha}}{m!} \sup |D^m z(\eta)||\delta r|^{m-m\alpha} \leq c\|z\|^*_n$$

If $j = 0$, then from (4.5) and (4.4), we have

$$[a(\boldsymbol{Z})]^*_\alpha \leq c\|a(\boldsymbol{Z})\|^*_n \leq \begin{cases} cK, \\ cK\|\boldsymbol{Z}\|^*_{n,\alpha} \end{cases}$$

For $j \geq 1$, consider

$$r^j[t]^*_\alpha \leq c[D^k_z a]^*_\alpha \|\boldsymbol{Z}\|^*_{n,\alpha}$$
$$\text{By (4.5)} \leq c\|D^k_z a\|^*_n \|\boldsymbol{Z}\|^*_{n,\alpha}.$$

Of course, (4.4) also holds for $D^k_z a$, we have

$$r^j[t]^*_\alpha \leq cK\|\boldsymbol{Z}\|^*_{n,\alpha}$$

Combine all the inequality, we finally have

$$r^j[D^j a]^*_\alpha \leq cK\|\boldsymbol{Z}\|^*_{n,\alpha}, \quad 1 \leq j < n$$

the result follows. □

5. Potential Theoretic Lemmas

In the previous seciton, we get the estmation of $a(\boldsymbol{Z})$ whenever $\|\boldsymbol{Z}\|^*_{n,\alpha} \leq 1$. In this section, we prove the properties of the integral operator T^j. Our main goal in this section is to show the next Theorem

Theorem 5.1. *There exists a constant $C = C(n, \alpha)$ such that*

$$\|\boldsymbol{TF}\|^*_{n,\alpha} \leq Cr\|\boldsymbol{F}\|^*_{n-1,\alpha}.$$

Before states next lemma, we observe that as (3.2), we also have there exists $c = c(n, \alpha)$

(5.1)
$$\sup|T^j y| + r^\alpha \sup|\delta_j T^j y| + r\sup|D_j T^j y| + r^{1+\alpha}\sup|\delta_j D_j T^j y| \leq cr\left[\sup|y| + r^\alpha \sup|\delta_j y|\right],$$

for y defined in the cylinder.

Lemma 5.2.

$$\|T^j D_j f\|^{*,l}_{n-1,\alpha} \leq c\|f\|^{*,l}_{n-1,\alpha}, \quad j, l = 1, \cdots, n \text{ with } j \neq l$$

Proof. It suffice to give such a bound of the functions $r^{k+m\alpha}\delta^m D^{k,l} T^j D_j f$, $0 \leq k \leq n-1$ and $0 \leq m \leq n$. For $j_1, \cdots, j_m$ distinct and different from j, and $i_1, \cdots, i_k$ distinct and different from j, l, consider

$$y = \delta_{j_1} \cdots \delta_{j_m} D_{i_1} \cdots D_{i_k} T^j D_j f.$$

We see that it suffice to derive such a bound for the functions $\eta = r^{k+m\alpha} T^j y$, $r^\alpha \delta_j \eta$, $rD_j\eta$, and $r^{1+\alpha}\delta_j D_j \eta$. From (5.1), we see that these four functions are bounded by

$$cr^{k+1+m\alpha}\left[\sup|y| + r^\alpha \sup|\delta_j y|\right] \leq c\|f\|^{*,l}_{n-1,\alpha}$$

and the result follows. □

Similar argument we may prove

Lemma 5.3.

$$\|T^j(gh)\|^{*,l}_{n-1,\alpha} \leq cr\|g\|^{*,j}_{n-1,\alpha}\|h\|^{*,l}_{n-1,\alpha} \quad j, l = 1, \cdots, n.$$

Corollary 5.4.

$$\|T^j g\|^{*,l}_{n-1,\alpha} \le \begin{cases} cr\|g\|^{*,j}_{n-1,\alpha} \\ cr\|g\|^{*,l}_{n-1,\alpha} \end{cases} \qquad j,l = 1,\cdots,n.$$

Lemma 5.5.

$$\|T^j f\|^{*}_{n-1,\alpha} \le cr\|f\|^{*,j}_{n-1,\alpha}$$

Proof. From Corollary 5.4 we already have $\|T^j f\|^*_{n,\alpha} \le cr\|f\|^{*,j}_{n-1,\alpha}$. It remains to show the functions $r^n D^n T^j f$ and $r^{n+l\alpha}\delta^l D^n T^j f$ are bounded by $cr\|f\|^{*,j}_{n-1,\alpha}$. However, the proof is similar as to Theorem 5.2 by using (5.1). □

It is clear that Theorem 5.1 follows immediately from Lemma 5.2 and 5.5.

6. Existence Theorem

In this section we will prove

Theorem 6.1. *If the coeffcents a^i_j have bounded derivative up to order $2n$ for r small, the system* (3.9) *has a unique solution $\boldsymbol{Z}$ in $\boldsymbol{B}$ satisfying also* (3.7) *such that the transformation from the ζ coordinates to z coordiantes has non-vanishing Jacobian.*

We simplify (3.9) by $\boldsymbol{Z} = \mathscr{T}[\boldsymbol{Z}]$. Our strategy is

Lemma 6.2. *For the integral system* (3.9), *if $\|\boldsymbol{Z}\|_{n,\alpha}, \|\widetilde{\boldsymbol{Z}}\|_{n,\alpha} \le 4r$, then*

$$\begin{aligned} &\|\mathscr{T}[\boldsymbol{Z}]\|_{n,\alpha} \le 4r \\ (6.1)\qquad &\|\mathscr{T}[\boldsymbol{Z}] - \mathscr{T}[\widetilde{\boldsymbol{Z}}]\|_{n,\alpha} \le \frac{1}{2}\|\boldsymbol{Z} - \widetilde{\boldsymbol{Z}}\|_{n,\alpha}\end{aligned}$$

for then the usual iteration scheme yields a unique fixed point of $\mathscr{T}[\boldsymbol{Z}]$ in $\|\boldsymbol{Z}\| \le 4r$ by contraction mapping Theorem.

Recall our defintion of $f^i_j[\boldsymbol{Z}] = -a^i_k \bar{d}_j \bar{z}^k$ and a^i_k vanish at $\boldsymbol{Z} = \boldsymbol{0}$ and have bounded derivative up to order $2n$, say by K.

Lemma 6.3. *If $\|\boldsymbol{Z}\|^*_{n,\alpha}, \|\widetilde{\boldsymbol{Z}}\|^*_{n,\alpha} \le 4r$, then for all $1 \le i,j \le n$*

$$\begin{aligned} \|f^i_j[\boldsymbol{Z}]\|^{*,j}_{n-1,\alpha} &\le cKr \\ \|f^i_j[\boldsymbol{Z}] - f^i_j[\widetilde{\boldsymbol{Z}}]\|^{*,j}_{n-1,\alpha} &\le cK\|\boldsymbol{Z} - \widetilde{\boldsymbol{Z}}\|^{j,*}_{n-1,\alpha}\end{aligned}$$

Proof. We first observe that

$$\|D_j f\|^{*,j}_{n-1,\alpha} \le \frac{c}{r}\|f\|^*_{n,\alpha} \tag{6.2}$$

By the multiplication property of our norms we have

$$\begin{aligned} \|f^i_j[\boldsymbol{Z}]\|^{*,j}_{n-1,\alpha} &\le \sum_{k=0}^{n-1} \|a^i_k\|^{*,j}_{n-1,\alpha}\|\bar{d}_j\bar{z}^k\|^*_{n-1,\alpha} \\ \text{by (6.2)} &\le \frac{c}{r}\sum_k \|a^i_k\|^{*,j}_{n-1,\alpha}\|\bar{z}^k\|_{n,\alpha} \\ &\le 4c\sum_k \|a^i_k\|_{n-1,\alpha} \\ \text{by (4.2)} &\le 4c(ncK4r). \end{aligned} \tag{6.3}$$

This proves the first part of lemma.

Similarly, by Lemma 4.1 and (6.2), we find that

$$\begin{aligned}\|f_j^i[Z] - f_j^i[\widetilde{Z}]\|_{n-1,\alpha}^{*,j} &\leq \sum_{k=0}^{n-1} \|a_k^i(Z)\bar{d}_j(\bar{z}^k - \bar{\tilde{z}}^k)\|_{n-1,\alpha}^{*,j} + \sum_{k=0}^{n-1} \|\bar{d}_j\bar{z}^k(a_k^i(Z) - a_k^i(\widetilde{Z}))\|_{n-1,\alpha}^{*,j} \\ &= \sum \|a_k^i(Z)\|_{n-1,\alpha}^{*,j}\|d_j(z^k - \tilde{z}^k)\|_{n-1,\alpha}^{*,j} + \sum_{k=0}^{n-1} \|\bar{d}_j\bar{z}^k\|_{n-1,\alpha}^{*,j}\|a_k^i(Z) - a_k^i(\widetilde{Z})\|_{n-1,\alpha}^{*,j} \\ &\leq ncK4c\|Z - \widetilde{Z}\|_{n-1,\alpha}^{*} + 4c\sum \|a_k^i(Z) - a_k^i(\widetilde{Z})\|_{n-1,\alpha}^{*,j}\end{aligned}$$

By mean-value Theorem we have there exists Z' with $\|Z'\|_{n-1,\alpha}^{j} \leq 4r$ such that

$$\|a_k^i(Z) - a_k^i(\widetilde{Z})\|_{n-1,\alpha}^{*,j} \leq \left[\sum_{m=1}^{n} \|\partial_m a_k^i(Z')\|_{n-1,\alpha}^{*,j} + \sum_{m=1}^{n} \bar{\partial}_m a_k^i(Z')\|_{n-1,\alpha}^{*,j}\right] \|Z - \widetilde{Z}\|_{n-1,\alpha}^{*,j}$$

By Lemma 5.2, we have

$$\|a_k^i(Z) - a_k^i(\widetilde{Z})\|_{n-1,\alpha}^{*,j} \leq cK\|Z - \widetilde{Z}\|_{n,\alpha}.$$

and second part of theorem follows. □

Proof of Lemma 6.2. Set $F^i[Z] = \{f_1^i[Z], \cdots, f_n^i[Z]\}$ and denote the i-th component of $\mathscr{T}[Z]$ by $y^i[Z]$ i.e.

$$y^i[Z] = \zeta^i + TF^i[Z] - z_0^i[Z]$$

Observe that

$$\begin{aligned}\|y^i[Z]\|_{n,\alpha}^{*} &\leq \|\zeta\|_{n,\alpha}^{*} + 2\|TF^i[Z]\|_{n,\alpha}^{*} \\ \text{by Theorem 5.1 } &\leq (2 + 2^{1-\alpha})r + 2Cr\|F^i[Z]\|_{n-1,\alpha}^{*} \\ \text{by Lemma 6.3 } &\leq (2 + 2^{1-\alpha})r + 2CrcKr \\ &\leq 4r \quad \text{for } r \text{ sufficient small.}\end{aligned} \tag{6.4}$$

and this prove the first part of Lemma 6.2. To prove second part, directly calculate

$$\begin{aligned}\|y^i[Z] - y^i[\widetilde{Z}]\|_{n,\alpha}^{*} &\leq 2\|T(F^i[Z] - F^i[\widetilde{Z}])\|_{n,\alpha}^{*} \\ \text{by Theorem 5.1 } &\leq 2Cr\|F^i[Z] - F^i[\widetilde{Z}]\|_{n-1,\alpha}^{*} \\ \text{by Lemma 6.3 } &\leq 2CrcK\|Z - \widetilde{Z}\|_{n,\alpha}^{*} \\ &\leq \frac{1}{2}\|Z - \widetilde{Z}\|_{n,\alpha}^{*} \quad \text{for } r \text{ sufficient small}\end{aligned}$$

□

Proof of Theorem 6.1. By contraction mapping Theorem, there exists a fixpoint Z for r sufficient small. As we show that $\|TF^i[Z]\|_{n,\alpha}^{*} \leq cCKr^2$, it follows that for r sufficient small, the Jacobian of the z^i with respect to ζ^i variable is different from zero and that the matrix $[\bar{d}_j\bar{z}^k]$ is closed to identity.

So far, the integrablility condition is not used. It comes out for the proof that the solution Z is also a solution of (3.7).

Differetiate (3.9) with respect to $\bar{\zeta}^j$ and by (3.6), we have

$$\bar{d}_j z^i = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!}\sum^{j} T^{j_1}\bar{d}_{j_1}\cdots T^{j_s}\bar{d}_{j_s}\cdot T^k(\bar{d}_j f_k^i - \bar{d}_k f_j^i) - a_m^i\bar{d}_j\bar{z}^m$$

So if we set $g_j^i = \bar{d}_j z^i + a_m^i\bar{d}_j\bar{z}^m$, then

$$g^i_j = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum\nolimits^j T^{j_1}\bar{d}_{j_1} \cdots T^{j_s}\bar{d}_{j_s} \cdot T^k(\bar{d}_j f^i_k - \bar{d}_k f^i_j) \tag{6.5}$$

Put the definition of $f^i_j = -a^i_k \bar{d}_j \bar{z}^k$ into (6.5) and chain rule, we get

$$\begin{aligned}
\bar{d}_i f^i_k - \bar{d}_k f^i_j &= \bar{d}_k a^i_m \bar{d}_j \bar{z}^m - \bar{d}_j a^i_m \bar{d}_k \bar{z}^m \\
&= [\bar{d}_k z^p \partial_p a^i_m \bar{d}_j \bar{z}^m + \bar{d}_k \bar{z}^p \bar{\partial}_p a^i_m \bar{d}_j \bar{z}^m] - [\bar{d}_j z^p \partial_p a^i_m \bar{d}_k \bar{z}^m + \bar{d}_j \bar{z}^p \bar{\partial}_p a^i_m \bar{d}_k \bar{z}^m] \\
&= [\partial_p a^i_m (\bar{d}_k z^p \bar{d}_j \bar{z}^m - \bar{d}_j z^p \bar{d}_k \bar{z}^m)] + [(\bar{\partial}_p a^i_m - \bar{\partial}_m a^i_p) \bar{d}_k \bar{z}^p \bar{d}_j \bar{z}^m]
\end{aligned}$$

For first part, directly compute

$$\begin{aligned}
\bar{\partial}_p a^i_m (\bar{d}_j \bar{z}^m g^p_k - \bar{d}_k \bar{z}_m g^p_j) &= \bar{\partial}_p a^i_m [\bar{d}_j \bar{z}^m (\bar{d}_k z^p - a^p_s \bar{d}_k \bar{z}^s) - \bar{d}_k \bar{z}^m (\bar{d}_j z^p - a^p_s \bar{d}_j \bar{z}^s)] \\
&= \bar{\partial}_p a^i_m [(\bar{d}_k z^p \bar{d}_j \bar{z}^m - \bar{d}_j z^p \bar{d}_k \bar{z}^m) + (a^p_s \bar{d}_j \bar{z}^s \bar{d}_k \bar{z}^m - a^p_s \bar{d}_k \bar{z}^s \bar{d}_j \bar{z}^m)] \\
&= \bar{\partial}_p a^i_m (\bar{d}_k z^p \bar{d}_j \bar{z}^m - \bar{d}_j z^p \bar{d}_k \bar{z}^m)
\end{aligned}$$

For second part, use integrability condition, we have

$$(\bar{\partial}_p a^i_m - \bar{\partial}_m a^i_p) \bar{d}_k \bar{z}^p \bar{d}_j \bar{z}^m = (a^s_p \partial_s a^i_m - a^s_m \partial_s a^i_p) \bar{d}_k \bar{z}^p \bar{d}_j \bar{z}^m = 0$$

Put all of these into (6.5), we have

$$g^i_j = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum\nolimits^j T^{j_1}\bar{d}_{j_1} \cdots T^{j_s}\bar{d}_{j_s} \cdot T^k [\partial_p a^i_m (\bar{d}_j \bar{z}^m g^p_k - \bar{d}_k \bar{z}^m g^p_j)] \tag{6.6}$$

By Lemma 5.3, 5.5 and Lemma 5.2, we get

$$\begin{aligned}
\sum_{i,j} \|g^i_j\|^{*,j}_{n-1,\alpha} &\le cc'r \sum_{i,j,m} \sum_{k \neq j} \left[\|\partial_p a^i_m \bar{d}_j \bar{z}^m\|^{*,j}_{n-1,\alpha} \|g^p_k\|^{*,k}_{n-1,\alpha} + \|\partial_p a^i_m \bar{d}_k \bar{z}^m\|^{*,k}_{n-1,\alpha} \|g^p_j\|^{*,j}_{n-1,\alpha} \right] \\
&\le cr \max_{i,j,m,p} \|\partial_p a^i_m\|^{*,j}_{n-1,\alpha} \max_{m,k} \|d_k z^m\|^{*,k}_{n-1,\alpha} \left[\sum_{p,j} \|g^p_j\|^{*,j}_{n-1,\alpha} \right]
\end{aligned}$$

by (4.1) and (6.2) $\le cKr \sum_{p,j} \|g^p_j\|^{*,j}_{n-1,\alpha}$

If we further choose r small such that $cKr < 1$, then $g^i_j \equiv 0$ and completes the proof. □

7. Proof of Theorem 2.1

We have construct a solution $\boldsymbol{Z} = \{z^1, \cdots, z^n\}$ of (3.7) in the polycylinder $|\zeta^j| < r$ for small r; this solution vanishes at origin and has nonvanishing Jacobian with respect to ζ. So the functions $z^1, \cdots, z^n$ maps the polycylinder homeomorphically onto a neighborhood of U of origin in the z space. In U, the coordinates $\zeta^1, \cdots, \zeta^n$ are solutions of (2.3).

To complete the proof, it remains to show that the differentiability of ζ^j. Observe that since r small(hence a^k_m is small), so (3.7) is elliptic system of functions $z^1, \cdots, z^k$ of variable ζ^j, $\bar{\zeta}^j$. These functions are $C^{1,\alpha}$ with respect to ζ^j, $\bar{\zeta}^j$. Since $\boldsymbol{Z} \in \boldsymbol{B}$, so z^k is C^2 and by inverse function theorem ζ^k is also C^2. Moreover, ζ^j satisfy the eqation

$$\partial_j \bar{\partial}_j w - \partial_j a^k_j \partial_k w = 0$$

which is elliptic for r small. Then Theorem 2.1 follows the well-known regularity theorem

Theorem 7.1 (Theorem 6.17 in [5]). *Let L be a second order elliptic operator and $u \in C^2(\Omega)$ be a solution of $Lu = f$ where f and the coefficient of L lie in $C^{k,\alpha}$. Then $u \in C^{k+2,\alpha}$.*

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§3. Existence of Isothermal Coordinates on Surfaces. 許綺云

Let

(1) $ds^2 = E(x,y)\,dx^2 + 2F(x,y)\,dx\,dy + G(x,y)\,dy^2$. $EG-F^2>0$, $E>0$,
be a Riemannian metric defined on a neighborhood of a surface with local coordinates x,y. By isothermal parameters we mean local coordinates u, v relative to which the metric becomes

(2) $ds^2 = \lambda(u,v)(du^2+dv^2)$, $\lambda>0$.

I. Sufficiency of E, F, G satisfying a Hölder condition of order λ, $0<\lambda<1$.

(1) is positive definite, so we may write $ds^2 = \theta_1^2+\theta_2^2$, where $\theta_1 = a_1dx+b_1dy$, $\theta_2 = a_2dx+b_2dy$ are real linear differential forms. One example of θ_1, θ_2 is provided by

$$E\,dx^2+2F\,dx\,dy+G\,dy^2 = E\left(dx+\frac{F}{E}dy\right)^2+\frac{EG-F^2}{E}dy^2$$

Since $\begin{cases} a_1^2+a_2^2 = E, \\ a_1b_1+a_2b_2 = F \\ b_1^2+b_2^2 = G \end{cases}$ if θ_1', θ_2' also satisfies $ds^2 = \theta_1'^2+\theta_2'^2$,

we have $\begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \end{pmatrix} = A\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ for some orthogonal A. $\Rightarrow \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = A\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$

If we assume that $a_1b_2-a_2b_1>0$, then $\det(A)=1$, ie. A is a rotation. Hence, putting $\phi = \theta_1+i\theta_2$ ($\Rightarrow ds^2 = \phi\bar{\phi}$), ϕ is determined up to multiplication by a complex number of absolute value 1.

Now the determination of isothermal coordinates u, v is equivalent to that of a complex-valued function $w = u+iv$ s.t. $dw = \frac{1}{\rho}\phi$

For we have $ds^2 = \phi\bar{\phi} = |\rho|^2 dw\,d\bar{w} = |\rho|^2(du^2+dv^2)$,

and conversely $\phi\bar{\phi} = ds^2 = \lambda(du^2+dv^2) = \lambda\,dw\,d\bar{w} \Rightarrow dw = \frac{1}{\sqrt{\lambda}}\phi$

Def A function $f(x,y)$ in a domain $D \subseteq \mathbb{R}^2$ is said to satisfy a Hölder condition of order λ, $0<\lambda\le 1$, denoted by $f \in C^{0,\lambda}(D)$, if

(3) $\quad |f(x,y) - f(x',y')| \le C r^{\lambda}, \quad \forall (x,y), (x',y') \in D,$

where C is a constant and $r = \sqrt{(x-x')^2+(y-y')^2}$.

Key Lemma. Let $D = \{ z \in \mathbb{C} \mid |z| \le R \}$, $f(z,\bar z)$ be a complex-valued continuous function in D satisfying

$$|f(z_1,\bar z_1) - f(z_2,\bar z_2)| \le B r_{12}^{\lambda}, \quad r_{12} = |z_1 - z_2|, \quad \forall z_1, z_2 \in D,$$

where λ, B are constants, $0<\lambda<1$. Let $F(\zeta,\bar\zeta)$ be defined by

$$-2\pi i\, F(\zeta,\bar\zeta) = \iint_D \frac{f(z,\bar z)\, d\bar z\, dz}{z-\zeta} = 2i \iint_D \frac{f(z,\bar z)\, dx\, dy}{z-\zeta}, \quad \zeta \in D.$$

Then (1) F_ζ, $F_{\bar\zeta}$ exist and $F_{\bar\zeta} = f$,

(2) If $|f(z,\bar z)| \le A$, $\forall z \in D$, then

$$|F(\zeta,\bar\zeta)| \le 4RA, \qquad |F_\zeta(\zeta,\bar\zeta)| \le \frac{2^{\lambda+1}}{\lambda} R^{\lambda} B,$$

$$|F(\zeta_1,\bar\zeta_1) - F(\zeta_2,\bar\zeta_2)| \le 2\left(A + \frac{2^{\lambda+1}}{\lambda} R^{\lambda} B\right) r_{12}$$

$$|F_\zeta(\zeta_1,\bar\zeta_1) - F_\zeta(\zeta_2,\bar\zeta_2)| \le \mu(\lambda) B r_{12}^{\lambda}, \text{ for } \zeta_1,\zeta_2 \in B_0\left(\tfrac{R}{5}\right).$$

Thm 1. In D, let $Zw = a(z,\bar z) w_{\bar z} + b(z,\bar z) w_z$ be a differential operator with $a, b \in C^{0,\lambda}(D)$, $0<\lambda<1$, and $a(0,0) = b(0,0) = 0$. Let $\alpha(z,\bar z) \in C^{0,\lambda}(D)$ with the same λ, $\sigma(z)$ be complex analytic with $\sigma(0)=0$. Then

$$2\pi i\, w(\zeta,\bar\zeta) + \iint_D \frac{(Zw+\alpha w)(z,\bar z)}{z-\zeta}\, d\bar z\, dz = \sigma(\zeta), \quad \zeta \in D$$

has a unique solution w s.t. $(Z+\alpha) w \in C^{0,\lambda}(D)$, provided that R is sufficiently small.

Then the existence of isothermal coordinates follows as a corollary.

Cor. Suppose in a domain D of the (x,y)-plane, $E,F,G \in C^{0,\lambda}(D)$, $0<\lambda<1$. Then every point of D has a neighborhood in which there exist isothermal coordinates.

(pf.) May assume $(0,0) \in D$ and only consider $(0,0)$.

Assume $E(0,0)=G(0,0)=1$, $F(0,0)=0$, and

$(\theta_1)_{(0,0)} = dx$, $(\theta_2)_{(0,0)} = dy$ ($\Rightarrow \phi_{(0,0)} = dx+idy = dz$)

i.e. $\phi = (1-a(z,\bar z))dz + b(z,\bar z)d\bar z$ with $a(0,0)=b(0,0)=0$

$dw = \frac{1}{\rho}\phi \Leftrightarrow w_{\bar z} = \frac{b}{\rho}$, $w_z = \frac{1-a}{\rho} \Leftrightarrow (1-a)w_{\bar z} - bw_z = 0$

$\Leftrightarrow w_{\bar z} = Zw$, where $Zw = aw_{\bar z} + bw_z$

By Thm 1., there exists a solution $w(z,\bar z)$ of

$$2\pi i\, w(\zeta,\bar\zeta) + \iint_D \frac{Zw(z,\bar z)}{z-\zeta}\, d\bar z\, dz = \sigma(\zeta), \quad \zeta \in D, \quad \sigma(0)=0$$

Since $Zw \in C^{0,\lambda}(D)$, $w_{\bar z} = Zw$ by Key Lemma (1). $\square$

To prove Key Lemma, we first have

Lemma 1. Let D be a domain of the (x,y)-plane bounded by a curve C, (ξ,η) be a point s.t. the vector joining (ξ,η) to (x,y) reverses $(k-1)$ times. Then

$$\left|\iint_D \frac{r^\lambda\, dx\, dy}{r^2}\right| \le \frac{2k\pi}{\lambda}\Delta^\lambda, \text{ if } \lambda>0;$$

$$\left|\iint_D \frac{r^\lambda\, dx\, dy}{r^2}\right| \le \frac{2k\pi}{-\lambda}\delta^\lambda, \text{ if } \lambda<0, \ (\xi,\eta)\notin D.$$

where $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$, $\Delta = \max_{(x,y)\in C} r$, $\delta = \min_{(x,y)\in C} r$.

(pf.) If $g(r)$ is C^1, then $d\left(g(r)\frac{-(y-\eta)dx+(x-\xi)dy}{r^2}\right) = \frac{g'(r)}{r}\, dx\wedge dy$,

and thus for $(\xi,\eta)\notin D$, so that the integrands have no singularity,

$$\iint_D \frac{g'(r)}{r}\, dx\, dy = \int_C g(r)\frac{-(y-\eta)dx+(x-\xi)dy}{r^2}$$

The formula is still true when $(\xi,\eta)\in D\setminus C$, provided that the integral on the left converges, and $g(0)=0$. In fact, apply the formula to the domain $D\setminus B_\varepsilon((\xi,\eta))$, and note that $\iint_{B_\varepsilon((\xi,\eta))}\frac{g'(r)}{r}\,dxdy\to 0$ as $\varepsilon\to 0$ by the convergence of the integral on the left, and that

$$\left|\int_{\partial B_\varepsilon((\xi,\eta))} g(r)\frac{-(y-\eta)dx+(x-\xi)dy}{r^2}\right|$$

$$\le \int_{\partial B_\varepsilon((\xi,\eta))}|g(r)|\,\frac{\sqrt{((y-\eta)^2+(x-\xi)^2)(dx^2+dy^2)}}{r^2}$$

$$=|g(\varepsilon)|\cdot\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon((\xi,\eta))}\varepsilon\,ds=|g(\varepsilon)|\,\frac{2\pi\varepsilon^2}{\varepsilon^2}\to 0 \text{ as } \varepsilon\to 0$$

When $g(r)=r^\lambda$, $\lambda\ne 0$, then $\iint_D\frac{g'(r)}{r}\,dxdy=\iint_D \lambda r^{\lambda-2}\,dxdy$ converges if $\lambda-2+1>-1$, i.e. $\lambda>0$.

Hence, $\left|\iint_D\frac{r^\lambda dxdy}{r^2}\right|=\frac{1}{|\lambda|}\left|\iint_D\frac{g'(r)}{r}\,dxdy\right|$

$$\le\frac{1}{|\lambda|}\int_C|g(r)|\left|\frac{-(y-\eta)dx+(x-\xi)dy}{r^2}\right|$$

$=\frac{1}{|\lambda|}\int_C r^\lambda|d\theta|$, where θ is the angle between the x-axis and the vector joining (ξ,η) to (x,y).

$$\le\begin{cases}\frac{2k\pi}{\lambda}\Delta^\lambda, & \text{if } \lambda>0\\ \frac{2k\pi}{-\lambda}\delta^\lambda, & \text{if } \lambda<0,\ (\xi,\eta)\notin D.\end{cases}\qquad\square$$

(pf. of Key Lemma)

$$F_{\bar\zeta}(\zeta,\bar\zeta)=\frac{F_x+iF_y}{2}=\lim_{h\to0}\frac{1}{h}\left(F(x+\tfrac{h}{2},y)-F(x,y)+iF(x,y+\tfrac{h}{2})-iF(x,y)\right)$$

$$=\lim_{h\to0}\frac{1}{h}\left(F(\zeta+\tfrac{h}{2},\bar\zeta+\tfrac{h}{2})-F(\zeta,\bar\zeta)+iF(\zeta+\tfrac{ih}{2},\bar\zeta-\tfrac{ih}{2})-iF(\zeta,\bar\zeta)\right).$$

$$=\frac{1}{-2\pi i}\lim_{h\to0}\frac{1}{h}\iint_D\left(\frac{f(z,\bar z)}{z-\zeta-\frac{h}{2}}-\frac{f(z,\bar z)}{z-\zeta}+i\frac{f(z,\bar z)}{z-\zeta-\frac{ih}{2}}-i\frac{f(z,\bar z)}{z-\zeta}\right)d\bar z dz$$

$$=-\frac{1}{2\pi i}\lim_{h\to0}\frac{1}{2}\iint_D\left(\frac{f(z,\bar z)}{(z-\zeta)(z-\zeta-\frac{h}{2})}-\frac{f(z,\bar z)}{(z-\zeta)(z-\zeta-\frac{ih}{2})}\right)d\bar z dz.$$

5.

$$= -\frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{2}\iint_D (f(z,\bar z)-f(\zeta,\bar\zeta))\left(\frac{1}{(z-\zeta)(z-\zeta-\frac{h}{2})}-\frac{1}{(z-\zeta)(z-\zeta-\frac{ih}{2})}\right)d\bar z dz$$

$$-\frac{1}{2\pi i} f(\zeta,\bar\zeta)\frac{\partial}{\partial\bar\zeta}\iint_D \frac{d\bar z dz}{z-\zeta}$$

By the assumption that $f\in C^{0,\lambda}(D)$, the limit and the integral can exchange, and thus the first term is 0.

On the other hand,

$$\iint_D \frac{d\bar z dz}{z-\zeta} = 2i\iint_D \frac{dxdy}{z-\zeta} = 2i\int_0^R\int_0^{2\pi}\frac{r d\theta dr}{z-\zeta} \underset{\uparrow}{=} 2i\int_0^R\int_{\partial B_0(r)}\frac{-i dz}{z(z-\zeta)}dr$$

$z=re^{i\theta}$, $dz=iz d\theta$

$$= 2\int_0^R\int_{\partial B_0(r)}\left(\frac{1}{z-\zeta}-\frac{1}{z}\right)dz\cdot\frac{r}{\zeta}dr$$

$$= 2\int_0^{|\zeta|} 2\pi i\cdot(-1)\cdot\frac{r}{\zeta}dr$$

$$= -2\pi i\frac{|\zeta|^2}{\zeta}$$

$$= -2\pi i\bar\zeta$$

From the formula, we conclude that $F_{\bar\zeta}(\zeta,\bar\zeta) = f(\zeta,\bar\zeta)$.

By the same argument, we obtain $F_\zeta = \frac{1}{-2\pi i}\iint_D \frac{f(z,\bar z)-f(\zeta,\bar\zeta)}{(z-\zeta)^2}d\bar z dz$.

By <u>Lemma 1</u>., $|F| = \left|\frac{2i}{-2\pi i}\iint_D\frac{f(z,\bar z)}{z-\zeta}dxdy\right| \le \frac{2A}{2\pi}\iint_D\frac{|z-\zeta|}{|z-\zeta|^2}dxdy$

$\le \frac{2A}{2\pi}\cdot 2\pi\cdot 2R = 4RA$, since $|z-\zeta|\le|z|+|\zeta|\le 2R$.

Similarly, $|F_\zeta| \le \frac{2B}{2\pi}\iint_D\frac{|z-\zeta|^\lambda}{|z-\zeta|^2} \le \frac{2B}{2\pi}\cdot\frac{2\pi}{\lambda}(2R)^\lambda = \frac{2^{\lambda+1}}{\lambda}R^\lambda B$.

$$|F(\zeta_1,\bar\zeta_1)-F(\zeta_2,\bar\zeta_2)| \le |F_\zeta||\zeta_1-\zeta_2| + |F_{\bar\zeta}||\bar\zeta_1-\bar\zeta_2| \le \left(\frac{2^{\lambda+1}}{\lambda}R^\lambda B + A\right)r_{12}$$

For the last inequality, let $D_0 = D\cap B_{\zeta_2}(2r_{12})$ $\left(\zeta_1,\zeta_2\in B_0(\frac{R}{5}) \Rightarrow B_{\zeta_2}(2r_{12})\subseteq D\right)$

$$-2\pi i\left(F_\zeta(\zeta_1,\bar\zeta_1)-F_\zeta(\zeta_2,\bar\zeta_2)\right) =$$

$$= \underbrace{\iint_{D_0}\frac{f(z,\bar z)-f(\zeta_1,\bar\zeta_1)}{(z-\zeta_1)^2}d\bar z dz}_{(\mathrm{I})} - \underbrace{\iint_{D_0}\frac{f(z,\bar z)-f(\zeta_2,\bar\zeta_2)}{(z-\zeta_2)^2}d\bar z dz}_{(\mathrm{II})}$$

$$+\underbrace{\iint_{D-D_0}(f(z,\bar z)-f(\zeta_1,\bar\zeta_1))\left(\frac{1}{(z-\zeta_1)^2}-\frac{1}{(z-\zeta_2)^2}\right)d\bar z dz}_{(\mathrm{III})}$$

$$+\underbrace{(f(\zeta_1,\bar\zeta_1)-f(\zeta_2,\bar\zeta_2))\frac{\partial}{\partial\zeta_2}\iint_{D-D_0}\frac{d\bar z dz}{z-\zeta_2}}_{(\mathrm{IV})}$$

By Lemma 1, $|(I)| \leq 2B\iint_{D_0} \frac{|z-\zeta_1|^\lambda}{|z-\zeta_1|^2} \leq \frac{4\pi B}{\lambda}(3r_{12})^\lambda$. since

$|z-\zeta_1| \leq |z-\zeta_2| + |\zeta_1 - \zeta_2| \leq 2r_{12} + r_{12} = 3r_{12}$;

$|(II)| \leq 2B\iint_{D_0} \frac{|z-\zeta_2|^\lambda}{|z-\zeta_2|^2} \leq \frac{4\pi B}{\lambda}(2r_{12})^\lambda$.

$$|(III)| = \left|\iint_{D-D_0} (f(z,\bar z) - f(\zeta_1,\bar\zeta_1)) \int_{\zeta_2}^{\zeta_1} \frac{2\,d\zeta}{(z-\zeta)^3}\, 2i\,dx\,dy\right|$$

$$\leq 4\int_{\zeta_2}^{\zeta_1}\left(\left|\iint_{D-D_0} \frac{f(z,\bar z) - f(\zeta,\bar\zeta)}{(z-\zeta)^3}dx\,dy\right| + \left|\iint_{D-D_0} \frac{f(\zeta_1,\bar\zeta_1) - f(\zeta,\bar\zeta)}{(z-\zeta)^3}dx\,dy\right|\right)|d\zeta|$$

$$\leq 4\int_{\zeta_2}^{\zeta_1}\left(B\iint_{D-D_0} \frac{|z-\zeta|^{\lambda-1}}{|z-\zeta|^2}dx\,dy + B\iint_{D-D_0} \frac{|\zeta_1-\zeta|^\lambda}{|z-\zeta|^3}dx\,dy\right)|d\zeta|$$

$$\leq 4\int_{\zeta_2}^{\zeta_1}\left(\frac{B\cdot 2\pi\cdot 2}{1-\lambda} r_{12}^{\lambda-1} + B r_{12}^\lambda \cdot \frac{2\pi\cdot 2}{1}\cdot r_{12}^{-1}\right)|d\zeta|$$

(since $|z-\zeta| \geq |z-\zeta_2| - |\zeta-\zeta_2| \geq 2r_{12} - r_{12} = r_{12}$)

$$\leq 16\pi B r_{12}^\lambda \left(\frac{1}{1-\lambda}+1\right)$$

$$\iint_{D-D_0} \frac{d\bar z\,dz}{z-\zeta_2} = \iint_D \frac{d\bar z\,dz}{z-\zeta_2} - \iint_{D_0}\frac{d\bar z\,dz}{z-\zeta_2} = -2\pi i\,\bar\zeta_2 + 0 \Rightarrow (IV) = 0$$

We may set $\mu(\lambda) = \frac{2}{\lambda}3^\lambda + \frac{2}{\lambda}2^\lambda + 8\left(\frac{1}{1-\lambda}+1\right)$. □

Now me make more estimates to prepare for the proof of Thm 1.

Let $D, a, b, \alpha, \sigma, Z$ be as in Thm 1, ie. $D = \{z \in \mathbb{C} \mid |z| \leq R\}$;

$a, b, \alpha \in C^{0,\lambda}(D)$, $0 < \lambda < 1$, $a(0,0) = b(0,0) = 0$; $Z = a\frac{\partial}{\partial \bar z} + b\frac{\partial}{\partial z}$;

σ complex analytic with $\sigma(0) = 0$.

Since " $f \in C^{0,\lambda}(D)$, $0<\lambda<1$, $g \in C^1(D)$" implies

$$|fg(\zeta_1,\bar\zeta_1) - fg(\zeta_2,\bar\zeta_2)| \leq |(f(\zeta_1,\bar\zeta_1) - f(\zeta_2,\bar\zeta_2))g(\zeta_1,\bar\zeta_1)|$$
$$+ |f(\zeta_2,\bar\zeta_2)(g(\zeta_1,\bar\zeta_1) - g(\zeta_2,\bar\zeta_2))|$$
$$\leq C r_{12}^\lambda + C' r_{12} \leq (C+C')r_{12}^\lambda \text{ for } r_{12} \text{ small,}$$

ie. $fg \in C^{0,\lambda}(D)$ in a possibly smaller D,

we may assume that $(Z+\alpha)\sigma = b\sigma' + \alpha\sigma \in C^{0,\lambda}(D)$.

Hence, we may find a constant M s.t.

$1<M,\ |\alpha|\le M,\ |\sigma|\le M,$

$|h(\zeta_1,\bar\zeta_1)-h(\zeta_2,\bar\zeta_2)|\le M r_{12}^{\lambda}$, where $h=a,b,\alpha,\sigma$

$|(Z+\alpha)\sigma(\zeta_1)-(Z+\alpha)\sigma(\zeta_2)|\le \frac{M^2}{2^\lambda} r_{12}^{\lambda}$.

Since $a, b, (Z+\alpha)\sigma$ vanishes at 0,

$|a(\zeta,\bar\zeta)|\le M|\zeta|^\lambda\le MR^\lambda,\ |b|\le MR^\lambda,\ |(Z+\alpha)\sigma|\le\frac{M^2}{2^\lambda}R^\lambda\le M^2R^\lambda$

Consequently, for the f and F as in Key Lemma 1,

$$|(Z+\alpha)F| = |aF_{\bar\zeta}+bF_\zeta+\alpha F|\le MR^\lambda\cdot A+MR^\lambda\cdot\frac{2^{\lambda+1}}{\lambda}R^\lambda B+M\cdot 4RA$$
$$= MR^\lambda\left((1+4R^{1-\lambda})A+\frac{2^{\lambda+1}}{\lambda}R^\lambda B\right)$$

$$|(Z+\alpha)F(\zeta_1,\bar\zeta_1)-(Z+\alpha)F(\zeta_2,\bar\zeta_2)|$$
$$=|af(\zeta_1,\bar\zeta_1)-af(\zeta_2,\bar\zeta_2)+bF_\zeta(\zeta_1,\bar\zeta_1)-bF_\zeta(\zeta_2,\bar\zeta_2)+\alpha F(\zeta_1,\bar\zeta_1)-\alpha F(\zeta_2,\bar\zeta_2)|$$
$$\le|(a(\zeta_1,\bar\zeta_1)-a(\zeta_2,\bar\zeta_2))f(\zeta_1,\bar\zeta_1)|+|a(\zeta_2,\bar\zeta_2)(f(\zeta_1,\bar\zeta_1)-f(\zeta_2,\bar\zeta_2))|$$
$$+|(b(\zeta_1,\bar\zeta_1)-b(\zeta_2,\bar\zeta_2))F_\zeta(\zeta_1,\bar\zeta_1)|+|b(\zeta_2,\bar\zeta_2)(F_\zeta(\zeta_1,\bar\zeta_1)-F_\zeta(\zeta_2,\bar\zeta_2))|$$
$$+|(\alpha(\zeta_1,\bar\zeta_1)-\alpha(\zeta_2,\bar\zeta_2))F(\zeta_1,\bar\zeta_1)|+|\alpha(\zeta_2,\bar\zeta_2)(F(\zeta_1,\bar\zeta_1)-F(\zeta_2,\bar\zeta_2))|.$$
$$\le Mr_{12}^\lambda\cdot A+MR^\lambda\cdot Br_{12}^\lambda+Mr_{12}^\lambda\cdot\frac{2^{\lambda+1}}{\lambda}R^\lambda B+MR^\lambda\cdot\mu(\lambda)Br_{12}^\lambda$$
$$+Mr_{12}^\lambda\cdot 4RA+M\cdot 2\left(A+\frac{2^{\lambda+1}}{\lambda}R^\lambda B\right)r_{12}$$
$$=Mr_{12}^\lambda\left(A+A(4R+2r_{12}^{1-\lambda})+BR^\lambda\left(1+\frac{2^{\lambda+1}}{\lambda}+\mu(\lambda)+\frac{2^{\lambda+2}}{\lambda}r_{12}^{1-\lambda}\right)\right)$$
$$\le Mr_{12}^\lambda\left(A+A(4R+2^{2-\lambda}R^{1-\lambda})+B\left(\left(1+\frac{2^{\lambda+1}}{\lambda}+\mu(\lambda)\right)R^\lambda+\frac{8}{\lambda}R\right)\right)$$
$$=:Mr_{12}^\lambda\left(A+Ag_1(R)+Bg_2(R)\right).$$

(pf. of Thm 1)

Assume that $4R\le1,\ 2^{2-\lambda}\frac{\lambda+2}{\lambda}R^{1-\lambda}\le1$,

and choose a constant $c(\lambda)$ s.t.

$$\frac{\lambda+2}{\lambda}+\frac{\lambda}{\lambda+2}2^\lambda\le c$$
$$1+\mu(\lambda)+\frac{3\lambda+2}{\lambda}2^\lambda\le c.\quad(\Rightarrow c>1)$$

We construct the solution by successive approximation.

Define $2\pi i\, w_0(\zeta,\bar\zeta) = \sigma(\zeta)$,

$$2\pi i\, w_{n+1}(\zeta,\bar\zeta) = -\iint_D \frac{(Zw+\alpha w)(z,\bar z)}{z-\zeta}\, d\bar z\, dz, \quad n \geq 0,$$

and $w(\zeta,\bar\zeta) = \sum_{n=0}^{\infty} w_n(\zeta,\bar\zeta)$.

Formally, $w(\zeta,\bar\zeta)$ is a solution to

$$2\pi i\, w(\zeta,\bar\zeta) + \iint_D \frac{(Zw+\alpha w)(z,\bar z)}{z-\zeta}\, d\bar z\, dz$$

We justify the definition of w_n and w by the following inequalities:

$|w_n| \leq M(cMR^\lambda)^n$,

$|(Z+\alpha)w_n| \leq M(cMR^\lambda)^{n+1}$,

$|w_n(\zeta_1,\bar\zeta_1) - w_n(\zeta_2,\bar\zeta_2)| \leq M(cMR^\lambda)^n r_{12}^\lambda$,

$|(Z+\alpha)w_n(\zeta_1,\bar\zeta_1) - (Z+\alpha)w_n(\zeta_2,\bar\zeta_2)| \leq \frac{cM^2}{2^\lambda}(cMR^\lambda)^n r_{12}^\lambda$.

By the last inequality, we know $(Z+\alpha)w_n \in C^{0,\lambda}(D)$, justifying the integral defining w_{n+1}.

And the first inequality implies that the infinite series defining w converges absolutely and uniformly provided that R is sufficiently small s.t. $cMR^\lambda < 1$, and thus w is actually a solution s.t. $(Z+\alpha)w \in C^{0,\lambda}(D)$.

We prove these inequalities by induction on n.

For $n=0$, the inequalities have already been made true on P.7, since $c>1$

By Key Lemma 1. and the induction hypothesis,

$|w_{n+1}| \leq 4R \cdot M(cMR^\lambda)^{n+1} \leq M(cMR^\lambda)^{n+1}$ $(\because 4R \leq 1)$

$$\begin{aligned}|w_{n+1}(\zeta_1,\bar\zeta_1) - w_{n+1}(\zeta_2,\bar\zeta_2)| &\leq 2\left(M(cMR^\lambda)^{n+1} + \frac{2^{\lambda+1}}{\lambda} R^\lambda \cdot \frac{cM^2}{2^\lambda}(cMR^\lambda)^n\right) r_{12} \\ &\leq M(cMR^\lambda)^{n+1} r_{12}^\lambda \left(2\left(1+\frac{2}{\lambda}\right) r_{12}^{1-\lambda}\right) \\ &\leq M(cMR^\lambda)^{n+1} r_{12}^\lambda \cdot 2\,\frac{\lambda+2}{\lambda}(2R)^{1-\lambda} \\ &\leq M(cMR^\lambda)^{n+1} r_{12}^\lambda\end{aligned}$$

From the estimations on P.7,

$$|(Z+\alpha)W_{n+1}| \le MR^{\lambda}\left((1+4R^{1-\lambda})\cdot M(cMR^{\lambda})^{n+1} + \frac{2^{\lambda+1}}{\lambda}R^{\lambda}\cdot\frac{cM^2}{2^{\lambda}}(cMR^{\lambda})^{n}\right)$$
$$= M(cMR^{\lambda})^{n+2}\cdot c^{-1}\left(1+4R^{1-\lambda}+\frac{2}{\lambda}\right)$$
$$\le M(cMR^{\lambda})^{n+2}c^{-1}\left(\frac{\lambda+2}{\lambda}+4\cdot\frac{\lambda}{\lambda+2}2^{\lambda-2}\right)$$
$$\le M(cMR^{\lambda})^{n+2}$$

$$|(Z+\alpha)W_{n+1}(\zeta_1,\bar\zeta_1)-(Z+\alpha)W_{n+1}(\zeta_2,\bar\zeta_2)|$$
$$\le M r_{12}^{\lambda}\left(M(cMR^{\lambda})^{n+1}(1+g_1(R)) + \frac{cM^2}{2^{\lambda}}(cMR^{\lambda})^{n}g_2(R)\right)$$
$$= \frac{cM^2}{2^{\lambda}}(cMR^{\lambda})^{n+1}r_{12}^{\lambda}\cdot c^{-1}\left(2^{\lambda}(1+g_1(R)) + R^{-\lambda}g_2(R)\right)$$

$$2^{\lambda}(1+g_1(R))+R^{-\lambda}g_2(R) = 2^{\lambda}(1+4R+2^{2-\lambda}R^{1-\lambda}) + R^{-\lambda}\left(\left(1+\frac{2^{\lambda+1}}{\lambda}+\mu(\lambda)\right)R^{\lambda}+\frac{8}{\lambda}R\right)$$
$$= 1+\mu(\lambda)+2^{\lambda}\left(1+4R+\frac{2}{\lambda}\right)+R^{1-\lambda}\left(4+\frac{8}{\lambda}\right)$$
$$\le 1+\mu(\lambda)+2^{\lambda}\left(2+\frac{2}{\lambda}\right)+\frac{\lambda}{\lambda+2}2^{\lambda-2}\cdot\frac{4(\lambda+2)}{\lambda}$$
$$= 1+\mu(\lambda)+\frac{3\lambda+2}{\lambda}2^{\lambda}$$
$$\le c.$$

For uniqueness of the solution, let $W'(z,\bar z)$ be another solution s.t. $(Z+\alpha)W' \in C^{0,\lambda}(D)$.

Then $\bar W = W - W'$ satisfies

$$-2\pi i\,\bar W(\zeta,\bar\zeta) = \iint_D \frac{(Z+\alpha)\bar W(z,\bar z)}{z-\zeta}\,d\bar z\,dz \qquad (*)$$

Let $A_R = \sup_{\zeta\in D}|(Z+\alpha)\bar W|$, $B_R = \sup_{\substack{\zeta_1,\zeta_2\in D\\ \zeta_1\ne\zeta_2}} \dfrac{|(Z+\alpha)\bar W(\zeta_1,\bar\zeta_1)-(Z+\alpha)\bar W(\zeta_2,\bar\zeta_2)|}{r_{12}^{\lambda}}$

Again from the estimation on P.7,

$$A_R \le MR^{\lambda}\left((1+4R)^{1-\lambda}A_R + \frac{2^{\lambda+1}}{\lambda}R^{\lambda}B_R\right)$$
$$\le MR^{\lambda}\left((1+4R)^{1-\lambda}A_{R'} + \frac{2^{\lambda+1}}{\lambda}R^{\lambda}B_{R'}\right) \quad \text{for } R<R'$$
$$\to 0 \text{ as } R\to 0$$

From the equation $(*)$, $\bar W \equiv 0$. $\square$

II. Insufficiency of continuous E, F, G.

We first observe that u, v are isothermal coordinates of (1) if and only if u, v satisfy the Cauchy-Riemann-Beltrami equations

(4) $D v_x = F u_x - E u_y$, $D v_y = G u_x - F u_y$, where $D = \sqrt{EG - F^2}$

In fact,

$$E dx^2 + 2F dx dy + G dy^2 = E\left(\frac{dz + d\bar z}{2}\right)^2 + 2F\left(\frac{dz + d\bar z}{2}\right)\left(\frac{dz - d\bar z}{2i}\right) + G\left(\frac{dz - d\bar z}{2i}\right)^2$$

$$= \frac{1}{4}\left((E - G - 2iF)dz^2 + 2(E+G)dz d\bar z + (E - G + 2iF)d\bar z^2\right)$$

$$= \alpha\left(dz + \beta d\bar z\right)\left(\bar\beta dz + d\bar z\right) = \alpha\left|dz + \beta d\bar z\right|^2$$

$$\begin{cases} \dfrac{E - G + 2iF}{4} = \alpha\beta \\ \dfrac{E+G}{2} = \alpha(1 + |\beta|^2) = \alpha\left(1 + \dfrac{(E-G)^2 + (2F)^2}{16\alpha^2}\right) \end{cases}$$

$$\Rightarrow \alpha = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right), \quad \beta = \frac{E - G + 2iF}{4\alpha}$$

$$\lambda(du^2 + dv^2) = \lambda|dw|^2 = \lambda|w_z dz + w_{\bar z} d\bar z|^2 = \lambda|w_z|^2\left|dz + \frac{w_{\bar z}}{w_z} d\bar z\right|^2$$

So we must have $w_{\bar z} = \beta w_z$.

After some lengthy computation, $w_{\bar z} = \beta w_z$ can be converted into (4).

Next, we state a lemma on inhomogeneous Cauchy-Riemann equations and use it to prove the existence of a continuous metric not admitting isothermal coordinates.

Lemma 2. Let α, β be continuous functions on $D = \overline{B_0(R)}$.

Then the system

$$u_x - v_y = \alpha(x,y), \quad u_y + v_x = \beta(x,y)$$

having C^1 solutions u, v on some $B_0(R')$ implies the existence of

$$\lim_{\varepsilon \to 0^+} \int_0^{2\pi}\int_\varepsilon^{R'} r^{-1}\left(\alpha(r\cos\theta, r\sin\theta)\cos 2\theta + \beta(r\cos\theta, r\sin\theta)\sin 2\theta\right) dr\, d\theta$$

<u>Thm 2.</u> There exists continuous metric (1) on, say $D = \overline{B_0(R)}$, $R<1$, such that in every neighborhood of $(0,0)$, there does not exist C^1 u, v. which transform (1) into (2).

(pf.) Let $\begin{cases} h(x,y) = \dfrac{x^2}{r^2 \log r^2} = \dfrac{1+\cos 2\theta}{2\log r^2}, & \text{where } r=\sqrt{x^2+y^2},\ \theta=\tan^{-1}\frac{y}{x}. \\ h(0,0) := 0. \end{cases}$

Then $h \in C(D)$

Let $E(x,y)=1$, $F(x,y)=0$, $G(x,y) = (1+h(x,y))^2 =: g(x,y)^2$

$G(x,y) > 0$ for R sufficiently small, so this defines a continuous metric on $D = \overline{B_0(R)}$ for some small R.

The Cauchy-Riemann-Beltrami equations become

(5) $\quad v_x = -g^{-1} u_y, \quad v_y = g u_x$

We claim that there is no C^1 solutions (u,v) of (5) s.t. $u_x(0,0)=1$, $u_y(0,0)=0$:

Let (u,v) be a C^1 solution to (5)

Then for $\alpha = (1-g)u_x$, $\beta = (1-g^{-1})u_y$ continuous, (u,v) is a solution to $u_x - v_y = \alpha$, $u_y + v_x = \beta$.

$u_x(0,0)=1$, $u_y(0,0)=0 \Rightarrow u_x = 1+o(1)$, $u_y = o(1)$

$\Rightarrow \alpha = (1-g)(1+o(1)) = -h + h\cdot o(1) = -\dfrac{1+\cos 2\theta}{2\log r^2} + o\left(\dfrac{1}{\log r^2}\right)$

$\beta = (1-g^{-1})o(1) = (1-(1+h)^{-1})o(1) = h\cdot o(1) = o\left(\dfrac{1}{\log r^2}\right)$.

$\Rightarrow \displaystyle\int_0^{2\pi}\int_\varepsilon^b r^{-1}(\alpha\cos 2\theta + \beta\sin 2\theta)\,dr\,d\theta = \int_\varepsilon^b (r\log r^2)^{-1}\left(-\tfrac{\pi}{2}+o(1)\right)dr$

$\rightarrow \infty$ as $\varepsilon \to 0$

By <u>Lemma 2.</u>, $u_x - v_y = \alpha$, $u_y + v_x = \beta$ cannot have C^1 solutions.

If the so-defined metric admits isothermal coordinates u, v s.t. $u(x,y), v(x,y) \in C^1(D)$, then every usual analytic function $s(u,v)+it(u,v)$ is transformed into $s(x,y)+it(x,y)$ st. $s(x,y)$ and $t(x,y)$ satisfy (5), but this contradicts the claim. $\square$

(pf. of Lemma 2.)

The system $u_x - v_y = \alpha$, $u_y + v_x = \beta$ can be written as $w_{\bar{z}} = \frac{\gamma}{2}$,

where $w = u + iv$, $\gamma = \alpha + i\beta$.

By Green's theorem,

$$\int_{\partial D - \partial B_\zeta(\varepsilon)} \frac{w(z)}{\zeta - z}\, dz = \iint_{D - B_\zeta(\varepsilon)} \left(-\left(\frac{w}{\zeta - z}\right)_y + i\left(\frac{w}{\zeta - z}\right)_x \right) dx\,dy$$

$$= i \iint_{D - B_\zeta(\varepsilon)} \frac{\gamma(z)}{\zeta - z}\, dx\,dy$$

Let $\varepsilon \to 0+$. $w(\zeta, \eta) + \frac{1}{2\pi i} \int_{\partial D} \frac{w(z)}{\zeta - z}\, dz = \frac{1}{2\pi} \iint_D \frac{\gamma(z)}{\zeta - z}\, dx\,dy$

Hence, $f(\zeta, \eta) = \iint_D \frac{\gamma(z)}{\zeta - z}\, dx\,dy \in C'(D)$

$$\Rightarrow \infty > \iint_{B_z(R')} \frac{\gamma(z)}{(\zeta - z)^2}\, dx\,dy = \int_0^{2\pi} \int_0^{R'} \frac{\alpha + i\beta}{(r\cos\theta + ir\sin\theta)^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{R'} r^{-1} (\alpha + i\beta)(\cos\theta - i\sin\theta)^2\, dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{R'} r^{-1} (\alpha(z)\cos 2\theta + \beta(z)\sin 2\theta)\, dr\,d\theta$$

$$+ i \int_0^{2\pi} \int_0^{R'} r^{-1} (-\alpha(z)\sin 2\theta + \beta(z)\cos 2\theta)\, dr\,d\theta$$

In particular,

$$\int_0^{2\pi} \int_0^{R'} r^{-1} \left(\alpha(r\cos\theta, r\sin\theta)\cos 2\theta + \beta(r\cos\theta, r\sin\theta)\sin 2\theta\right) dr\,d\theta < \infty \quad \square$$

The Riemann-Roch-Hirzebruch Theorem and Kodaira Vanishing Theorem

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ABSTRACT. This is the final report for MATH7302 (Differential Geometry (II)) in NTU. In this report, we develop basic properties of a holomorphic vector bundle E on a complex manifold M. Then we proceed to associate a Clifford structure on the bundle $\Lambda(T^{0,1}M)^* \otimes E$. The index of the Dirac operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ coincides with the Euler number of E. Then we derive a Lichnerowicz formula, called the Bochner-Kodaira formula, to get the Kodaira vanishing theorem. Finally, we apply the index theorem to derive the Riemann-Roch-Hirzebruch theorem. This report follows mainly the contents in sections 3.6 and 4.1 of Getzler's book, *Heat Kernels and Dirac Operators*.

1. HOLOMORPHIC VECTOR BUNDLES

Definition 1. *Let M be a complex manifold. A holomorphic vector bundle $\pi: E \to M$ is a complex vector bundle together with the structure of a complex manifold on E, such that for all $x \in M$, there exist an open neighborhood U of x and a trivialization $\phi_U: E_U \to U \times \mathbb{C}^k$ that is a biholomorphic map of complex manifolds. Such a trivialization is called a holomorphic trivialization.*

In the rest of this report, we always assume that M is a compact complex manifold and that E is a holomorphic vector bundle over M. Before proceeding, we set up some notations:

Notation 2.
$A(M,E) = \Gamma(M, \Lambda(TM)^* \otimes E)$
$A^{p,q}(M,E) = \Gamma(M, \Lambda(T^{1,0}M)^* \otimes \Lambda(T^{0,1}M)^* \otimes E)$

Thus $A(M,E) = \sum_{k=0}^{\infty} \sum_{p+q=k} A^{p,q}(M,E)$. Recall that $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ and $T^{0,1}M$ are the $\pm i$ eigenspaces of the complex structure $J \in \mathrm{End}(TM)$, respectively.

On a holomorphic vector bundle, we can define the Dolbeault operator $\bar{\partial}$: $A^{p,q}(M, E) \to A^{p,q+1}(M,E)$. Let $\{e_1, \cdots, e_k\}$ be a local holomorphic frame of E. If $\alpha \in A^{p,q}(M,E)$, then locally we can write $\alpha = \sum_i \alpha_i \otimes e_i$. We define $\bar{\partial}$ by

$$\bar{\partial}\alpha = \sum_i (\bar{\partial}\alpha_i) \otimes e_i.$$

If $\{e'_1, \cdots, e'_k\}$ is another local holomorphic frame and $e_i = g_{ij}e'_j$, then $\alpha = \sum_i g_{ij}\alpha_i \otimes e'_j$, and we compute:

$$\bar{\partial}\alpha = \sum_i \partial(g_{ij}\alpha_i) \otimes e'_j = \sum_i g_{ij}\bar{\partial}\alpha_i \otimes e'_j = \sum_i (\bar{\partial}\alpha_i) \otimes e_i,$$

since g_{ij} is holomorphic. We see that the Dolbeault operator is well-defined. It's clear from definition that $\bar{\partial}\bar{\partial} = 0$. So we obtain the following chain complex

$$0 \to A^{0,0}(M,E) \xrightarrow{\bar{\partial}} A^{0,1}(M,E) \xrightarrow{\bar{\partial}} A^{0,2}(M,E) \xrightarrow{\bar{\partial}} \cdots$$

This is called the Dolbeault chain complex. Its cohomology is called the Dolbealt cohomology. By Dolbeault's theorem (see Griffiths&Harris, p. 45), this cohomology is isomorphic to the sheaf cohomology space $H^{\cdot}(M, \mathcal{O}(E))$. In order to apply Hodge's theorem, we need to define $\bar{\partial}^*: A^{p,q}(M, E) \to A^{p,q-1}(M,E)$. Before doing this, we introduce the hermitian metric on E.

Definition 3. *A hermitian vector bundle is a holomorphic vector bundle $E \to M$ together with a hermitian metric on it. More precisely, for each $x \in M$, there associates a hermitian inner product $(\cdot,\cdot)_x$ on E_x such that this inner product varies smoothly on M.*

We adopt the convention that the hermitian inner product is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-conjugate linear in the second variable. If E is a hermitian vector bundle, then there exists a unique connection ∇ on E such that ∇ is compatible with the metric and the complex structure of M:

Proposition 4. *Let E be a hermitian vector bundle. Then there exists a unique connection ∇ such that*

1. $d(u,v)=(\nabla u,v)+(u,\nabla v)$, *and*
2. $\nabla^{0,1}u=\bar\partial u$

for all $u,v\in\Gamma(M,E)$. *Here* $\nabla^{0,1}u$ *denotes the* $(0,1)$*-part of the* E*-valued* 1*-form* ∇u.

Proof. Suppose such ∇ exists. Let $\{e_i\}$ be a local holomorphic frame of E and let $h_{ij}=(e_i,e_j)$. Denote the connection 1-form of ∇ as θ. Since $\nabla^{0,1}=\bar\partial$, we see that θ only has the $(1,0)$ part. Now we compute

$$\begin{aligned} dh_{ij} &= (\nabla e_i,e_j)+(e_i,\nabla e_j)\\ &= (\theta_{ik}e_k,e_j)+(e_i,\theta_{jk}e_k)\\ &= h_{kj}\theta_{ik}+h_{ik}\bar\theta_{jk}. \end{aligned}$$

By comparing types, we have

$$\begin{aligned} \partial h_{ij} &= h_{kj}\theta_{ik} \quad \textit{i.e.}\ \partial h=\theta h,\\ \bar\partial h_{ij} &= h_{ik}\bar\theta_{jk} \quad \textit{i.e.}\ \bar\partial h=h\bar\theta^{\,t}, \end{aligned}$$

and we see that $\theta=\partial h\cdot h^{-1}$ is the unique solution. Since θ is determined by the conditions of compatibility, θ is well-defined globally. □

From now on, we let M be a compact Kahler manifold. The Levi-Civita connection $\nabla^{L.C.}$ on $\Lambda(TM)^*$ restricting to $\Lambda(T^{0,1}M)^*$ is also a connection on $\Lambda(T^{0,1}M)^*$, since $\nabla^{L.C.}$ preserves the types. Denote the connection on E obtained from the previous proposition as ∇^E. We aquire a connection $\nabla=\nabla^{L.C.}\otimes 1+1\otimes\nabla^E$ on the hermitian vector bundle $\Lambda(T^{0,1}M)^*\otimes E$. The following proposition shows that $\bar\partial$ on $A^{p,q}(M,E)$ can be written in terms of the composition of covariant derivatives and exterior products.

Lemma 5. *Let* $\{e_i\}$ *be a holomorphic frame of* $T^{1,0}M$ *and let* $\{e^i\}$ *be its dual frame on* $(T^{1,0}M)^*$. *Then*

$$\bar\partial=\epsilon(\bar e^{\,i})\nabla_{\bar e_i}.$$

Proof. We already know that on $A^{p,q}(M)$, $d=\epsilon(\bar e^{\,i})\nabla^{L.C}_{\bar e_i}+\epsilon(e^i)\nabla^{L.C}_{e_i}$. Thus on $A^{p,q}(M)$, $\bar\partial=\epsilon(\bar e^{\,i})\nabla_{\bar e_i}$. Let $\{\sigma_j\}$ be a holomorphic frame on E. If $\alpha\in A^{p,q}(M,E)$, then locally $\alpha=\alpha_j\otimes\sigma_j$. Therfore

$$\begin{aligned} \epsilon(\bar e^{\,i})\nabla_{\bar e_i}\alpha &= (\epsilon(\bar e^{\,i})\nabla^{L.C}_{\bar e_i}\alpha_j)\otimes\sigma_j+\epsilon(\bar e^{\,i})\alpha_j\otimes\nabla^E_{\bar e_i}\sigma_j\\ &= (\bar\partial\alpha_j)\otimes\sigma_j\\ &= \bar\partial\alpha \end{aligned}$$

since $\nabla^E_{\bar e_i}\sigma_j=\bar\partial\sigma_j(\bar e_i)$ and $\bar\partial\sigma_j=0$, for $\{\sigma_j\}$ is a holomorphic frame. □

The adjoint of $\bar\partial$ with respect to the L^2 Hermitian inner product on $A^{0,\cdot}(M,E)$ is denoted as $\bar\partial^*$. We have a similar expression for $\bar\partial^*$:

Lemma 6. *Let* $\{e_i\}$ *be a unitary frame of* $T^{1,0}M$ *and let* $\{e^i\}$ *be its dual frame on* $(T^{1,0}M)^*$. *Then*

$$\bar\partial^*=-\iota(e^i)\nabla^E_{e_i}.$$

Proof. Let $\alpha\in A^{p,q-1}(M,E)$ and let $\beta\in A^{p,q}(M,E)$. Let $x\in M$. Then

$$\begin{aligned} (\bar\partial\alpha,\beta)_x &= (\epsilon(\bar e^{\,i})\nabla_{\bar e_i}\alpha,\beta)_x\\ &= (\nabla_{\bar e_i}\alpha,\iota(e^i)\beta)_x\\ &= \bar e_i(\alpha,\iota(e^i)\beta)_x-(\alpha,\nabla_{e_i}\iota(e^i)\beta)_x\\ &= \bar e_i(\alpha,\iota(e^i)\beta)_x+(\alpha,-\iota(e^i)\nabla_{e_i}\beta)_x. \end{aligned}$$

Here we use the fact that $\nabla_{e_i}\iota(e^i)=\nabla_{e_i}\iota(\bar{e}_i)=\iota(\nabla^{L.C.}_{e_i}\bar{e}_i)+\iota(\bar{e}_i)\nabla_{e_i}$ and the fact that M is Kahler. It remains to show that the integral of $\bar{e}_i(\alpha,\ \iota(e^i)\beta)_x$ vanishes. Denote $f_{\bar{i}}(x)=(\alpha,\ \iota(e^i)\beta)_x$ We may assume that $e_i=\frac{\partial}{\partial z^i}$ for computational convenience. The volume form at x is $dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge d\bar{z}^n$. Then

$$\begin{aligned}\bar{e}_i f_{\bar{i}} dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge d\bar{z}^n &= \bar{\partial}\Big((-1)^{n+i-1}f_{\bar{i}}dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge\widehat{d\bar{z}^i}\wedge\ldots\wedge d\bar{z}^n\Big)\\ &= d\Big((-1)^{n+i-1}f_{\bar{i}}dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge\widehat{d\bar{z}^i}\wedge\ldots\wedge d\bar{z}^n\Big)\\ &= d\omega.\end{aligned}$$

We must show that ω is globally defined. If $\frac{\partial}{\partial w^i}$ is another unitary basis, then

$$\begin{aligned}(\alpha,\iota(dz^i)\beta)_x &= \left(\alpha,\iota\Big(\frac{\partial z^i}{\partial w^j}dw^j\Big)\beta\right)_x\\ &= \frac{\partial\bar{z}^i}{\partial\bar{w}^j}(\alpha,\iota(dw^j)\beta)_x.\end{aligned}$$

On the other hand

$$\begin{aligned}dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge\widehat{d\bar{z}^i}\wedge\ldots\wedge d\bar{z}^n &= \frac{\partial z^1}{\partial w^{j_1}}dw^{j_1}\wedge\ldots\wedge\frac{\partial z^n}{\partial w^{j_n}}dw^{j_n}\wedge\\ &\quad \frac{\partial\bar{z}^1}{\partial\bar{w}^{k_1}}d\bar{w}^{k_1}\wedge\ldots\wedge\widehat{\frac{\partial\bar{z}^i}{\partial\bar{w}^{k_i}}d\bar{w}^{k_i}}\wedge\ldots\wedge\frac{\partial\bar{z}^n}{\partial\bar{w}^{k_n}}d\bar{w}^{k_n}.\end{aligned}$$

Hence

$$\begin{aligned}&(-1)^{n+i-1}(\alpha,\iota(dz^i)\beta)_x dz^1\wedge\ldots\wedge dz^n\wedge d\bar{z}^1\wedge\ldots\wedge\widehat{d\bar{z}^i}\wedge\ldots\wedge d\bar{z}^n =\\ &(-1)^{n+i-1}\frac{\partial z^1}{\partial w^{j_1}}\cdots\frac{\partial z^n}{\partial w^{j_n}}\frac{\partial\bar{z}^1}{\partial\bar{w}^{k_1}}\cdots\frac{\partial\bar{z}^n}{\partial\bar{w}^{k_n}}(\alpha,\iota(dw^{k_i})\beta)_x dw^{j_1}\wedge\ldots\wedge dw^{j_n}\wedge d\bar{w}^{k_1}\ldots\wedge\widehat{d\bar{w}^{k_i}}\wedge\ldots\wedge d\bar{w}^{k_n} =\\ &(-1)^{n+i-1}(\alpha,\iota(dw^i)\beta)_x dw^1\wedge\ldots\wedge dw^n\wedge d\bar{w}^1\ldots\wedge\widehat{d\bar{w}^i}\wedge\ldots\wedge d\bar{w}^n,\end{aligned}$$

since the change of basis is unitary.

□

By Hodge's theorem, for each cohomology class in $H^{\cdot}(M,\mathcal{O}(E))$, there exist a unique representative $\alpha\in[\alpha]$ such that $\bar{\partial}\alpha=\bar{\partial}^*\alpha=0$.

Definition 7. *The Euler number of a holomorphic vector bundle E is defined as*

$$\mathrm{Eul}(E)=\sum_{i=0}^{\dim_{\mathbb{C}}M}(-1)^i\dim(H^i(M,\mathcal{O}(E))).$$

2. The Clifford Structure on $\Lambda(T^{0,1}M)^*\otimes E$

Recall that $c(M)=\otimes(TM)^*/\{v\otimes w+w\otimes v+2\langle v,w\rangle\,|\,v,w\in(TM)^*\}$ is the Clifford algebra on M. In order to apply the powerful index theorem of Dirac operators, first we need to define a Clifford action $c\colon c(M)\to\mathrm{End}(\Lambda(T^{0,1}M)^*\otimes E)$. Let $f\in(TM)^*$. We decompose f as $f=f^{1,0}+f^{0,1}$. We define the Clifford action of f on $\Lambda(T^{0,1}M)^*\otimes E$ by

$$c(f)=\sqrt{2}(\epsilon(f^{0,1})-\iota(f^{1,0})).$$

To see that this really defines a Clifford action, we compute in local coordinates. Recall that $dx^i=\frac{1}{2}(dz^i+d\bar{z}^i)$ and $dy^i=\frac{1}{2i}(dz^i-d\bar{z}^i)$. Then

$$\begin{aligned}c(dx^i)c(dx^i) &= \left(\sqrt{2}\Big(\epsilon\Big(\frac{1}{2}d\bar{z}^i\Big)-\iota\Big(\frac{1}{2}dz^i\Big)\Big)\right)^2\\ &= -\frac{1}{2}(\epsilon(d\bar{z}^i)\iota(dz^i)+\iota(dz^i)\epsilon(d\bar{z}^i))\\ &= -\frac{1}{2}g^{i\bar{i}}\\ &= -\langle dx^i,dx^i\rangle,\end{aligned}$$

and

$$\begin{aligned} c(dy^i)c(dy^i) &= \left(\sqrt{2}\left(\epsilon\left(-\frac{1}{2i}d\bar{z}^i\right) - \iota\left(\frac{1}{2i}dz^i\right)\right)\right)^2 \\ &= -\frac{1}{2}g^{i\bar{i}} \\ &= -\langle dy^i, dy^i\rangle. \end{aligned}$$

Let ∇ be the connection on the Clifford bundle $\Lambda(T^{0,1}M)^* \otimes E$ defined in the previous section. It is easy to see that ∇ is a Clifford connection with respect to the Clifford action just defined. Now we obtain the Dirac operator on $A^{0,\cdot}(M,E)$ associated to ∇:

$$\begin{aligned} D &= c\circ\nabla \\ &= c(dx^i)\nabla_{\frac{\partial}{\partial x^i}} + c(dy^i)\nabla_{\frac{\partial}{\partial y^i}} \\ &= c(d\bar{z}^i)\nabla_{\frac{\partial}{\partial \bar{z}^i}} + c(dz^i)\nabla_{\frac{\partial}{\partial z^i}} \\ &= \sqrt{2}\left(\epsilon(d\bar{z}^i)\nabla_{\frac{\partial}{\partial \bar{z}^i}} - \iota(dz^i)\nabla_{\frac{\partial}{\partial z^i}}\right) \\ &= \sqrt{2}(\bar{\partial} + \bar{\partial}^*), \end{aligned}$$

where the last equality assumes that our frame is unitary. From the last equality, we see that the index of the Dirac operator D is

$$\mathrm{ind}(D) = \mathrm{Eul}(E).$$

Now we prove some vanishing theorems. Before we proceed, we define the generalized Laplacian $\Delta^{0,\cdot}$ on $A^{0,\cdot}(M,E)$ by the formula

$$\int_M (\Delta^{0,\cdot}s, s)dx = \int_M (\nabla^{0,1}s, \nabla^{0,1}s)dx.$$

Lemma 8. *Let e_i be a local unitary frame of $T^{1,0}M$. Then locally we have*

$$\Delta^{0,\cdot} = -\sum_i \nabla_{e_i}\nabla_{\bar{e}_i}.$$

Proof. By definition, $(\nabla^{0,1}s, \nabla^{0,1}s)_x = (\bar{e}^i \otimes \nabla_{\bar{e}_i}s, \bar{e}^j \otimes \nabla_{\bar{e}_j}s)_x = \sum_i (\nabla_{\bar{e}_i}s, \nabla_{\bar{e}_i}s)_x$, where we use the fact that e_i is a unitary frame. Then, by the property of metric connections,

$$\begin{aligned} (\nabla^{0,1}s, \nabla^{0,1}s)_x &= \sum_i (\nabla_{\bar{e}_i}s, \nabla_{\bar{e}_i}s)_x \\ &= \sum_i e_i(\nabla_{\bar{e}_i}s, s)_x - (\nabla_{e_i}\nabla_{\bar{e}_i}s, s)_x. \end{aligned}$$

Integrating both sides over M, the lemma follows. □

The canonical line bundle of a Kahler manifold is the holomorphic line bundle $K = \Lambda^n(T^{1,0}M)^*$ on M. The curvature of $K^* = \Lambda(T^{1,0}M)$ with respect the Levi-Civita connection is the $(1,1)$-form $F^{K^*} = \sum \langle R(,)\partial_i, \partial_{\bar{i}}\rangle$, where R is the Riemannian curvature of M.

The Lichnerowicz formula for the square of the operator $\bar{\partial} + \bar{\partial}^*$ is called the Bochner-Kodaira formula:

Lemma 9. *Let E be a Hermitian holomorphic vector bundle over the Kahler manifold M. In a local unitary coordinate system,*

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \Delta^{0,\cdot} + \sum_{i,j} \epsilon(d\bar{z}^i)\iota(dz^j)F^{E\otimes K^*}(\partial_{z^j}, \partial_{\bar{z}^i}).$$

Proof. The right hand side is invariant under any change of unitary basis. Thus we can choose the normal local coordinates such that the connection form vanishes at one point. We compute:

$$\begin{aligned}\bar\partial\bar\partial^* &= -\epsilon(d\bar z^i)\nabla_{\bar i}(\iota(dz^j)\nabla_j)\\ &= -\epsilon(d\bar z^i)\iota(dz^j)\nabla_{\bar i}\nabla_j\\ &= \epsilon(d\bar z^i)\iota(dz^j)(\nabla_j\nabla_{\bar i}-\nabla_{\bar i}\nabla_j)-\epsilon(d\bar z^i)\iota(dz^j)\nabla_j\nabla_{\bar i},\\ \bar\partial^*\bar\partial &= \iota(dz^j)\nabla_j(-\epsilon(d\bar z^i)\nabla_{\bar i})\\ &= -\iota(dz^j)\epsilon(d\bar z^i)\nabla_j\nabla_{\bar i}.\end{aligned}$$

Therefore

$$\begin{aligned}\bar\partial\bar\partial^*+\bar\partial^*\bar\partial &= \sum_{ij}(\epsilon(d\bar z^i)\iota(dz^j)(\nabla_j\nabla_{\bar i}-\nabla_{\bar i}\nabla_j)-(\epsilon(d\bar z^i)\iota(dz^j)+\iota(dz^j)\epsilon(d\bar z^i))\nabla_j\nabla_{\bar i})\\ &= \sum_{ij}\epsilon(d\bar z^i)\iota(dz^j)(R^+(\partial_j,\partial_{\bar i})+F^E(\partial_j,\partial_{\bar i}))+\Delta^{0,\cdot}\end{aligned}$$

where R^+ is the curvature of $\Lambda(T^{0,1}M)^*$. It remains to show that

$$\sum_{ij}\epsilon(d\bar z^i)\iota(dz^j)R^+(\partial_j,\partial_{\bar i})=\sum_{ij}\epsilon(d\bar z^i)\iota(dz^j)F^{K^*}(\partial_j,\partial_{\bar i}).$$

(The above equation seems to be invalid. But $R^+(,)$ on the line bundle $\Lambda^n(T^{0,1}M)^*$ is just a 2-form with value in $\mathbb{C}$. Therefore the equation is valid by this identification.)

To prove the above equation, we write the right hand side as

$$\begin{aligned}\sum_{ij}\epsilon(d\bar z^i)\iota(dz^j)R^+(\partial_j,\partial_{\bar i}) &= \sum_{ijkl}\epsilon(d\bar z^i)\iota(dz^j)\epsilon(d\bar z^l)\iota(dz^k)\langle R(\partial_j,\partial_{\bar i})\partial_k,\partial_{\bar l}\rangle\\ &= \sum_{ijk}\epsilon(d\bar z^i)\iota(dz^k)\langle R(\partial_j,\partial_{\bar i})\partial_k,\partial_{\bar j}\rangle.\end{aligned}$$

By the first Bianchi identity, we see that

$$R(\partial_j,\partial_{\bar i})\partial_k+R(\partial_k,\partial_j)\partial_{\bar i}+R(\partial_{\bar i},\partial_k)\partial_j=0.$$

But $R(\partial_k,\partial_j)=0$. Thus $R(\partial_j,\partial_{\bar i})\partial_k=-R(\partial_k,\partial_{\bar i})\partial_j$. It follows that

$$\begin{aligned}\sum_{ijk}\epsilon(d\bar z^i)\iota(dz^k)\langle R(\partial_j,\partial_{\bar i})\partial_k,\partial_{\bar j}\rangle &= \sum_{ijk}\epsilon(d\bar z^i)\iota(dz^k)\langle R(\partial_k,\partial_{\bar i})\partial_j,\partial_{\bar j}\rangle\\ &= \sum_{ik}\epsilon(d\bar z^i)\iota(dz^k)F^{K^*}(\partial_k,\partial_{\bar i}).\end{aligned}$$

This proves the lemma. □

Definition 10. *Let L be a Hermitian holomorphic line bundle. We say that L is positive if L has the curvature form $F=\sum F_{i\bar j}dz^i\wedge d\bar z^j$ such that the Hermitian form $v\mapsto F(v,\bar v)$ on $T^{1,0}M$ is positive.*

Warning 11. The above definition is different from the usual one. The usual definition says that L is positive if $\sqrt{-1}F$ is positive. Also note that the curvature obtained from the Chern connection is a purely imaginary (1,1)-form.

Theorem 12. *(Kodaira) (1) If L is a Hermitian holomorphic line bundle on a compact Kahler manifold M such that $L\otimes K^*$ is positive, then*

$$H^i(M,\mathcal{O}(L))=0\quad\forall i>0.$$

(2) If L is a positive Hermitian holomorphic line bundle and E is a Hermitian holomoprhic vector bundle over M, then for m sufficiently large,

$$H^i(M,\mathcal{O}(L^m\otimes E))=0\quad\forall i>0.$$

Proof. (1) Let $\alpha \in A^{0,i}(M,L)$. Denote the curvature of $L\otimes K^*$ by $F^{L\otimes K^*} = F_{ij}^{L\otimes K^*} dz^i \wedge d\bar z^j$. Let $\lambda(F^{L\otimes K^*})$ be the endomorphism $\sum_{ij} \epsilon(d\bar z^i)\iota(dz^j) F_{ij}^{L\otimes K^*}$ on $\Lambda^i(T^{0,1}M)^*$. By Bochner-Kodaira formula,

$$\begin{aligned} \int_M ((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\alpha, \alpha) &= \int_M (\nabla^{0,1}\alpha, \nabla^{0,1}\alpha) + \int_M (\lambda(F^{L\otimes K^*})\alpha, \alpha) \\ &\geqslant \int_M (\lambda(F^{L\otimes K^*})\alpha, \alpha). \end{aligned}$$

The right hand side is strictly postive if $i>0$ and $\alpha \neq 0$. Thus the conditions $\bar\partial\alpha = 0$ and $\bar\partial^*\alpha = 0$ imply that $\alpha = 0$.

(2) For the bundle $L^m \otimes E$, the Bochner-Kodaira formula gives

$$\bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \Delta^{0,\cdot} + m\lambda(F^L) + \text{other curvature terms independent of } m.$$

For m sufficiently large the terms $m\lambda(F^L)$ dominates the other curvature terms, and we obtain (2). □

3. THE RIEMANN-ROCH-HIRZEBRUCH THEOREM

In this section, we apply the local index theorem of Dirac operators to obtain the Riemann-Roch-Hirzebruch theorem. This theorem, together with the vanishing theorem in the previous section, gives us some information of the dimension of holomorphic sections on a Hermitian holomorphic vector bundle. The combination of an index theorem with a vanishing theorem is a powerful technique in differential geometry.

We quote the index theorem:

Theorem 13. *The index of a Dirac operator D on a Clifford module E over a compact oriented even-dimensional manifold is given by the cohomological formula*

$$\mathrm{ind}(D) = (2\pi i)^{-n/2} \int_M \hat A(M)\mathrm{ch}(E/S),$$

where the A-hat genus $\hat A(M) = \det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$ *and the Chern character is defined as* $\mathrm{ch}(E) = \mathrm{Str}_E(\exp(-F^E))$.

Let M be a compact Kahler manifold, and consider the Clifford bundle $E = \Lambda(T^{0,1}M)^*$. Let ∇^E be the Levi-Civita connection on E. Let $(z^1, \cdots, z^n)$ be a local coordinate system such that ∂_j is a unitary basis of $T^{1,0}M$ at one point. Then we can write the curvature of E as

$$(\nabla^E)^2 = \sum_{kl} \langle R(,)\partial_k, \partial_{\bar l}\rangle \epsilon(d\bar z^l)\iota(dz^k),$$

where R is the Riemannian curvature. Recall that $(\nabla^E)^2 = R^E + F^{E/S}$, where R^E is the action of Riemannian curvature on E. In fact, we have

$$\begin{aligned} R^E &= \frac{1}{4}\sum_{kl} \langle R(,)\partial_k, \partial_{\bar l}\rangle c(d\bar z^l)c(dz^k) + \\ &\quad \frac{1}{4}\sum_{kl} \langle R(,)\partial_l, \partial_{\bar k}\rangle c(d\bar z^l)c(dz^k). \end{aligned}$$

Therefore we have

$$(\nabla^E)^2 = R^E + \frac{1}{2}\sum_k \langle R(,)\partial_k, \partial_{\bar k}\rangle.$$

If W is a holomorphic Hermitian vector bundle and ∇ is the Clifford connection on $E = \Lambda(T^{0,1}M)^* \otimes W$ as in section 1, then we obtain

$$F^{E/S} = \frac{1}{2}\mathrm{Tr}_{T^{1,0}M}(R^+) + F^W,$$

where R^+ now denotes the curvature of the holomorphic bundle $T^{1,0}M$. Since $TM\otimes_{\mathbb{R}}\mathbb{C}=T^{1,0}M\oplus T^{0,1}M$, we have $R=\begin{pmatrix} R^+ & 0 \\ 0 & R^- \end{pmatrix}$, where R^- is the curvature of $T^{0,1}M$. In fact, $R^-=-(R^+)^T$ as matrices and we have

$$\hat{A}(M)=\det\left(\frac{R^+}{e^{R^+/2}-e^{-R^+/2}}\right).$$

(This is seen by the splitting principle. We may assume that $R=\begin{pmatrix} R^+ & 0 \\ 0 & R^- \end{pmatrix}$ is a diagonal matrix. Then we get $\hat{A}(M)=\det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)=\det^{1/2}\left(\frac{R^+/2}{\sinh(R^+/2)}\right)\det^{1/2}\left(\frac{R^-/2}{\sinh(R^-/2)}\right)$.)

Thus

$$\begin{aligned}\hat{A}(M)\mathrm{ch}(E/S) &= \hat{A}(M)\exp(-\mathrm{Tr}(R^+/2))\mathrm{ch}(W)\\ &= \mathrm{Td}(M)\mathrm{ch}(W),\end{aligned}$$

where $\mathrm{Td}(M)$ is the Todd genus of the complex manifold M:

$$\mathrm{Td}(M)=\det\left(\frac{R^+}{e^{R^+}-1}\right)=\det\left(\frac{R^+}{e^{R^+/2}-e^{-R^+/2}}\right)\exp(-\mathrm{Tr}(R^+/2)).$$

We obtain the following theorem:

Theorem 14. *(Riemann-Roch-hirzebruch) The Euler number of the holomorphic bundle W is given by the formula*

$$\mathrm{Eul}(W)=(2\pi i)^{-n/2}\int_M \mathrm{Td}(M)\mathrm{ch}(W).$$

As an exmaple, we consider the case when M is a compact Riemann surface with $R^+\in A^2(M,\mathbb{C})$ and L is a line bundle with curvature $F\in A^2(M,\mathbb{C})$. Then $\mathrm{Td}(M)=1-R^+/2$ and $\mathrm{ch}(L)=1-F$. We obtain the classical Riemann-Roch theorem

$$\dim H^0(M,\mathcal{O}(L))-\dim H^1(M,\mathcal{O}(L))=\frac{-1}{4\pi i}\int_M R^+ + 2F.$$

If we take $L=\mathbb{C}$, we see that

$$\dim H^0(M)-\dim H^{0,1}(M)=1-g=\frac{-1}{4\pi i}\int_M R^+,$$

where $g=\dim H^{0,1}(M)=\dim H^{1,0}(M)=\frac{1}{2}\dim H^1(M)$ is the genus of M. If we define the degree $\deg(L)\in\mathbb{Z}$ of the line bundle L by the formula

$$\deg(L)=\frac{-1}{2\pi i}\int_M F,$$

we may restate the classical Riemann-Roch theorem

$$\mathrm{Eul}(L)=1-g+\deg(L).$$

Report on the Atiyah-Singer index Theorm 郭文德

Recall for a elliptic operator $P: \Gamma(E) \to \Gamma(F)$, its index ind P is defined by $\operatorname{ind} P := \ker P - \operatorname{coker} P$ (which we shall call it the analytical in the in this report.) Here, we will give it a topological description in terms of K-theory. So the report will begin with a review of K-theory and then define the "tological index". Our goal will be showing that the two index coincide.

I. Review of the K-theory.

Def'n (1.1): Let X be a cpt. topological space, $V(X) := \{$ the equivalent class of complex vector bundle over $X\}$. $(V(X), \oplus, \otimes)$ is a semi-ring with $\oplus$ as addition and $\otimes$ as multiplication. We define $K(X)$ be the universal group $\ni$ $\exists$ semi-gp. homo. $\alpha: V(X) \to K(X)$ $\ni$ $\forall$ semi-gp. homo. $f: V(X) \to H$ where H is an abelian gp. $\exists!$ $\bar{f}: K(X) \to H$ gp. homo. $\ni$ $f = \bar{f} \circ \alpha$. (For any semi-gp. such universal gp. always exists, we can construct by considering the free abelian gp. of it then idetify the "sum" of the semi-gp. with the sum of the free abelian gp.)

Fact (1.1): let X, Y be cpt. $f_0, f_1: X \to Y$ be homotopic to each other. Then $f_0^* E \cong f_1^* E$ $\forall$ vector bundle over Y so $f_0^* = f_1^*: K(Y) \to K(X)$ as the homomorphis betweem K-groups.

Elements in $K(X)$ can be repsesented by $[E]-[F]$.

Def'n (1.2): To make K-theory a cohomology theory, we now consider X as a pointed space, (X, pt). Let $i: pt \to X$ be the inclusion, it gives a exact seq. $0 \to \hat{K}(X) \to K(X) \xrightarrow{i^*} K(pt) \to 0$. $\hat{K}(X) := \ker i^*$ is called the reduced K-theory.

Def'n (1.3): For $Y \underset{\text{closed}}{\subseteq} X$, regard X/Y as a pointed top. space with base pt. Y/Y. The relative K-theory $K(X,Y) := \hat{K}(X/Y)$.

Def'n (1.4): We define a "L-type" K-theory. Let $\mathcal{L}_n(X,Y)$ denote the set $V = (V_0, \cdots, V_n; \sigma_1, \dots, \sigma_n)$ where $V_0, \cdots, V_n$ are vector bundle over X + $0 \to V_0|_Y \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_n} V_n|_Y \to 0$ is exact. V, V' is isomophic if $\forall i \ \exists \ \varphi_i: V_i \to V_i'$ +

$$\begin{tikzcd} V_{i-1}|_Y \arrow[r, "\sigma_i"] \arrow[d, "\varphi_{i-1}", "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & V_i|_Y \arrow[d, "\wr"] \\ V'_{i-1}|_Y \arrow[r, "\sigma'_i"] & V'_i|_Y \end{tikzcd}$$

on X.

An element $V = (V_0, \cdots, V_n; \sigma_1, \cdots, \sigma_n)$ is said to be elementary if $\exists \ i$ + (a) $V_i = V_{i-1}$, $\sigma_i = Id$ (b) $V_j = \{0\}$ for $j \neq i$ or $i-1$.

$\oplus$ on $\mathcal{L}_n(X,Y)$ is clear. $V, V' \in \mathcal{L}_n(X,Y)$ is said to be equivalent if $\exists$ elementary $E_1, \cdots, E_k, F_1, \cdots, F_l \in \mathcal{L}_n(X,Y)$ + $V \oplus E_1 \oplus \cdots \oplus E_k \cong V' \oplus F_1 \oplus \cdots \oplus F_l$.

$L_n(X,Y) :=$ The equivalent class of $\mathcal{L}_n(X,Y)$ and its element is denoted by $[V_0, \cdots, V_n; \sigma_1, \cdots, \sigma_n]$.

Fact (1.2): For $n \geq 1$, $L_n(X,Y) \simeq L_{n+1}(X,Y)$ by $(V_0, \cdots, V_n; \sigma_1, \cdots, \sigma_n) \mapsto (V_0, \cdots, V_n, 0; \sigma_1, \cdots, \sigma_n, 0)$, so define $L(X,Y) := \varinjlim_n L_n(X,Y)$. The inclusion $L_n(X,Y) \hookrightarrow L(X,Y)$ is an isom.

pf): (skipped)

Fact (1.2): $\exists !$ equivalence of functors $\chi : L(X,Y) \to K(X,Y)$ with

$$\chi([V_0,\cdots,V_n]) = \sum_{k=0}^{n} (-1)^k [V_k] \quad \text{when } Y=\emptyset.$$

pf) We will determines the equivalence on $L_1(X,Y)$. Given $V=[V_0,V_1;\sigma] \in L_1(X,Y)$. We construct $\chi(V) \in K(X,Y)$ in the following way. Set $X_k := X \times \{k\}$, $k=0,1$.

Define an equivalent relation on $X_0 \amalg X_1$ by $(x,k) \sim (x',k')$ if $x=x' \in Y$.

$Z = X_0 \cup_Y X_1 := \frac{X_0 \amalg X_1}{\sim}$. $i: X_1 \hookrightarrow Z$, $j:(Z,\emptyset) \hookrightarrow (Z, X_1)$ gives a nat. seq. $0 \to K(Z,X_1) \xrightarrow{j^*} K(Z) \xrightarrow{i^*} K(X_1) \to 0$ which is split exact $\because$ $\rho: Z \to X_1$ the retraction induce $K(X_1) \xrightarrow{\rho^*} K(Z)$ + $\rho^* \circ i^* = id_{K(Z)}$.

Furthermore, $Z/X_1 \cong X/Y$ induces $\varphi: K(Z,X_1) \xrightarrow{\simeq} K(X,Y)$.

For $V=[V_0,V_1;\sigma]$, we define a vector bundle W over Z by setting $W|_{X_k} = V_k$ and indetifies them by σ over Y ($\because \sigma|_Y$ is an isom.) Set $W_1 := \rho^*(V_1) \in K(Z)$. We have $[W]-[W_1] \in \ker i^* = K(Z,X_1) \simeq K(X,Y)$. $\chi(V) := \varphi^{-1}([W]-[W_1])$

This defines the homo. χ. (We skip the uniqueness part.) $\square$

For our later discussion we note multiplication in $K(X,Y)$ can be realized explicitly in $L_1(X,Y)$. Let $V=(V_0,V_1,\sigma)$, $W=(W_0,W_1,\tau) \in L_1(X,Y)$. Construct metric on each bundles. σ^*, τ^* denote the adjoint of σ and τ which are isom when σ and τ are. Then $U:=(U_0,U_1;\rho) \in L_1(X,Y)$ where $U_0 := V_0 \otimes W_0 \oplus (V_1 \otimes W_1)$, $U_1 := (V_1 \otimes W_0) \oplus (V_0 \otimes W_1)$, and $\rho = \begin{pmatrix} \sigma\otimes 1 & -1\otimes\tau^* \\ 1\otimes\tau & \sigma^*\otimes 1 \end{pmatrix}$.

$\Rightarrow \rho^* = \begin{pmatrix} \sigma^*\otimes 1 & 1\otimes\tau^* \\ -1\otimes\tau & \sigma\otimes 1 \end{pmatrix} \Rightarrow \rho\rho^* = \begin{pmatrix} \sigma\sigma^*\otimes 1 + 1\otimes\tau^*\tau & 0 \\ 0 & \sigma^*\sigma\otimes 1 + 1\otimes\tau\tau^* \end{pmatrix}$

On Y,

let $w \in V_1\otimes W_0$, $(\sigma\sigma^*\otimes 1 + 1\otimes\tau^*\tau)w = 0 \Rightarrow \langle(\sigma\sigma^*\otimes 1 + 1\otimes\tau^*\tau)w, w\rangle = 0$

$\Rightarrow 0 = |(\sigma^*\otimes 1)w|^2 + |(\tau\otimes 1)w|^2 \Rightarrow (\sigma^*\otimes 1)w = (\tau\otimes 1)w = 0 \Rightarrow w=0$

so $\rho\rho^*|_{V_1\otimes W_0}$ is 1-1. Similarly, $\rho\rho^*$ and $\rho^*\rho$ are 1-1. $\Rightarrow \rho, \rho^*$ are both 1-1

ρ^* is 1-1 $\Rightarrow \rho$ is onto. $\Rightarrow \rho$ is an isom. on Y. $\Rightarrow U \in L_1(X,Y)$

Def'n (1.5): For a locally compact top. space X, we define $K_{cpt}(X) := \hat{K}(X^+)$ where X^+ is the one point compactification of X. This is called the K-theory of X with cpt. supp.

Any element of $K_{cpt}(X)$ can be represented by $[E]-[F]$ where E, F are trivail near ∞, i.e., outside a cpt. set in X since they belongs to $\ker i^*$ $(i: \infty \hookrightarrow K^+)$ so $[i^*E] = [i^*F] = [\tau^N]$ for some $N \in \mathbb{N}$ where τ^N is the trivail bundle over ∞. ($\because K(\infty) = K(pt) \cong \mathbb{Z}$.)

In fact, if $O \overset{i}{\underset{open}{\subseteq}} X$ $\exists$ nat. extension homomorphism $i_! : K_{cpt}(O) \to K_{cpt}(X)$ induced by $X^+ \xrightarrow{\text{quotient}} X^+/(X^+-U) \cong O^+$. Actually, this is gien by extending bundle E over U trivially to the whole X (since it is already trivail out a cpt. set of U.)

Def'n (1.6): We can, similar to the cpt. case, define $L_n(X)_{cpt}$ to be the equivalence classes $[V_0, \cdots, V_n; \sigma_1, \cdots, \sigma_n]$ where $V_0, \cdots, V_n$ are bundles on X & $0 \to V_0 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_n \to 0$ is exact seq. defined outside a cpt. set of X.

Fact (1.3): Similar to the cpt. case, $L_n(X)_{cpt} \cong L(X)_{cpt} := \varinjlim_n L_n(X)_{cpt}$ and $L(X)_{cpt} \cong K(X)_{cpt}$ by $[V_0, V_1; \sigma] \mapsto [V_0]-[V_1]$ so elements in $K(X)_{cpt}$ can by represented by such tuple with $\sigma: V_1 \xrightarrow{\sim} V_0$ defined in a nbd. of ∞.

We recall an important theorem, the Thom Isomorphism

Thm (1.1): Let $\pi: E \to X$ be a complex hermitian vector bundle over cpt space X. Then $\Lambda_{-1}(E) := [\pi^*\Lambda_{\mathbb{C}}^{even}E, \pi^*\Lambda_{\mathbb{C}}^{odd}E; \sigma] \in K_{cpt}(E)$ where $\sigma_e(\varphi) := e \wedge \varphi - e^* \llcorner \varphi$, and $i_!: K(X) \to K_{cpt}(E)$ by $i_!(a) = (\pi^* a) \cdot \Lambda_{-1}(E)$ is an isom.

II. The topological index.

Def'n (2.1): A differential operator of order m on X (now a C^∞-mfd) is a linear map $P : \Gamma(E) \to \Gamma(F)$ where $E, F \in V(X)$ having the following property: $\forall\, p \in X$ $\exists$ chart of p, say $U \subseteq X$ $\ni$ $E|_U \cong U \times \mathbb{C}^p$ and $F|_U \cong U \times \mathbb{C}^q$ $\ni$ $P = \sum'_{|\alpha| \le m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ on U and $A^\alpha(x) \ne 0$ for some $|\alpha| = m$.

If we make a change of localization of $E|_U$ and $F|_U$ by smooth map $g_E : U \to GL(n, p)$, $g_F : U \to GL_q(\mathbb{C})$. $P = g_F\left(\sum_{\alpha \le m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}\right) g_E^{-1} = \sum_{|\alpha| \le m} \hat{A}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$.
Note $\hat{A}^\alpha = g_F A^\alpha g_E^{-1}$.

If we change the local coord. $\hat{x} = \hat{x}(x)$ on U, then by $\frac{\partial}{\partial x_j} = \frac{\partial \hat{x}_k}{\partial x_j} \frac{\partial}{\partial \hat{x}_k}$,
$P = \sum_{|\alpha| \le m} \hat{A}^\alpha(\hat{x}) \frac{\partial^\alpha}{\partial \hat{x}^\alpha}$ where $\hat{A}^\alpha = \sum_{|\beta| = m} A^\beta \left[\frac{\partial \hat{x}}{\partial x}\right]^\alpha_\beta$ for $|\alpha| = m$ and where $\left[\frac{\partial \hat{x}}{\partial x}\right]^*_*$ denotes the symmetrization of the m^{th} tensor power of the Jacobian matrix $\left(\frac{\partial \hat{x}_k}{\partial x_j}\right)$.

So $\{i^m A^\alpha\}_{|\alpha| = m}$ defines an section $\sigma(P)$ of bundle $(\odot^m TX) \otimes \mathrm{Hom}(E, F)$ where $\odot$ denotes symmetric tensor product.

Def'n (2.2): $\sigma(P) \in \Gamma\left((\odot^m TX) \otimes \mathrm{Hom}(E, F)\right)$ is called the principal symbol of differential operator P.

We recall $\odot^m V \simeq$ homogeneous polynomal function of degree m on V^*, canonically.
$\Rightarrow$ For $\xi \in T_x^* X$, $\sigma_\xi(P) : E_x \to F_x$ and writen in the trivialization $\sigma_\xi(P) = i^m \sum_{|\alpha| = m} A^\alpha(x) \xi^\alpha$.

Def'n (2.3): Let P be a differential operator of order m over X. P is elliptic if $\forall$ $\xi \ne 0$, $\xi \in T^* X$, $\sigma_\xi(P) : E_x \to F_x$ is invertible.

Fact (2.1): Let $P : \Gamma(E) \to \Gamma(F)$, $P' : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(L)$ $\ni$ P, P' has the same order. Then $\forall\, \xi \in T^* X$, $t, t' \in \mathbb{R}$, one has that $\sigma_\xi(tP + t'P') = t\sigma_\xi(P) + t'\sigma_\xi(P')$ and $\sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P)$.

Given $P:\Gamma(E)\to\Gamma(F)$, we pull back the bundles to T^*X via $\pi: T^*X\to X$ and consider $\sigma(P)$ as $\sigma(P):\pi^*E\to\pi^*F$. If P is elliptic, it is isom. outside the zero section. Fix a metric on X, $DX := \{\xi\in T^*X \mid \|\xi\|\le 1\}$. The symbol of P defines a class $i(P) := [\pi^*E, \pi^*F; \sigma(P)] \in K(DX, \partial DX)$. Since $DX/\partial DX \cong TX^+$, may consider $\sigma(P)$ as in $K_{cpt}(TX)$.

There's only few steps to the def'n of the top. index but before that we consider $f: X\hookrightarrow Y$, proper embedding between manifolds. Let N be a tubular nbd. of $f(X)$ and we assume it has cpx. str. (so $\dim Y - \dim X$ is even). Recall the Thom isom. gives $i_!: K_{cpt}(X)\to K_{cpt}(N)$. Composite it with $K_{cpt}(N)\to K_{cpt}(Y)$ in the discussion followed Def'n (1.5), it gives $f_!: K_{cpt}(X)\to K_{cpt}(Y)$.

Now for such $f: X\hookrightarrow Y$. The normal bundle associated to embedding $f_*: TX\hookrightarrow TY$ has a canonical cpx. str.. This normal bundle is just the pull back to TX of $N\oplus N$ where the first in "manifold-directions", the second in "fiber-directions"; cpx. str. given by $T=\begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}$. So for any proper embedding $f: X\hookrightarrow Y$, there is $f_!: K_{cpt}(TX)\to K_{cpt}(TY)$

Now we can define the top. index. Let X, P, E, F. Choose any $f: X\hookrightarrow \mathbb{R}^n$, it associates $f_!: K_{cpt}(TX)\to K_{cpt}(T\mathbb{R}^N)$ followed by the above discussion.

$q: T\mathbb{R}^n\to pt$ which can be regared as $q:\mathbb{C}^N\to pt$ $\because$ $T\mathbb{R}^N = \mathbb{R}^N\oplus\mathbb{R}^N \cong \mathbb{C}^N$

gives $q_!: K_{cpt}(T\mathbb{R}^N)\to K(pt)\cong\mathbb{Z}$ which is just $(i_!)^{-1}$ where $i_!: K(pt)\to K_{cpt}(\mathbb{C}^N)$ is the map given by the Thom isom.

Def'n (2.4): $\text{top-ind}(P) \equiv q_! f_! \sigma(P)$ is the top. index.

Fact (7.1): top-ind(P) is indep. of choose of f.

pf): Consider $\tilde{f} = j \circ f$ where $j: \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+N'}$ is a linear inclusion.

$\rightsquigarrow$ $j_!: K_{cpt}(T\mathbb{R}^N) \xrightarrow{\cong} K_{cpt}(T\mathbb{R}^{N+N'})$ by Thom ism. Let $\hat{q}: T\mathbb{R}^{N+N'} \to pt$.

Then $\hat{q}_! \cdot \tilde{f}_! = q_! \circ f_!$. Given two embedding $f_i: X \to \mathbb{R}^{N_i}$, $i=0,1$. Then, $j_i \circ f_i: X \hookrightarrow \mathbb{R}^{N_0+N_1}$ are isotopic, $i=0,1$. That is, $F_t := t j_1 \circ f_1 + (1-t) j_0 \circ f_0$ is a smooth family of embeddings. Since K_{cpt} has homotopy invariance, top-ind. is the same for f_0, f_1.

Theorem 13.2 (The Atiyah-Singer Index Theorem): For elliptic P on cpt mfd X, ind(P) = top-ind(P). This is the main theorem on this report and we need some preparation.

Recall a pseudodifferential Operators P on $\mathbb{R}^n$ is defined by $Pu(x) = (2\pi)^{-\frac{n}{2}} \int e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\,d\xi$ where $\hat{u}$ is the Fourier trans. of u and $p(x,\xi)$ is some "good" fuction (called total symbol of P)

Def'n (2.5): Fix $m \in \mathbb{R}$. $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a symbol of order m if for each α, α' $\exists$ $C_{\alpha\alpha'}$ $\ni$ $|D_x^{\alpha} D_{\xi}^{\alpha'} p(x,\xi)| \le C_{\alpha\alpha'}(1+|\xi|)^{m-|\alpha'|}$. Let Sym^m denote the space of these symbo

Fact (2.2): To each $p \in \mathrm{Sym}^m$, it defines a $P: \mathcal{S} \to \mathcal{S}$ where $\mathcal{S}$ is the Schwartz space. If p has cpt. x-supp, it can extend continuously to $P: L^2_{s+m} \to L^2_s$ $\forall s$ where L^2_s denotes the sobolev space. Such P is called a pseudodifferential operators of order m on $\mathbb{R}^n$, this space is denoted as ΨDO_m. pf): Lawson 3.3.2 standard estimate.

To proceed our discussion, we consider $P \in \Psi DO_m(E,F)$ with the symbol $p \in \mathrm{Sym}^m$.

Then we define the pricipal simbol $\sigma(P) := [p] \in \mathrm{Sym}^m / \mathrm{Sym}^{m-1}$. It can be showed that $\sigma(P)$ transforms under diffeomorphism like a function on the cotangent bundle of $\mathbb{R}^n$. (Lawson Cor. 3.3.14.). We consider a special class of $P \in \Psi DO_m(E,F)$ on cpt. C^∞ mfd. We call P classical if its pricipal symbol is homogeneous of degree m in ξ outside some cpt. set of T^*X, i.e., P is classic if $\exists$ $C>0$ $\ni$ $\sigma_{t\xi}(P) = t^m \sigma_{\xi}(P)$ $\forall \xi \in T^*X$ with $|\xi| \ge C$ and $t \ge 1$. If X is non-cpt., we define it to be classical if it satisfies the above property over every cpt. subset of X. $\Psi CO_m(E,F)$:= the set of all classical operator.

Given an operator $P \in \Psi DO_m(E,F)$, $\hat{\sigma}_\xi(P) := \lim_{t\to\infty} \frac{\sigma_{t\xi}(P)}{t^m}$ is called its asymptotic pricipal symbol which is defined $\forall\, \xi \in \partial DX := \{\xi \in T^*X \mid \|\xi\|=1\}$

Fact (2.3): $0 \to \Psi DO_{m-1}(E,F) \to \Psi CO_m(E,F) \xrightarrow{\sigma} \Gamma(Hom(\pi^*E, \pi^*F)) \to 0$ is exact where $\pi : \partial DX \to X$ is the projection, X is a cpt. C^∞-mfd.

pf: We prove the surjection. For $s \in \Gamma(Hom(\pi^*E, \pi^*F))$, extend s smoothly to T^*X homogeneously of degree m in ξ for $\|\xi\| \geq 1$. Given local trivialization of E and F over a chart U, we can regard it as in the Euclidean space so it's easy to construct P in U with $\sigma(P)=s$. Let $\{U_j\}$ be a finite covering of X of such charts and let $\{\chi_j\}$ be P.O.U. subord. to it. $P := \sum \chi_j P_{\chi_j} \in \Psi CO_m(E,F)$ has $\sigma(P)=s$.

Fact (2.4) top.-ind(P) is indep. of ① The asymptotic symbol ② The homotopy classes of representative (π^*E, π^*F, σ) of u ③ The order m

so ind can be define as homo. ind: $K_{cpt}(T^*X) \to \mathbb{Z}$.

pf: The proof is based on the fact that ind P depends on the homotopy class of its pricipal symbol $\sigma(P)$ which is the thm 7.10, §3 of Lawson's book. Then for each case we construct a required homotopy. The detial is skipped here which can be find in the discussion on P.246, Lawson.

III. Proof of the Atiyah-Singer Index Theorem.

Theorem (3.1): For any index fuction, ind, if ① when $X = T^*X = pt$, $\text{ind} \equiv \text{id}_{\mathbb{Z}}: K(pt) \to \mathbb{Z}$ and ② if X, Y are cpt. mfd. and $f: X \hookrightarrow Y$, embedding, then $\text{ind}(u) = \text{ind}(f_! u)$.
We will have ind = top-ind. (①:= property 1, ②:= property 2)

pf): Let X be an cpt. mfd, $f: X \hookrightarrow S^N$ and $j: pt \hookrightarrow S^N$.
By ②, $\text{ind}\, u = \text{ind}(f_! u) = \text{ind}(j_! j_!^{-1} f_! u) = \text{ind} \circ j_!^{-1} \circ f_! u$. By ① $q_! := j_!^{-1} \triangleq \text{ind} \circ j_!^{-1}$
$\Rightarrow \text{ind}\, u = (\text{ind} \circ j_!^{-1}) \circ f_! u = q_! f_! u = \text{top-ind}\, u$.

So our main work will be showing the analytic index (we will still denote it ind.) satisfies property 1 and property 2.

Fact (3.1): Ind has property 1.
pf): Vector bundles over pt are just $\mathbb{C}^n, n \in \mathbb{N}$, so elements in $K(pt)$ has the form $[\mathbb{C}^r] - [\mathbb{C}^s]$ where $(\mathbb{C}^r, \mathbb{C}^s) \in \mathcal{L}_1(pt)$ is a pair of v.s.. An elliptic operator $P: \mathbb{C}^r \to \mathbb{C}^s$ is just a linear trans. $\Rightarrow r = \dim\ker P + \dim \text{im} P$, $s = \dim \text{im} P + \dim \text{coker} P$
$\Rightarrow r - s = \dim\ker P - \dim\text{coker} P = \text{ind}\, P$. $\square$

For the proof of property 2, we follow Atiyah and Singer to replace it by some simpler property:

Lemma (3.1): The Excision Property: Let O be an open mfd, and let $f: O \hookrightarrow X$, $f': O \hookrightarrow X'$ be two embeddings into cpt. mfds X and X'. Then $\text{ind} \circ f_! = \text{ind} \circ f'_!$ on $K_{cpt}(T^*O)$.

Lemma (3.2): The Multiplicative Property: Let X, Y be cpt. mfd. Then for $u \in K_{cpt}(T^*X)$, $v \in K_{cpt}(T^*Y)$ we have $\text{ind}(u \cdot v) = (\text{ind}\, u)(\text{ind}\, v)$.

Before proving these two properties we need to have a better understanding of how elements in $K_{cpt}(T^*X)$ looks like.

Lemma (3.3): Let $\pi : B \to X$ be a smooth, real vector bundle over X. Then every element in $K_{cpt}(B)$ can be represented by a triple of the form $(\pi^*E, \pi^*F, \sigma) \in \mathcal{L}_1(B)_{cpt}$ where E and F are vector bundles on X which are trivial outside a cpt. set, and where $\sigma : \pi^*E \to \pi^*F$ is homogeneous of degree 0 on the fibres of B. (wherever it is defined.)

pf): By fact (1.3), $\forall u \in K_{cpt}(B)$, $\exists$ $\sigma_0 : E_0 \to F_0$ which is a bundle equi. outside a cpt. subset $K \subseteq B$ s.t. $u = [E_0, F_0; \sigma_0]$. We apply a fact that $\forall$ vector bundle $\pi : E \to B$, $\exists$ $E^{\perp}$ + $E \oplus E^{\perp}$ is trivial to E_0. $(\hat{E}, \hat{F}, \hat{\sigma}) := (E_0 \oplus E_0^{\perp}, F_0 \oplus E_0^{\perp}; \sigma \oplus id)$ where $E_0 \oplus E_0^{\perp}$ is trivial.

$\exists$ trivialization $\tau_{\hat{E}} : \hat{E}|_{B-K} \xrightarrow{\sim} (B-K) \times \mathbb{C}^m$, $\tau_F : \hat{F}|_{B-K} \xrightarrow{\sim} (B-K) \times \mathbb{C}^m$ + $\tau_{\hat{E}}$ is the trivialization on B and $\tau_{\hat{F}} := \tau_{\hat{E}} \circ \hat{\sigma}^{-1}$ on $B-K$.

Choose a cpt. $\Omega \subset X$ so that $K \subset B|_{\Omega}$. Set $E = i^*\hat{E}$, $F = i^*\hat{F}$ where $i : X \hookrightarrow B$, the zero section, and let $\tau_E := \tau_{\hat{E}}|_X$, $\tau_F := \tau_{\hat{F}}|_X$. Claim: over B $\exists$ bundle isom. $f_E : \hat{E} \xrightarrow{\sim} \pi^*E$ and $f_F : \hat{F} \xrightarrow{\sim} \pi^*F$. + $f_E = \tau_E^{-1} \circ \tau_{\hat{E}}$ and $f_F = \tau_F^{-1} \circ \tau_{\hat{F}}$ over $B|_{X-\Omega}$.

Let $h : B \times [0,1] \to B$ by $h(b,t) = tb$, then $h(\cdot, 1) = id_B$, $h(\cdot, 0) := \pi$. Set $\mathcal{E} := h^*\hat{E}$, $\mathcal{F} := h^*\hat{F}$. Note that $\mathcal{E}|_{B\times\{0\}} = \pi^*E$, $\mathcal{E}|_{B\times\{1\}} = \hat{E}$, $\mathcal{F}|_{B\times\{0\}} = \pi^*F$, $\mathcal{F}|_{B\times\{1\}} = \hat{F}$.

$\because$ $\hat{E}, \hat{F}$ is trivial over $B|_{X-K}$ $\therefore$ We can construct connection on $\mathcal{E}, \mathcal{F}$ which extend the canonical flat connections (compatible with the trivializations). Parallel trans. along $b \times [0,1]$ gives what we want. $\sigma := f_F \circ \hat{\sigma} \circ f_E^{-1} : \pi^*E \to \pi^*F$ is an isom. defined on $B-K$ and const. on $B - \pi^{-1}(\Omega)$. Fix $r > 0$ so that $K \subset \{b \in B \mid \|b\| \le r\}$. We now redefine σ in $\|b\| \ge r$ by extending homogeneously with degree zero. $\square$

- proof of lemma (3.1): let $u \in K_{cpt}(T^*\mathcal{O})$. By lemma (3.3), $u = [\pi^*E, \pi^*F, \sigma]$ where E, F are bundle over $\mathcal{O}$ which are trivial outside a cpt. set of $\mathcal{O}$ and σ is homogeneous of degree 0 outside a cpt set of $T^*\mathcal{O}$. In particular, outside a cpt. $\Omega \subseteq \mathcal{O}$, $\exists$ trivialization $\tau_E : E|_{\mathcal{O}-\Omega} \xrightarrow{\sim} (\mathcal{O}-\Omega)\times\mathbb{C}^m$ and $\tau_F : F|_{\mathcal{O}-\Omega} \xrightarrow{\sim} (\mathcal{O}-\Omega)\times\mathbb{C}^m$ w.r.t.
$\sigma_{x,\xi} = \underset{\uparrow \text{deg. } 0}{\sigma_x} = (\tau_F)_x^{-1} \circ \tau_E|_x$ at all pt. $(x,\xi) \in T^*(\mathcal{O}-\Omega)$. $\Rightarrow$ Over $T^*(\mathcal{O}-\Omega)$ σ comes from a bundle map $\sigma_0 : E \to F$ over the basis. Moreover, we the trivialization given above σ_0 becomes the identity map, i.e., $\sigma_0(z_1,\cdots,z_m) = (z_1,\cdots,z_m)$ for all $x \in \mathcal{O}-\Omega$. ($\because \tau_{F|_{\mathcal{O}}} \circ \sigma_0 \circ \tau_{E|_{\mathcal{O}}}^{-1}$)

$\Rightarrow$ $\sigma_0 \in \Gamma(\mathrm{Hom}(E,F))$ is a differential operator of order zero.

By fact (2.3), chose $P \in \Psi(\mathcal{O}; E, F)$ with $\sigma(P) = \sigma$ outside a cpt. set in $T^*\mathcal{O}$ and which is $\sigma_0 = id$ in $\mathcal{O}-\Omega$.

Now given an open embedding, $f : \mathcal{O} \hookrightarrow X$. Extending E, F trivial over $X - f(\mathcal{O})$ by the above trivialization and extends operator P by identity, $\Rightarrow$ This defines elliptic operator $f_! P$ on X with $[\sigma(f_! P)] = f_! [\sigma(P)] = f_! u$.

elements in $\ker f_! P$ has supp in Ω ($\because P$ is extended by id.) and hence $\in \ker P$. and $\exists$ a nat. embedding $\ker P \hookrightarrow \ker f_! P$ given by extending by zero.

$\Rightarrow$ $\dim \ker P = \dim \ker f_! P$. Similar for $(f_! P)^*$. $\Rightarrow$ $\mathrm{ind}(f_! u) = \mathrm{ind}(f_! P) = \dim(\ker P) - \dim(\mathrm{coker}\, P)$

The right hand side is indep. of f, $\square$

proof of lemma (1.2)

We represent u, v by 1-st order elliptic operators $P: \Gamma(E) \to \Gamma(F)$, $Q: \Gamma(E) \to \Gamma(F)$ over X and Y respectively. We define a "graded tensor product"

$D: \Gamma((E\otimes E')\oplus(F\otimes F')) \longrightarrow \Gamma((F\otimes E')\oplus(E\otimes F'))$ by $D = \begin{pmatrix} P\otimes 1 & -1\otimes Q^* \\ 1\otimes Q & P^*\otimes 1 \end{pmatrix}$

where we fix metrics on E, F and P^*, Q^* are the corresponding adjoints. $E\otimes E'$, etc., denotes the exterior tensor product over $X\times Y$.

$$D^*D = \begin{pmatrix} P^*P\otimes 1 + 1\otimes Q^*Q & 0 \\ 0 & PP^*\otimes 1 + 1\otimes QQ^* \end{pmatrix}, \quad DD^* = \begin{pmatrix} PP^*\otimes 1 + 1\otimes Q^*Q & 0 \\ 0 & P^*P\otimes 1 + 1\otimes QQ^* \end{pmatrix}$$

, $\hat{P}^{(*)} := P^{(*)}\otimes 1$, $\hat{Q}^{(*)} := 1\otimes Q^{(*)}$.

- $\ker D = \ker\hat{P}\cap\ker\hat{Q}$, $\ker D^* = \ker\hat{P}^*\cap\ker\hat{Q}^*$

$D^*D\varphi = 0 \Rightarrow (D^*D\varphi, \varphi) = 0 \Leftrightarrow |D\varphi|^2 = 0 \Leftrightarrow D\varphi = 0$. $\Rightarrow \ker D^*D \subseteq \ker D \Rightarrow$ "$=$"

$\because$ D^*D is diagonal may consider on each part of the direct sum.

let $\varphi \in \Gamma(F\otimes E')$, $D^*D\varphi = 0 \Rightarrow (\hat{P}^*\hat{P}\varphi, \varphi) + (\hat{Q}^*\hat{Q}\varphi, \varphi) = 0$

$\Rightarrow |\hat{P}\varphi|^2 + |\hat{Q}\varphi|^2 = 0 \Rightarrow \hat{P}\varphi = 0 = \hat{Q}\varphi \Rightarrow \ker D^*D = \ker\hat{P}\cap\ker\hat{Q}$. Similar for D^*.

- $\ker\hat{P}\cap\ker\hat{Q} \cong \ker P\otimes\ker Q$:

$\ker P\times\ker Q \to \ker\hat{P}\cap\ker\hat{Q}$ by $(p, q) \to p\otimes q$ bilinearly

so it induces $\ker P\otimes\ker Q \to \ker\hat{P}\cap\ker\hat{Q}$ by $p\otimes q \to p\otimes q$.

And clearly for $a\otimes b \neq 0$ in $E\otimes E'$, $a\otimes b \in \ker\hat{P}\cap\ker\hat{Q} \Rightarrow P(a) = Q(b) = 0$

so $\ker D \cong (\ker P\otimes\ker Q)\oplus(\ker P^*\otimes\ker Q^*)$, $\operatorname{coker} D \cong (\ker P^*\otimes\ker Q)\oplus(\ker P\otimes\ker Q^*)$

$(\operatorname{ind} P)(\operatorname{ind} Q) = ([\ker P] - [\operatorname{coker} P])([\ker Q] - [\operatorname{coker} Q])$

$= [\ker P][\ker Q] + [\operatorname{coker} P][\operatorname{coker} Q] - ([\operatorname{coker} P][\ker Q] + [\ker P][\operatorname{coker} Q])$

$= \operatorname{ind} D$.

Note although D is, by def'n, the tensor product of P, Q but it may not belong to $\Psi CD_1((E\otimes E')\oplus(F\otimes F'), (F\otimes E')\oplus(E\otimes F'))$ since it's pricipal symbol is not homogeneous outside a cpt. set in $T^*(X\times Y)$. It is only homogeneous outside a uniform nbd. of "T^*Y-axes" in $T^*(X\times Y)$

We claim we can construct $\{(P\otimes 1)_\varepsilon \in \Psi DO_1(E\otimes E', F\otimes E')$ for $\varepsilon>0\} \ni$

$\lim_{\varepsilon\to 0}(P\otimes 1)_\varepsilon = P\otimes 1$ where the lim is taken from $L_1^2(E\otimes E')$ to $L_0^2(E\otimes E')$

($L_k^2(E)$ denote the completion of $\Gamma(E)$ w.r.t. sobolev norm k)

Apply to each entry gives $\lim_{\varepsilon\to 0} D_\varepsilon = D$ on maps between sobolev spaces.

Cor. 7.4. on §3, Lawson ⇒ index is locally constant so $\operatorname{ind} D_\varepsilon = \operatorname{ind} D$.

On the other hand, given $K \subseteq_{cpt} T^*(X\times Y)$, $\exists\ \varepsilon_K>0 \ni \sigma(D_\varepsilon)=\sigma(D)$ on K for $\varepsilon\le\varepsilon_K$.

Since the a class in K_{cpt} is determined on its cpt. part, $[\sigma(D_\varepsilon)] = [\sigma(D)] = u\cdot v$ for ε smal. $\Rightarrow \operatorname{ind}(u\cdot v) = \operatorname{ind}(D_\varepsilon) = \operatorname{ind} D = (\operatorname{ind} P)(\operatorname{ind} Q) = (\operatorname{ind} u)(\operatorname{ind} v)$

Construction of $(P\otimes 1)_\varepsilon$: First construct, $\phi:\mathbb{R}^+\to[0,1] \ni \phi(t)=0$ for $t\le 1$, $\phi(t)=1$ for $t\ge 2$. For $\varepsilon>0$, $\psi_\varepsilon(r,s) := 1-\phi(\varepsilon\sqrt{r^2+s^2})\,\phi(\frac{\varepsilon s}{r})$. $(P\otimes 1)_\varepsilon := \psi_\varepsilon(|\xi|,|\eta|)(P\otimes 1)$

The check is straitforward. □

We will need the multiplicity property for fibre bundles.

Recall $\pi:P\to X$ is a pripal G-bundle if ∃ Lie gp. G acting on P and the action is compatible with the fibre str.. Let F be another space, Homeo(F) denote homeo. on F. $\forall\ \rho:G\to \text{Homeo}(F)$, we can construct a fibre bundle over X with fibre F as follows. For $(p,f)\in P\times F$, $g(p,f):=(pg^{-1},\rho(g)f)$. Define $P\times_\rho F := P\times F/G$

The projection $P\times F\to P\xrightarrow{\pi}X$ can be descends into $\pi_\rho: P\times_\rho F\to X$, it is called bundle associated to P by ρ.

Here we consider $G=O(n)$. Let $P\to X$ be an pricipal O_n-bundle

Let $O(n)$ acts on $\mathbb{R}^n$ by the nat. way and extending trivially to $S^n=\mathbb{R}^n\cup\{\infty\}$

$V:=P\times_{O(n)}\mathbb{R}^n$, $Z:=P\times_{O(n)}S^n$. $T^*Z=\pi^*T^*X\oplus T(Z/X)$ where T^*Z/π^*T^*X.

$\pi:Z\to X$. This gives $K_{cpt}(T^*X)\otimes K_{cpt}T(Z/X)\to K_{cpt}(T^*Z)$

(For $E\otimes E'$ on X, $K_{cpt}(E)\otimes K_{cpt}(E')$ is defined by $\Delta^*(E\otimes E')$ where $\Delta:X\to X\times X$)

Also, we have $K_{O(n)}(T^*S^n)_{cpt} \to K_{O(n)}(P\times T^*S^n)_{cpt} \to K_{cpt}(P\times_{O(n)} T^*S^n) = K_{cpt}(T(Z/X))$

Combine this gives us a multiplication $K_{cpt}(T^*X)\otimes K_{O(n)}(T^*S^n)_{cpt} \to K_{cpt}(T^*Z)$.

In the above construction, we associate a rep. $\rho: O(n)\to O(N)$ a vector bundle $V_\rho := P\times_\rho \mathbb{R}^N$. So it gives a homo. $\alpha_\rho: R(O(n)) \to K(X)$.

Since $K_{cpt}(T^*X)$ is nat. a $K(X)$-module. Therefore, it becomes a $R(O(n))$-module by α_ρ.

($R(G)$ is the rep. ring of G, $K_{O(n)}$ is the "K-theory of $O(n)$-bundle)

lemma (13.4) Multiplicative property for Sphere Bundle. Let Z be an S^n-bundle over a cpt. manifold X. Then $\mathrm{ind}(u\cdot v) = \mathrm{ind}(u\cdot \mathrm{ind}_{O(n)} v)$ for all $u\in K_{cpt}(T^*X)$, $v\in K_{O(n)}(T^*$

pf:) The spirite of the proof is similar to lemma (13.2) but need modifications.

The proof is given by Lawson in 13.6, §3.

With the above reductions, we still need to calculate the very simple case.

lemma (13.5): Consider the standard rep. $S^n \hookrightarrow \mathbb{R}^n\oplus\mathbb{R}$ (i.e., by rotations about an axis). Let $i: pt \hookrightarrow S^n$ denote one of the fixed pt. Then $\mathrm{ind}_{O(n)}(i_! 1) = 1 \in R(O(n))$

(S^n is regarded as $O(n)$-mfd).

pf); This is a consequence of Hodge Theory. The detail is in lemma 13.7, Lawson.

Now we assume all the above lemma.

proof of Property 2; That is, we want to prove $\mathrm{ind} = \mathrm{ind}\circ f_!$, $f: X\hookrightarrow Y$.

By Excision Property we may replace Y by a tubular nbd. N of X which is open in Y and is diffeomorphic to the normal bundle of X. So it's sufficient to show $\forall u\in K_{cpt}(T^*X)$, $\mathrm{ind}\, u = \mathrm{ind}(f_! u)$ where V is a vector bundle over X, $f: X\hookrightarrow V$, the zero section.

Again by Excision Property, compactify V to sphere bundle (as $Z = P\times_{O(n)} S^n$).

We apply the multiplicative Property for sphere bundle (lemma 13.4)), with $v := i_! 1$.

By lemma (13.5): $\mathrm{ind}(u\cdot i_! 1) \underset{(13.4)}{=} \mathrm{ind}(u\cdot \mathrm{ind}_{O(n)}(i_! 1)) \underset{(13.5)}{=} \mathrm{ind}(u)$.

However $f_! u = u\cdot i_! 1$, so $\mathrm{ind}(f_! u) = \mathrm{ind}(u)$, $\square$.

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Topic : Estimate first eigenvalue

周祐正

Notation : M be a Riemannian mfd

Given $\varphi \in C^\infty(M)$, let $\|\varphi\|^2 := \int_M \varphi^2 + \int_M |\nabla\varphi|^2$

The completion of $C^\infty(M)$ w.r.t $\|\cdot\|$ denoted by $L_1^2(M)$

of $C_0^\infty(M)$ w.r.t. $\|\cdot\|$ denoted by $L_{1,0}^2(M)$

Min - Max principle :

(1) if $\partial M = \phi$, then $H := \{ f \in L_1^2(M) \mid \int_M f = 0 \}$

(2) if $\partial M \neq \phi$, and with Dirichlet BC ($\varphi \in C^\infty(M)$, $\varphi|_{\partial M} = 0$)

then $H := L_{0,1}^2(M)$

(3) if $\partial M \neq \phi$, and with Neumann BC ($\varphi \in C^\infty(M)$, $\frac{\partial\varphi}{\partial n}|_{\partial M} = 0$, n is the outer normal of ∂M)

then $H := \{ f \in L_1^2(M) \mid \int_M f = 0 \}$

Then Δ is a self-adjoint elliptic operator on H

by the spectral theory, we can find ONB $\{f_i\}$ of H

with $\Delta f_i = -\lambda_i f_i$, $f_i \in C^\infty(M) \cap H$

s.t. $\lambda_1 = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M |f|^2} \,\middle|\, f \in H \right\}$

$\lambda_i = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M |f|^2} \,\middle|\, f \in H,\ \int_M f \cdot f_j = 0,\ j = 1, \cdots, i-1 \right\}$

Let M be a complete Riemannian mfd

Denote $B(x_0, r_0)$ be a geodesic ball on M, with heat kernel $H(x,y,t)$

and $V_n(k, r_0)$ be a geodesic ball of the space form of curvature k.

with heat kernel $\mathcal{E}(r(x,y),t)$ (has constant sectional curvature k)

$\mathcal{E}(r(x,y),t)$ can be thought of as a function on $B(x_0, r_0)$.

Thm 1 (Cheeger - Yau). M be a complete Riemannian mfd.

$Ric(M) \geq (n-1)k$, $n = \dim M$ and with the assumption above

we have $\mathcal{E}(r(x,y),t) \leq H(x,y,t)$

(with Dirichlet or Neumann BC)

(pf) compute $H(x,y,t) - \mathcal{E}(r(x,y),t) = \int_0^t \frac{d}{ds}\Big(\mathcal{E}(x,y,t-s) H(x,y,t) \Big) ds$

$$= \int_0^t \int_{B(x_0,r_0)} \frac{d}{ds}\big(\mathcal{E}(x,z,t-s) H(z,y,s)\big) dz\, ds$$

$$= -\int_0^t \int_{B(x_0,r_0)} \frac{d}{ds}\left[\mathcal{E}(r(x,z),t-s)\right] \cdot H(z,y,s)\, dz\, ds + \int_0^t \int_{B(x_0,r_0)} \mathcal{E}(r(x,z),t-s) \frac{\partial H}{\partial s}(z,y,s)\, dz\, ds$$

$$= -\int_0^t \int_{B(x_0,r_0)} \tilde{\Delta} \mathcal{E}(r(x,z),t-s) H(z,y,s)\, dz\, ds + \int_0^t \int_{B(x_0,r_0)} \mathcal{E}(r(x,z),t-s) \Delta H(z,y,s)\, dz\, ds$$

where $\tilde{\Delta}, \Delta$ respectively the laplacian operators on the space form M

also $\int_{B(x_0,r_0)} \mathcal{E}\Delta H - \Delta\mathcal{E} H = \int_{\partial B(x_0,r_0)} \mathcal{E}\frac{\partial H}{\partial n} - \frac{\partial \mathcal{E}}{\partial n} H = 0$. (Dirichlet or Neumann BC)

Hence $H(x,y,t) - \mathcal{E}(r(x,y),t) = \int_0^t \int_{B(x_0,r_0)} (-\tilde{\Delta}\mathcal{E} + \Delta\mathcal{E}) H\, dz\, dt$

by the fact that $H > 0$, it suffice to prove $(\Delta - \tilde{\Delta})\mathcal{E} > 0$

Concider in local coordinate near x, $r(y) = r(x,y)$

$\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + \hat{m}(r)\frac{\partial}{\partial r}$, where $\hat{m}(r) = \frac{\partial \log\sqrt{\tilde{g}}}{\partial r}$

$\Delta = \frac{\partial^2}{\partial r^2} + m(r)\frac{\partial}{\partial r}$, where $m(r) = \frac{\partial \log\sqrt{g}}{\partial r}$

therefore $(\Delta - \tilde{\Delta})\mathcal{E} = (m(r) - \hat{m}(r))\frac{\partial \mathcal{E}}{\partial r}$

(By Bishop Volume comparison Theorem, M be a complete Riemannian mfd

$Ric(M) \geq (n-1)k$, $n = \dim M$ Then for any $x \in M$, $R > 0$

$\frac{Vol(B(x,R))}{Vol(V_n(k,R))}$ is non-increasing in R)

since $\lim_{r\to 0} \frac{Vol(B(x,R))}{Vol(V_n(k,R))} = 1$, we have $Vol(B(x,R)) \leq Vol(V_n(k,R))$

$\Rightarrow m(r) - \hat{m}(r) \leq 0$

(and the fact that : if $H(r(x,y),t)$ be the heat kernel of a geodesic ball in a space form, then we have $\frac{\partial H}{\partial r} < 0$).

then we have $\frac{\partial \mathcal{E}}{\partial r} < 0$ $\Rightarrow$ $H(x,y,t) - \mathcal{E}(r(x,y),t) = \int_0^t \int_{B(x_0,r_0)} (-\tilde{\Delta} + \Delta)\mathcal{E}\cdot H\, dz\, dt \geq 0$ ✱

Thm 2 (Cheng) M be a complete Riemannian mfd

$Ric(M) \geq (n-1)k$, $n = \dim M$, $B(x_0, r_0)$ be a geodesic ball on M.

and $V_n(k, r_0)$ be a geodesic ball of the space form of curvature k

Then $\lambda_1(B(x_0,r_0)) \leq \lambda_1(V(k,r))$ w.r.t. Dirichlet BC

(pf) let $H(x,y,t)$, $\mathcal{E}(r(x,y),t)$ be the heat kernel of $B(x_0,r_0)$, $V_n(k,r_0)$ respectively

by thm 1 $\Rightarrow$ $H(x,x,t) \geq \mathcal{E}(0,t)$

write $H(x,x,t) = \sum_i e^{-\lambda_i t}\phi_i^2(x)$ where $\lambda_i = \lambda_i(B(x_0,r_0))$, $\hat{\lambda}_i = \hat{\lambda}_i(V_n(k,r_0))$

$\mathcal{E}(0,t) = \sum_i e^{-\hat{\lambda}_i t}\hat{\phi}_i^2(x)$ $\phi_i, \hat{\phi}_i$ are the corresponding eigenfunction.

$$\Rightarrow e^{-\lambda_1 t}\left[\varphi_1^2(x) + e^{-(\lambda_2-\lambda_1)t}\varphi_2^2(x) + \cdots\right] \geq e^{-\hat{\lambda}_1 t}\left[\hat{\varphi}_1^2(0) + e^{-(\hat{\lambda}_2-\hat{\lambda}_1)t}\hat{\varphi}_2^2(0)\cdots\right].$$

$$\Rightarrow \varphi_1^2(x) + e^{-(\lambda_2-\lambda_1)t}\varphi_2^2(x) + \cdots \geq e^{(\lambda_1-\hat{\lambda}_1)t}\left[\hat{\varphi}_1^2(0) + e^{-(\hat{\lambda}_2-\hat{\lambda}_1)t}\hat{\varphi}_2^2(0) + \cdots\right]$$

Notice that λ_1 is simple and $\varphi_1^2(x) > 0$. $\hat{\lambda}_1$ and $\hat{\varphi}_1^2(x)$ is the same.

(pf). first show φ_1 is the same sign on $B(x_0,r_0)$.

if not, let $B^+(x_0,r_0) = \{x \in B(x_0,r_0) \mid \varphi_1(x) > 0\}$, let $\varphi_1^+ := \varphi_1|_{B^+(x_0,r_0)}$

and $\varphi_1^+ \equiv 0$ on $B(x_0,r_0) \setminus B^+(x_0,r_0)$. the similar def of $B^-(x_0,r_0)$ and φ_1^-.

then observe that
$$\min\left\{\frac{\int_{B(x_0,r_0)}|\nabla\varphi_1^+|^2}{\int_{B(x_0,r_0)}|\varphi_1^+|^2}, \frac{\int_{B(x_0,r_0)}|\nabla\varphi_1^-|^2}{\int_{B(x_0,r_0)}|\varphi_1^-|^2}\right\} \leq \int_{B(x_0,r_0)}\frac{|\nabla\varphi_1|^2}{|\varphi_1|^2}$$

Say φ_1^- is the eigenfunction and $\varphi_1^- \leq 0 \Rightarrow \Delta\varphi_1^- \geq 0$

by Hopf maximum principle, φ_1^- can not achieves its maximum on $B(x_0,r_0)$

or it will be constant $\Rightarrow \varphi_1^- < 0$ ※

second show λ_1 is simple, if u_1, u_2 eigenfunctions of λ_1 and $u_1 \neq cu_2$ $\forall c \in \mathbb{R}$

then $\exists\, c' \in \mathbb{R}$ s.t. $c'u_1 + u_2$ not the same sign on $B(x_0,r_0)$ $\to\!\!\!\!\times$ ※

let $t \to \infty$, the inequality holds only when $\lambda_1 - \hat{\lambda}_1 \leq 0$.

Thm 3 (Cheng) Let M be a compact Riemannian manifold, $\partial M = \phi$. $Ric(M) \geq (n-1)K$, then $\lambda_m(M) \leq \lambda_1(V(k, \frac{d}{2m}))$ where $d = diam(M)$

(pf) We can find $x_1, x_2, \cdots, x_{m+1} \in M$ s.t. $B(x_i, \frac{d}{2m})$ pairwise disjoint

Let φ_i be the first eigenfunction on $B(x_i, \frac{d}{2m})$ with Dirichlet BC.

by thm 1 $\Rightarrow \int_{B(x_i, \frac{d}{2m})} |\nabla \varphi_i|^2 = \lambda_1(B(x_i, \frac{d}{2m})) \int_{B(x_i, \frac{d}{2m})} |\varphi_i|^2 \leq \lambda_1(V(K, \frac{d}{2m})) \int_{B(x_i, \frac{d}{2m})} |\varphi_i|^2$

let $\{\psi_i\}$ be the eigenfunctions on M, with $\Delta \psi_i = -\lambda_i \psi_i$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$

There exists constants $a_1, \cdots, a_{m+1}$

s.t. $\sum_{i=1}^{m+1} a_i \varphi_i \not\equiv 0$, and $\sum_{i=1}^{m+1} a_i \varphi_i \perp \{\psi_1, \psi_2, \cdots, \psi_m\}$

(extend φ_i to be zero outside the $B(x_i, \frac{d}{2m})$)

then by the Min-Max principle

$$\lambda_m(M) \int_M \left(\sum_{i=1}^{m+1} a_i \varphi_i\right)^2 \leq \int_M \left|\sum_{i=1}^{m+1} a_i \nabla \varphi_i\right|^2 = \int_M \sum_{i=1}^{m+1} a_i^2 |\nabla \varphi_i|^2$$

$$\leq \lambda_1(V(K, \tfrac{d}{2m})) \int_M \sum_{i=1}^{m+1} a_i^2 |\varphi_i|^2 \leq \lambda_1(V(K, \tfrac{d}{2m})) \int \left(\sum_{i=1}^{m+1} a_i \varphi_i\right)^2$$

$$\Longrightarrow \lambda_m(M) \leq V_1(V(K, \tfrac{d}{2m}))$$

in particular if M be cpt Riemannian mfd, $\partial M = \phi$ $Ric(M) \geq 0$, $\dim M = 0$

estimate its λ_1

consider the function $f(x) = \frac{d^2}{4} - x^2$

then
$$\lambda_1(V(0, \tfrac{d}{2})) \leq \frac{\int_{B_n(0,\frac{d}{2})} |\nabla f|^2}{\int_{B_n(0,\frac{d}{2})} f^2} = \frac{\int_0^{\frac{d}{2}} 4r^2 \cdot r^{n-1} dr}{\int_0^{\frac{d}{2}} (\frac{d^2}{4} - r^2)^2 \cdot r^{n-1} dr} = \frac{1}{(\frac{d^2}{4})} \cdot \left(\frac{\frac{4}{n+2}}{\frac{1}{n} - \frac{2}{n+2} + \frac{1}{n+4}} \right) = \frac{2n(n+4)}{d^2}$$

let φ be the eigenfunction of λ_1 on M (cpt, $\partial M=\phi$) we have $\int_M \varphi = -\frac{1}{\lambda_1}\int_M \Delta\varphi = 0$, so by multiplying a constant, we can assume

$$a-1 = \inf_M \varphi, \quad a+1 = \sup_M \varphi, \text{ where } 0 \le a < 1$$

Thm4 (Li–Yau) M^n be a cpt Riemannian mfd, $\partial M=\phi$, $Ric(M) \ge 0$

then $\lambda_1 \ge \dfrac{\pi^2}{(1+a)d^2}$, where $d = \operatorname{diam} M$.

lemma : $|\nabla u|^2 \le \lambda(1+a)(1-u^2)$

(pf). let $u = \varphi - a$, then $\Delta u = -\lambda_1(u+a)$ and $|u| \le 1$

Let $P = |\nabla u|^2 + cu^2$, where $c = \lambda(1+a)$

Assume $P(x)$ takes it maximum at x_0,

If $|\nabla u(x_0)| \neq 0$ or the lemma is clearly true.

by rotate the frame so that $u_1(x_0) = |\nabla u(x_0)|$

by the maximum principle at x_0, $\nabla P = 0$, $\Delta P \le 0$.

we have $P_i = u_m u_{mi} + c u u_i$

$\Rightarrow$ $0 = u_1(u_{11} + cu)$ and $u_{ij}u_{ij} \ge u_{11}^2 = c^2u^2$.

$0 \ge \frac{1}{2}\Delta P = u_{mi}u_{mi} + u_m u_{mii} + c u_1^2 + c u \Delta u$. $\quad R_{klij} = \langle R(X_i, X_j)X_l, X_k\rangle$

first compute $\sum_{m,i} u_m u_{mii} = \sum u_m u_{imi} = \sum u_m u_{iim} + \sum u_m u_k R_{kimi}$

$= \sum u_m u_{iim} + Ric(\nabla u, \nabla u) \ge \sum_m u_m (\Delta u)_m$

$\Rightarrow 0 \ge \frac{1}{2}\Delta P \ge u_{mi}u_{mi} + u_m(\Delta u)_m + cu_1^2 + cu\Delta u$

$\ge c^2u^2 - \lambda_1 u_1^2 + cu_1^2 - c\lambda_1 u(u+a)$

$= (c-\lambda_1)(u_1^2 + cu^2) - ac\lambda_1 u \quad \ge a\lambda_1 P(x_0) - ac\lambda_1$

$\Rightarrow |\nabla u|^2 + cu^2 \le c \Rightarrow |\nabla u|^2 \le \lambda(1+a)(1-u(x))^2$

pf of thm 4

let $u(x_1) = \sup u$, $u(x_2) = \inf u$, γ be the shortest geodesic joining x_1 and

Then

$$\pi = \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} \leq \int_\gamma \frac{|\nabla u|}{\sqrt{1-u^2}}\, ds \leq \sqrt{\lambda(1+a)} \int_\gamma ds \leq \sqrt{\lambda(1+a)} \cdot d$$

$$\Rightarrow \lambda = \frac{\pi^2}{(1+a)d^2}$$

Lemma 1: The function $z(u) = \frac{2}{\pi}(\sin^{-1}(u) + u\sqrt{1-u^2}) - u$ defined on $[-1, 1]$

satisfies

$\dot{z}u + \ddot{z}(1-u^2) + u = 0$ ——— ①

$\dot{z}^2 - 2z\ddot{z} + \dot{z} \geq 0$ ——— ②

$2z + \dot{z}u + 1 \geq 0$ ——— ③

$(1-u^2) \geq 2|z|$ ——— ④

(pf) ① compute $\dot{z} = \frac{4}{\pi}\sqrt{1-u^2} - 1$, $\ddot{z} = \frac{-4u}{\pi\sqrt{1-u^2}}$

then $\dot{z}u + \ddot{z}(1-u^2) + u = \frac{4u}{\pi}\sqrt{1-u^2} - u + \frac{-4u}{\pi\sqrt{1-u^2}}(1-u^2) + u = 0$.

② $\dot{z}^2 - 2z\ddot{z} + \dot{z} = \frac{4}{\pi\sqrt{1-u^2}} \left\{ \frac{4}{\pi}(\sqrt{1-u^2} + u\sin^{-1}u) - (1+u^2) \right\}$

so suffice to show $\frac{4}{\pi}(\sqrt{1-u^2} + u\sin^{-1}u) - (1+u^2) \geq 0$.

compute $\frac{d}{du}\left\{ \frac{4}{\pi}(\sqrt{1-u^2} + u\sin^{-1}u) - (1+u^2) \right\} = \frac{4}{\pi}\sin^{-1}u - 2u$, which is nonpositive on $[0, 1]$.

Hence $\frac{4}{\pi}(\sqrt{1-u^2} + u\sin^{-1}u) - (1+u^2) \geq \left[\frac{4}{\pi}(\sqrt{1-u^2} + u\sin^{-1}u) - (1+u^2)\right]_{u=1} = 0$.

③ $2z - \dot{z}u + 1 = \frac{4}{\pi}\sin^{-1}(u) + 1 - u$ is clearly nonnegative.

④ let $f(u) = 1-u^2 - \frac{4}{\pi}(\sin^{-1}u + u\sqrt{1-u^2}) + 2u$

then compute $\dot{f} = -2u - \frac{4}{\pi}(2\sqrt{1-u^2}) + 2$,

$\ddot{f} = -2 + \frac{8u}{\pi\sqrt{1-u^2}}$, $\dddot{f} = \frac{8}{\pi(1-u^2)^{\frac{3}{2}}}$

case 1: $-1 \leq u \leq 0$, $\ddot{f} \leq 0$. Hence $f(u) \geq \min\{f(-1), f(0)\} = 0$

case 2: $0 \leq u \leq 1$, $\dddot{f} \geq 0$. Hence $\dot{f} \leq \max\{\dot{f}(0), \dot{f}(1)\} = 0$

Therefore $f(u) \geq f(1)$ ※

Lemma 2: let M be a cpt Riemannian mfd, $\partial M = \phi$, $Ric(M) \geq 0$.
φ be the eigenfunction of λ_1 on M, assume $a-1 = \inf_M \varphi$, $a+1 = \sup \varphi$, $0 \leq a < 1$

by setting $u = \phi - a$ we have

$$|\nabla u|^2 \leq \lambda(1-u^2) + 2a\lambda z(u), \quad z(u) = \frac{2}{\pi}(\sin^{-1}(u) + u\sqrt{1-u^2}) - u \quad (\text{lemma } 1)$$

(pf) let $u = \varepsilon(\phi - a)$ where $0 < \varepsilon < 1$

we have $\Delta u = -\lambda(u + \varepsilon a)$ and $-\varepsilon \leq u \leq \varepsilon$

Consider the function $Q = |\nabla u|^2 - c(1-u^2) - 2a\lambda z(u) \overset{④}{\leq} |\nabla u|^2 - (c + a\lambda)|z(u)|$

so we can find c large s.t. $\sup_M Q = 0$

if $c \leq \lambda$ then let $\varepsilon \to 1$. then done!, assume $c > \lambda$

claim: $|\nabla u(x_0)| > 0$, where x_0 be the maximum points of Q

if not, then $0 = Q(x_0) = -c(1-u^2)(x_0) - 2a z(u(x_0))$
$\leq -(c - a\lambda)(1-\varepsilon^2)$ —×—

by the maximum principle at x_0, $\nabla Q = 0$, $\Delta Q \leq 0$

$\Rightarrow \quad 0 = \frac{1}{2}Q_i = u_m u_{mi} + c u u_i - a\lambda \dot{z} u_i$

rotate the frame s.t. $u_1(x_0) = |\nabla u(x_0)|$

$\Rightarrow \quad u_{mi} u_{mi} \geq u_{11}^2 = (cu - a\lambda\dot{z})^2$

$$\begin{aligned}0 \geq \frac{1}{2}\Delta Q(x_0) &= u_{mi}u_{mi} + u_m u_{mii} + c u_i^2 + cu\Delta u - a\lambda\ddot{z}u_1^2 - a\lambda\dot{z}\Delta u \\ &\geq (cu - a\lambda\dot{z})^2 + u_m(\Delta u)_m + (c - a\lambda\ddot{z})u_1^2 + (cu - a\lambda\dot{z})\Delta u \\ &\geq (cu - a\lambda\dot{z})^2 + (c - \lambda - a\lambda\ddot{z})[c(1-u^2) + 2a\lambda z] - \lambda(cu - a\lambda\dot{z})(u + \varepsilon a)\end{aligned}$$

$$\begin{aligned}= &-ac\lambda\{(1-u^2)\ddot{z} + u\dot{z} + \varepsilon u\} + a^2\lambda^2\{-2z\ddot{z} + \dot{z}^2 + \varepsilon\dot{z}\} \\ &+ a\lambda(c-\lambda)\{-u\dot{z} + 2z + 1\} + (c-\lambda)(c - a\lambda)\end{aligned}$$

Lemma 1.

$$\Longrightarrow \quad 0 \ge ac\lambda(1-\varepsilon)u - a^2\lambda^2(1-\varepsilon)\dot{z} + (c-\lambda)(c-a\lambda)$$

$$\ge -ac\lambda(1-\varepsilon) - a^2\lambda^2(1-\varepsilon)\left(\frac{4}{\pi}-1\right) + (c-\lambda)(c-a\lambda)$$

$$\ge -(c-\lambda)\lambda(1-\varepsilon) + (c-\lambda)^2.$$

$$\Rightarrow \quad c \le \lambda\left\{\frac{2+(1-\varepsilon)+\sqrt{(1-\varepsilon)(9-\varepsilon)}}{2}\right\} \qquad \text{let } \varepsilon \to 1$$

#

Thm 5: M be a cpt Riemannian mfd, $\partial M = \phi$, $Ric(M) \ge 0$

then $\lambda_1 \ge \frac{\pi^2}{d^2}$ where $d = diam(M)$.

(pf) $$\lambda_1^{\frac{1}{2}} \cdot d \ge \lambda_1^{\frac{1}{2}} \cdot \int_\gamma ds \ge \int_\gamma \frac{|\nabla u|\, ds}{\sqrt{1-u^2+2az(u)}} \ge \int_{-1}^{1} \frac{du}{\sqrt{1-u^2+2az(u)}}$$

$$= \int_0^1 \left\{ \frac{1}{\sqrt{1-u^2+2az}} + \frac{1}{\sqrt{1-u^2-2az}} \right\} du = \int_0^1 \frac{1}{\sqrt{1-u^2}} \left\{ \left(1+\frac{2az}{1-u^2}\right)^{-\frac{1}{2}} + \left(1-\frac{2az}{1-u^2}\right)^{-\frac{1}{2}} \right\}$$

$$\ge \int_0^1 \frac{1}{\sqrt{1-u^2}} \left\{ 2 + \frac{3a^2z^2}{(1-u^2)^2} + \cdots \right\} \ge \pi$$

#

陳孝中

Theorem. The complement of the image of the Gauss map of a non-flat complete minimal surface M in $\mathbb{R}^3$ <u>contains</u> at most 6 points of S^2.

proof) Suppose the Gauss map N of M misses 7 points.

As in the proof of Osserman, we may assume that $M = D$ or $\mathbb{C}$, and the metric is $\lambda(z)d|z|^2$ with $\lambda = |f|^2(1+|g|^2)^2$, where f and g are holomorphic, $|f|>0$, as in the Weierstrass representation. $g = p \circ N : M \longrightarrow \mathbb{C}$.
g has no pole means that the north pole N_0 in S^2 is omitted.
The theorem then becomes:

Let f, g be holomorphic functions on $M = D$ or $\mathbb{C}$. $|f|>0$. Suppose that $g(z) = a_i$ has no solution for $i = 1, \dots, 6$, where a_i's are 6 distinct complex number. Then the metric $|f|^2(1+|g|^2)^2$ on M is not complete.

Let $h = f^{\frac{-2}{p}} g' \prod_{i=1}^{6} (g-a_i)^{-\alpha}$, where $\frac{5}{6} < \alpha < 1$, $p = \frac{5}{6\alpha}$, $\alpha = 1 - \frac{1}{k}$ for some $k \in \mathbb{N}$

(Here $|f^{\frac{-2}{p}}|>0$, $|(g-a_i)|>0$, and M is simply connected, so $f^{\frac{-2}{p}}$, $(g-a_i)^{-\alpha}$ is well-defined.)

The Laplace-Beltrami operator Δ of this metric is given by $(\frac{1}{\lambda})\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$.
Hence the function $u = |h|$ satisfies $\Delta \log u = 0$ almost everywhere in M.
(g' may vanishes on a discrete set in M.)

Claim: $u \notin L^p(M)$.

If u is a constant, consider $L(\partial B_r(0))$, which is decreasing as r decreases.
So $A(B_a(0)) = \int_{r=0}^{a} L(\partial B_r(0))\,dr \nearrow \infty$ as a goes to ∞.
So $\int_M u^p = u^p A(M) = \infty$. (By Cartan Hadamard, the exponential map at 0, $\exp_0$ is an diffeomorphism of $\mathbb{R}^2$ and M, under the normal coordinate, $\partial B_r(0)$ is just the circle of center 0, radius r in $\mathbb{R}^2$).

If u is not constant, this follows from Yau's theorem:
Let M be a complete Riemannian manifold with infinite volume and u a non-negative function satisfying $\Delta \log u = 0$ a.e.. Then $\int_M u^p = \infty$ for $p > 0$.

Since the area element is $\lambda\, dxdy$, the condition $u \notin L^p(M)$ is equivalent to:

$$\int_M \frac{|g'|^p (1+|g|^2)^2}{\prod_{j=1}^6 |g-a_j|^{p\alpha}}\, dxdy = \infty .$$

It remands to show that the integral is actually finite.

Let $D_j = \{ z\in D \mid |g(z)-a_j| \le l \}$,

where $0 < l < \frac{1}{4} \min_{j\neq k \in \{1,\dots,6\}} \{ |a_j - a_k|, 4 \}$.

$M = D$: If $M = \mathbb{C}$, consider

$$\begin{array}{ccc} & \overset{\exists \tilde g}{\nearrow} & D \\ & & \big\downarrow \text{ the universal covering} \\ \underset{\text{simply connected}}{\mathbb{C}} & \xrightarrow{\ g\ } & \mathbb{C}\setminus\{a_1, a_2\} \end{array}$$

$\Rightarrow \tilde g$ is bounded $\Rightarrow \tilde g =$ constant $\Rightarrow M$ is a plane in $\mathbb{R}^3$.

Let $D' = D \setminus \bigcup_{j=1}^6 D_j$, $\quad H = \dfrac{|g'|^p (1+|g|^2)^2}{\prod_{j=1}^6 |g-a_j|^{p\alpha}}$.

Then $\int_D H\, dxdy = \sum_{j=1}^6 \int_{D_j} H\, dxdy + \int_{D'} H\, dxdy$

On D_j, $\quad H \le \dfrac{|g'|^p (1 + (|a_j| + l)^2)^2}{|g-a_j|^{p\alpha} (l - \frac{1}{4})^5} \le C \dfrac{|g'|^p}{(|g-a_j|^{\alpha} + |g-a_j|^{2-\alpha})^p}$

$(|g-a_j| \le 1,\ \alpha < 1 \Rightarrow |g-a_j|^{\alpha} \ge |g-a_j|^{2-\alpha})$

Lemma: Let g be a holomorphic function in D, $f \neq 0, a$. $\alpha = 1 - \frac{1}{k}$

Then $\dfrac{|f'|}{|f|^{\alpha} + |f|^{2-\alpha}} \in L^p(D)$ for every $0 < p < 1$.

So $\int_{D_j} H\, dxdy \le C \int_D \dfrac{|g'|^p}{(|g-a_j|^{\alpha} + |g-a_j|^{2-\alpha})^p}\, dxdy < \infty$, by the lemma.

$$\frac{(1+|g|^2)^2 |g|^{\frac{1}{6}}}{\prod_{j=1}^5 |g-a_j|^{p\alpha}} = \frac{(1+|g|^2)^2 |g|^{\frac{1}{6}}}{\prod_{j=1}^5 |g-a_j|^{\frac{5}{6}}} \text{ is bunded over } D'.$$

So $\int_{D'} H\, dxdy \le C \int_{D'} \dfrac{|g'|^p}{|g-a_6|^{p\alpha} |g|^{\frac{1}{6}}} \le C \int_{D'} \dfrac{|g'|^p}{(|g-a_6|^{\alpha} + |g-a_6|^{2-\alpha})^p} < \infty$.

(May take $p < \frac{11}{12}$)

Theorem (Yau) Suppose $\Delta \log u = 0$ a.e. M : complete.

Then $\int_M u^p = \infty$ for $p > 0$, unless u is a constant.

proof) We may assume $p = 1$ (Since $\Delta \log u^p = p \cdot \Delta \log u = 0$)

Compute $\Delta \log u$ locally, under a geodesic coordinate $\{x^i\}$:

$$\Delta \log u = \sum_{i=1}^n \partial_i \frac{\partial_i u}{u} = \sum_{i=1}^n \left(\frac{\partial_i^2 u}{u} - \frac{(\partial_i u)^2}{u^2} \right) = \frac{\Delta u}{u} - \frac{|du|^2}{u^2}.$$

Set $v_\varepsilon = (u+\varepsilon)^{\frac{1}{2}}$, then $\Delta \log v_\varepsilon = \frac{1}{2} \Delta \log (u+\varepsilon) = \frac{1}{2}\left(\frac{\Delta(u+\varepsilon)}{(u+\varepsilon)} - \frac{|d(u+\varepsilon)|^2}{(u+\varepsilon)^2} \right)$

$$= \frac{1}{2}\left(\frac{|du|^2}{u(u+\varepsilon)} - \frac{|du|^2}{(u+\varepsilon)^2} \right) = \frac{1}{2}\left(\frac{\varepsilon |du|^2}{u(u+\varepsilon)^2} \right) \geq 0, \text{ i.e.}$$

(1): $\dfrac{\Delta v_\varepsilon}{v_\varepsilon} - \dfrac{|dv_\varepsilon|^2}{v_\varepsilon^2} \geq 0 \Rightarrow v_\varepsilon \geq |dv_\varepsilon|^2$.

Let $0 < R_1 < R_2$, r be the distance function form a fixed point p.

Then there exists a Lipschitz continuous function w such that

for some constant $C > 0$, $|dw| \leq \dfrac{C}{R_2 - R_1}$, $0 \leq w(x) \leq 1$,

and $\begin{cases} w(x) = 1 & \text{for } x \in B(R_1) \\ w(x) = 0 & \text{for } x \in M \setminus B(R_2) \end{cases}$, where $B(R)$ denotes $B_R(p)$

(May set $w = \varphi\left[\dfrac{r + R_2 - 2R_1}{R_2 - R_1}\right]$, where φ is a smooth function

on $\mathbb{R}$ with $0 \leq \varphi(t) \leq 1$, $\varphi(t) = 1$ for $t \leq 1$, $\varphi(t) = 0$ for $t \geq 2$, $|\varphi'(t)| \leq C$.)

$$\Delta v_\varepsilon = d^* d v_\varepsilon = - * d * d v_\varepsilon$$

$$\int_{B(R_2)} dv_\varepsilon \wedge * d(w^2 v_\varepsilon) = -\int_{B(R_2)} * d v_\varepsilon \wedge d(w^2 v_\varepsilon) = -\int_{B(R_2)} d * d v_\varepsilon \wedge (w^2 v_\varepsilon)$$

Since $\int_{\partial B(R_2)} * dv_\varepsilon \wedge w^2 v_\varepsilon = 0$ ($w = 0$ out side $B(R_2)$).

So $\int_{B(R_2)} w^2 |dv_\varepsilon|^2 \underset{(1)}{\leq} \int_{B(R_2)} w^2 v_\varepsilon \Delta v_\varepsilon = -\int_{B(R_2)} dv_\varepsilon \wedge * d(w^2 v_\varepsilon)$

$$= -\int_{B(R_2)} dv_\varepsilon \wedge * \left((2 w v_\varepsilon dw) + w^2 dv_\varepsilon \right) = -\int_{B(R_2)} 2 v_\varepsilon dv_\varepsilon \wedge * w dw$$

$-\int_{B(R_2)} w^2 |dv_\varepsilon|^2$. Hence

$$2\int_{B(R_2)} w^2 |dv_\varepsilon|^2 \le -2\int_{B(R_2)} v_\varepsilon dv_\varepsilon \wedge * w\, dw \le 2\int_{B(R_2)} v_\varepsilon w |dv_\varepsilon||dw|$$

$$\le \int_{B(R_2)} \left(v_\varepsilon^2 |dw|^2 + w^2 |dv_\varepsilon|^2 \right)$$

Since $|dw| \le \frac{C}{R_2 - R_1}$,

$$\frac{1}{4}\int_{B(R_2)} \frac{w^2 |du|^2}{u+\varepsilon} = \int_{B(R_2)} w^2 |dv_\varepsilon|^2 \le \frac{C^2}{(R_2-R_1)^2}\int_{B(R_2)} v_\varepsilon^2$$

Let $\varepsilon \longrightarrow 0$; we have

$$\frac{1}{4}\int_{B(R_2)} \frac{w^2 |du|^2}{u} \le \frac{C^2}{(R_2-R_1)^2}\int_{B(R_2)} u$$

Suppose $\int_M u < \infty$. Let $R_2 = 2R_1 \longrightarrow \infty$.

Then $\int_M \frac{|du|^2}{u} = \lim_{R_1 \to \infty} \int_{B(R_1)} \frac{w^2 |du|^2}{u} \le \frac{4C^2}{(R_2-R_1)^2}\int_{B(R_2)} u < \infty.$

So $\left(\int_M |du|\right)^2 \le \int_M \frac{|du|^2}{u} \int_M u < \infty$

Let $u_\varepsilon = u+\varepsilon$, then

$$\Delta \log u_\varepsilon = 2\Delta \log v_\varepsilon = \frac{\varepsilon}{u(u+\varepsilon)^2} |du|^2$$

Since $\int_M |d \log u_\varepsilon| = \int_M \frac{|du|}{u+\varepsilon} \le \int_M \frac{|du|}{\varepsilon} < \infty$,

$*d \log u_\varepsilon$ is integrable.

So $0 = \lim_{i \to \infty} \int_{B_i} \Delta \log u_\varepsilon = \int_M \frac{\varepsilon}{u(u+\varepsilon)^2} |du|^2$, for all $\varepsilon > 0$.

Here B_i's are as in lemma 1.

So $\int_M \frac{|du|^2}{u(u+\varepsilon)^2} = 0 \Rightarrow du \equiv 0 \Rightarrow u$ must be a constant.

Lemma 1: Let w be a smooth integrable $n-1$ form defined on M^n. Then there exists a sequence of domain B_i in M^n such that $M^n = \bigcup_i B_i$, $B_i \subset B_{i+1}$ and $\lim_{i\to\infty} \int_{B_i} dw = 0$.

proof) Let r be the Lipschitz function defined on M^n be the function of the distance from a fixed point p. $B(R) \doteq B_R(p)$ as before. Then we can find a non-negative smooth function g_R such that

(1) For all but a finite number of $t < R$, $g_R^{-1}(t)$ is a compact regular hypersurface.

(2) $|dg_R| \leq \frac{3}{2}$ on $g_R^{-1}([0,R])$

(3) $g_R^{-1}(t) \subseteq B(t+1) \setminus B(t-1)$ for $t \leq R$

(In our case, $M = D$ is complete, with nonpositive curvature, by Cartan-Hadamard theorem, the tangent space of D at O is diffeomorphic to D by the exponential map at O, $\exp_0$. And under this coordinate, $r^{-1}(t)$ is just the circle centered at O with radius t; so we may take $g_R = r$ in the following)

$$\int_{g_R^{-1}([0,R])} |dg_R||w| = \int_0^R \left(\int_{g_R^{-1}(t)} |w|\right) dt$$

Then by (2), $\int_0^R \left(\int_{g_R^{-1}(t)} |w|\right) dt \leq \frac{3}{2}\int_M |w|$

Therefore, for some $\frac{R}{2} \leq t_R \leq R$, where $g^{-1}(t_R)$ is compact regular hypersurface,

$$\int_{g_R^{-1}(t_R)} |w| \leq \frac{3}{R}\int_M |w|.$$

By the Stokes' theorem, $\left|\int_{g_R^{-1}([0,t_R])} dw\right| \leq \int_{g_R^{-1}(t_R)} |w| \leq \frac{3}{R}\int_M |w|$.

By (3), $M^n = \bigcup_{i=1}^{\infty} g_i^{-1}([0,t_i])$

and $\lim_{i\to\infty} \int_{g_i^{-1}([0,t_i])} dw = 0$.

Definition. Let $f(z)$ be a holomorphic function on D. We say $f(z)$ is normal if the family $\{g(S(z)) \mid S :$ comformal transformation of D into $D\}$ is normal, i.e. for every sequence in it, there is a subsequence that converges uniformly on compact subsets of D.

Theorem (Montel) If $f(z)$ is bounded, $f(z)$ is normal.

proof) If $|f| < N$, $K \subseteq D$, let $\varepsilon > 0$ with $d(\partial D, K) > \varepsilon$.
Then $f'(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^2} dz$, where γ is the circle centered at a with radius ε. So $f'(z)$ is bounded on K, say, $|f'| < N'$.
So N' is a Lipchitz constant for f on K. Hence $\{f(S(z))\}$ is equicontinuous, bounded, and then by Arzela Ascoli theorem, $\{f(S(z))\}$ is normal on $K \Rightarrow f$ is normal. ($D = \bigcup_{i=1}^{\infty} K_i$ for some K_i : compact)

Theorem (Montel) If $f(z)$ omit two values, $f(z)$ is normal

proof) Consider

$$\begin{tikzcd} & D \arrow[d, "\text{universal covering}"] \\ D \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{a,b\} \end{tikzcd}$$

(D: simply connected)

$\tilde{f}$ is bounded $\Rightarrow \tilde{f}$ is normal $\Rightarrow f$ is normal.

Theorem 1. If $f(z)$ is normal, then $\left\{ \frac{|(f \circ S)'(z)|}{1 + |f(S(z))|^2} \right\}$ is uniformly bounded on any compact subset of D.

proof) Suppose not. Let E be a compact subset in D, $\{z_n\}$ be a sequence in E and $f_n \in \left\{ \frac{(f \circ S)'(z)}{1 + |f(S(z))|^2} \right\}$, such that

$$\frac{|f_n'(z_n)|}{1 + |f_n(z_n)|^2} \longrightarrow \infty, \text{ as } n \text{ increases.}$$

Since E is compact, by taking subsequences if necessary, we may assume that $z_n \longrightarrow z_0 \in E$, and $f_n(z_n) \longrightarrow w$ (w may $= \infty$)

f is normal, so there is a subsequence $f_{n_p}(z)$ converges uniformly to $g(z)$ in $\{z \mid |z - z_0| \leq \delta\}$, where $\delta \leq d(\partial D, z)$.

So that $f'_{n_p}(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(t)}{(t-z)^2}dt \longrightarrow g'(z)$ uniformly in $\{z \mid |z-z_0| \le \frac{1}{2}\delta\}$

Thus $f'_{n_p}(z) = f'(z) + o(1)$, $f_{n_p}(z) = f(z) + o(1)$ (the $o(1)$ goes to 0 uniformly on $\{z \mid |z-z_0| \le \frac{1}{2}\delta\}$

$$\text{So} \quad \frac{|f'_{n_p}(z)|}{1+|f_{n_p}(z)|^2} = \frac{|f'(z)| + o(1)}{1+(|f(z)|+o(1))^2} = \frac{|f'(z)|}{1+|f'(z)|^2} + o(1) = O(1)$$

uniformly as $p \to \infty$, on $\{z \mid |z-z_0| \le \frac{1}{2}\delta\}$. $\times$

By the theorem above, there is a constant B, such that

$$\frac{|f_n'(0)|}{1+|f_n(0)|^2} \le B, \quad \text{for } f_n(z) = f(S(z))$$

Let $f_n(z) = f\left(\frac{z_0+z}{1+\bar{z}_0 z}\right)$, then $\frac{|f'(z_0)|(1-|z_0|^2)}{1+|f(z_0)|^2} \le B$, for $z_0 \in D$.

Lemma: Let f be a holomorphic function in D, and $f \ne 0, a$.

Let $\alpha = 1 - \frac{1}{k}$, $k \in \mathbb{N}$. Then we have

$$\frac{|f'|}{|f|^\alpha + |f|^{2-\alpha}} \in L^p(D) \quad \text{for every } 0<p<1.$$

proof) Since f omits 0, D is simply connected, so $f^{\frac{1}{k}}$ can be defined.

$k \in \mathbb{N} \Rightarrow f^{\frac{1}{k}}$ must omit two points, so $f^{\frac{1}{k}}$ is normal.

By the above theorem, there is a constant C such that

$$\frac{|(f^{\frac{1}{k}})'|}{1+|f^{\frac{1}{k}}|^2} \le \frac{C}{1-|z|^2}$$

$$\text{so} \quad \frac{|f'|}{k|f|^{1-\frac{1}{k}}(1+|f|^{\frac{2}{k}})} \le \frac{C}{1-|z|^2} \Rightarrow \frac{|f'|}{|f|^\alpha + |f|^{2-\alpha}} \le \frac{kC}{1-|z|^2}$$

$$\int_D \frac{1}{(1-|z|^2)^p} = 2\pi \int_{r=0}^1 \frac{1}{(1-r^2)^p} r\,dr = \pi \int_{r=0}^1 \frac{1}{(1-r^2)^p} d r^2 = \frac{-\pi(1-r^2)^{1-p}}{1-p}\Bigg|_{r=0}^1 = \frac{\pi}{1-p} < \infty.$$

The Kazhdan-Warner problem on prescribing curvature on surfaces.

林冠宇

given (M, g) : a $n(\geq 2)$-dimensional smooth Riemannian manifold,

we say another Riemannian metric $\tilde{g}$ on M is pointwise conformal to g

if $\exists\ 0 < \rho \in C^\infty(M)$, s.t. $\tilde{g} = \rho g$;

denote by $C_g = \{ \rho g \mid \rho \in C^\infty(M), \rho > 0 \}$,

we are interested in the following question :

given arbitrary function $K \in C^\infty(M)$, does there exists $\tilde{g} \in C_g$ s.t.

the scaler curvature $\tilde{R}$ w.r.t. $\tilde{g}$ has $\tilde{R} = K$?

- some computations :

from $\tilde{g} = \rho g$, $\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})$,

and $R_{ij} = \sum_t (\partial_t \Gamma_{ij}^t - \partial_j \Gamma_{it}^t) + \sum_{t,s} (\Gamma_{ij}^s \Gamma_{st}^t - \Gamma_{it}^s \Gamma_{sj}^t)$, we'll have

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2} \left(\delta_{ik} \partial_j \log \rho + \delta_{jk} \partial_i \log \rho - g_{ij} \sum_l g^{kl} \partial_l \log \rho \right),$$

$$\tilde{R}_{ij} = R_{ij} - \frac{n-2}{2} \partial_{ij} \log \rho + \frac{n-2}{4} (\partial_i \log \rho)(\partial_j \log \rho) - \frac{1}{2} \left(\Delta \log \rho + \frac{n-2}{2} |\nabla \log \rho|^2 \right) g_{ij}$$

(where $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_i (\sqrt{g}\, g^{ij} \partial_j)$, $g = \det(g_{ij})$)

hence
$$\tilde{R} = \sum_{ij} \tilde{R}_{ij} \tilde{g}^{ij} = \sum_{ij} \rho^{-1} \tilde{R}_{ij} g^{ij}$$
$$= \frac{1}{\rho} R - \frac{n-1}{\rho^2} \Delta \rho - \frac{(n-1)(n-6)}{4\rho^3} |\nabla \rho|^2$$

- in the case $n=2$,

substitute $\rho = e^{2u}$, obtain $\tilde{R} = e^{-2u}(R - 2\Delta u)$;

also, since scalar curvature $R = 2\cdot K = 2\cdot($ Gaussian curvature $)$,

$$\therefore \quad \Delta u - K + \tilde{K} e^{2u} = 0 \quad \ldots\ldots\ldots \quad (*)$$

so our problem becomes solving the semilinear PDE $(*)$ with given $K, \tilde{K} \in C^\infty(M)$

in what follows, we assume M^2 is compact without boundary

- a necessary condition:

by Gauss-Bonnett, $2\pi\chi(M) = \int_M K\,d\mu = \int_M \tilde{K}\,d\tilde{\mu} = \int_M \tilde{K} e^{2u}\,d\mu$

(where $d\mu$, $d\tilde{\mu}$ are volume form w.r.t. metric g and $\tilde{g} = e^{2u} g$).

so a necessary condition on $\tilde{K}$ is given by:

$$\begin{cases} (1)\ \tilde{K} < 0 \text{ somewhere}, & \text{if } \chi(M) < 0 \\ (2)\ \tilde{K} \equiv 0 \text{ or } \tilde{K} \text{ changes sign}, & \text{if } \chi(M) = 0 \\ (3)\ \tilde{K} > 0 \text{ somewhere}, & \text{if } \chi(M) > 0 \quad (= 2\ (S^2) \text{ or } 1\ (\mathbb{RP}^2)) \end{cases}$$

are (1)(2)(3) sufficient?

Case I : $\chi(M) < 0$:

in this case, the existence problem of $(*)$ has not been completely solved, but we've a relatively good understanding for the problem.

<prop. 1> define $Tu := \Delta u + f(x,u)$ for some $f \in C^\infty(M \times \mathbb{R})$

(in our case, $f(x,u) = -K(x) + \tilde{K}(x) e^{2u}$)

if $\exists\ \phi, \psi \in C^2(M)$ s.t. $T\phi \geq 0$, $T\psi \leq 0$, $\phi \leq \psi$, then

$\exists\ u \in C^\infty(M)$, $\phi \leq u \leq \psi$ s.t. $Tu = 0$

if we assume <prop. 1>, then we'll have the followings :

<prop. 2> the equation $(*)$ can be solved if $\exists$ a sup-solution of $(*)$.

i.e., $\exists\ \psi \in C^2(M)$ s.t. $\Delta\psi - K + \tilde{K} e^{2\psi} \geq 0$

<pf> denote by $(K)_M$ the average of K in M, i.e. $(K)_M = \dfrac{\int_M K\, d\mu}{\int_M d\mu}$.

then $\because (K-(K)_M)_M = 0$, by Hodge decomposition, $\exists\ f \in C^\infty(M)$ s.t. $\Delta f = K - (K)_M$;

let $\phi := f - c$, for some constant c, then

(1) $\because M$ is compact, $\therefore \phi \leq \psi$ for some c large enough ;

(2) $T\phi = \Delta\phi - K + \tilde{K} e^{2\phi} = \Delta f - K + \tilde{K} e^{2f-2c} = \underbrace{-(K)_M}_{} + \tilde{K} e^{2f-2c} > 0$

$(<0)\ (\because \chi(M) < 0)$

for c large enough

by (1), (2), ϕ is a sub-solution of $(*)$ if c is chosen large enough.

hence by <prop. 1>, $\exists\ u \in C^\infty(M)$ s.t. u solves $(*)$ #

<Thm. 3> if $\tilde{K} \le 0$ but $\tilde{K} \not\equiv 0$, then (*) has a solution $u \in C^\infty(M)$

<pf> by <prop. 2>, we only need to construct a sup-solution ψ of (*);

let $f \in C^\infty(M)$ solves $\Delta f = (\tilde{K})_M - \tilde{K}$, and consider $\psi = af + b$;

(1) $\because (\tilde{K})_M < 0$, $\therefore \exists\, a > 0$ s.t. $a(\tilde{K})_M < K(x)$, $\forall x \in M$

(2) $\Delta \psi - K + \tilde{K} e^{2\psi} = a \Delta f - K + \tilde{K} e^{2af+2b}$

$$= \underbrace{\left(a(\tilde{K})_M - K\right)}_{<0} + \left(e^{2af+2b} - a\right) \underbrace{\tilde{K}}_{\le 0}$$

by choosing b large s.t. $e^{2af+2b} - a > 0$, we'll have $\Delta \psi - K + \tilde{K} e^{2\psi} < 0$.

hence $\psi = af + b$ is a sup-solution of (*) #

now we prove <prop. 1>

<pf> (1) find $A > 0$ s.t. $-A \le \phi \le \psi \le A$; then $M \times [-A, A]$ compact,

find $c > 0$ s.t. $F(x,t) = f(x,t) + ct \nearrow$ in t, $\forall$ fixed $x \in M$

(2) define $Lu := -\Delta u + cu$, then L is 1-1 with compact inverse $C^{0,\alpha} \xrightarrow{L^{-1}} C^{2,\alpha}$;

also $Lv_1 \ge Lv_2$ implies $v_1 \ge v_2$, $\therefore L$ is positive

(if $Lv \ge 0$, let $u := \min\{v, 0\}$, then $0 \ge \int_M \underbrace{(Lv)}_{\ge 0}\underbrace{(u)}_{\le 0}\, d\mu = \int_M (|\nabla u|^2 + cu^2)\, d\mu \ge 0 \Rightarrow u = 0$)

(3) $\phi_0 := \phi$, $\psi_0 := \psi$, and define ϕ_k, ψ_k by solving $L\phi_k = F(x, \phi_{k-1})$, $L\psi_k = F(x, \psi_{k-1})$.

then (i) $L\phi_1 - L\phi_0 = F(x, \phi_0) - (-\Delta\phi_0 + c\phi_0) = \Delta\phi_0 + f(x, \phi_0) = T\phi \ge 0 \Rightarrow \phi_1 \ge \phi_0$;

similarly, $L\psi_1 - L\psi_0 = T\psi \le 0 \Rightarrow \psi_1 \le \psi_0$,

$L\psi_1 - L\phi_1 = F(x, \psi_0) - F(x, \phi_0) \ge 0$ since $\psi_0 \ge \phi_0$ and the way we choose F

(ii) inductively, $L\phi_k - L\phi_{k-1} = F(x, \phi_{k-1}) - F(x, \phi_{k-2}) \ge 0$, and hence

$\phi \le \phi_1 \le \cdots \le \phi_k \le \psi_k \le \cdots \le \psi_1 \le \psi$, write $\phi_k \longrightarrow \underline{u}$, $\psi_k \longrightarrow \bar{u}$

(4) since $\{\phi_k\}_{k \ge 1}$ is bounded, M is compact, it's bounded in $W^{2,p}(M)$; then by Morrey's inequality, bounded in $C^{1,\alpha}(M)$, so $\phi_k \longrightarrow \underline{u}$ in $C^{1,\alpha}(M)$, hence

$L\underline{u} = F(x, \underline{u}) \Longleftrightarrow \Delta \underline{u} + f(x, \underline{u}) = 0$ and by regularity theorem, $\underline{u} \in C^\infty(M)$ #

Case II . $\chi(M) = 0$:

in this case, the existence problem of $(*)$ is completely solved.

we'll need the following lemma and corollary.

<Lemma 4> (Trüdinger)

(M^2, g) : compact w/o boundary. then $\exists\ \beta, C > 0$ s.t.

$u \in H^1(M)$, $\int_M u\, d\mu = 0$, $\int_M |\nabla u|^2 d\mu \le 1 \Rightarrow \int_M e^{\beta u^2} d\mu \le C$

<Cor. 5.> (1) $\exists\ C > 0$, $\eta > 0$ s.t. $u \in H^1(M) \Rightarrow \int_M e^u d\mu \le C e^\wedge \{\eta \|\nabla u\|_2^2 + (u)_M\}$

(2) $u \in H^1(M) \Rightarrow e^u \in L^p(M)$, $\forall p \ge 1$

(3) if $u_i \rightharpoonup u$ in $H^1(M)$, then $e^{u_i} \longrightarrow e^u$ in $L^p(M)$, $\forall p \ge 1$;

also, $\int_M \tilde{K} e^u d\mu$ is continuous w.r.t. $H^1(M)$-weak topology

<Thm. 6> $(*)$ has a smooth solution iff

$\tilde{K} \equiv 0$, or $\tilde{K}$ changes sign and $\int_M \tilde{K} e^{2f} d\mu < 0$, where f solves $\Delta f = K$

<pf> $(\Rightarrow)$ notice that $(K)_M = 0$, hence $\Delta f = K$ can be solved;

now $v := u - f$ has $\Delta v + \tilde{K} e^{2v + 2f} = \Delta u - K + \tilde{K} e^{2u} = 0$,

hence $\int_M \tilde{K} e^{2f} d\mu = -\int_M (\Delta v)(e^{-2v}) d\mu = -2 \int_M e^{-2v} |\nabla v|^2 \le 0$

with equality occurs iff v is constant, which implies $\tilde{K} \equiv 0$ #

<Thm. 6> <pf> ($\Leftarrow$)

(1) when $\tilde{K} \equiv 0$, then $(*)$ becomes $\Delta u - K = 0$, which can be solved since $(K)_M = 0$;

when $\tilde{K}$ changes sign, $S := \{ u \in H^1(M) \mid \int_M u \, d\mu = \int_M \tilde{K} e^{2u+2f} d\mu = 0 \}$

then $S \neq \phi$ since $\tilde{K}$ changes sign (if $\int_M \tilde{K} e^{2u_0+2f} d\mu = 0$, then [illegible] $u = u_0 + c \in S$)

consider $J(u) := \frac{1}{2} \int_M |\nabla u|^2 d\mu$, $(u \in S)$, which is weakly L.S.C.

($\because u_k \rightharpoonup u$ in $H^1(M) \Rightarrow Du_k \rightharpoonup Du$ in $L^2(M) \Rightarrow J(u) \leq \liminf_{k\to\infty} J(u_k)$)

(2) if $(u_k)_{k\geq 1}$ is a minimizing sequence s.t. $J(u_k) \searrow C_0 := \inf_{u\in S} J(u)$.

then $\|\nabla u_k\|_2^2$ is bounded, by Poincaré, $(u_k)_{k\geq 1}$ is bounded in $H^1(M)$.

assume $u_k \rightharpoonup u_0 \in H^1(M)$, then $\because J$ is weakly L.S.C., $\therefore J(u_0) \leq C_0$,

by <Cor.5> (3), $0 = \lim_{k\to\infty} \int_M \tilde{K} e^{2u_k+2f} d\mu = \int_M \tilde{K} e^{2u_0+2f} d\mu$,

and $0 = \lim_{k\to\infty} \int_M u_k \, d\mu = \int_M u_0 \, d\mu$ ($\because u_k \to u$ in $L^1(M)$), $\therefore u_0 \in S$

(3) by Thm. of Lagrange multiplier, $\exists$ constant $\alpha, \beta > 0$ s.t.

$u_0 \in H^1(M)$ is a weak solution to $\Delta u_0 + \alpha + \beta \tilde{K} e^{2u_0+2f} = 0$;

integrate over M, obtain $\alpha |M| = -\beta \int_M \tilde{K} e^{2u_0+2f} d\mu = 0 \Rightarrow \alpha = 0$; also,

$$\underbrace{\beta \int_M \tilde{K} e^{2f} d\mu}_{<0} \overset{(\Delta u_0 + \alpha + \beta \tilde{K} e^{2u_0+2f} = 0)}{=\!=} - \int_M e^{-2u_0} \Delta u_0 \, d\mu = -2 \int_M e^{-2u_0} |\nabla u_0|^2 < 0$$

$\Rightarrow \beta > 0$

(4) now let $v_0 = u_0 + \frac{1}{2} \log \beta \in H^1(M)$, then $\Delta v_0 + \tilde{K} e^{2v_0+2f} = 0$ (weak solution)

by <Cor 5.> (2), $v_0 \in H^1(M)$, $\therefore e^{2v_0} \in L^p(M)$, $\forall p \geq 1$

so by iteration of elliptic regularity theorem, $v_0 \in C^\infty(M)$,

then $u := v_0 + f \in C^\infty(M)$ solves $\Delta u - K + \tilde{K} e^{2u} = 0$ ... $(*)$

#

now we prove <Lemma 4> and <Cor. 5>

<pf> (of <Lemma 4>)

(1) let $(U_i, \phi_i)_{1\le i\le k}$ be a P.O.U. of M with each U_i diffeomorphic to unit disc D.

let $u_i = \phi_i u$. we first prove the case $M = D$

(2) <claim> $\|u_i\|_p \le C_0 \sqrt{p}\, \|\nabla u_i\|_2$ for some constant C_0. $\forall p \ge 2$

<pf> (i) first consider a function $v \in C_0^1(D)$ with the identity

$$v(x) = \frac{1}{2\pi}\int_D (\nabla v)(y)\cdot \frac{x-y}{|x-y|^2}\,dy \quad \text{(as in potential theory)} \quad \text{then}$$

$$|v(x)| \le \frac{1}{2\pi}\left(\int_D |\nabla v(y)|^2 |x-y|^{-\frac{2p}{p+2}}dy\right)^{\frac{1}{p}}\cdot\left(\int_D |x-y|^{-\frac{2p}{p+2}}dy\right)^{\frac{1}{2}}\left(\int_D |\nabla v|^2 dy\right)^{\frac{1}{2}-\frac{1}{p}}$$

$$\because \int_D |x-y|^{-\frac{p}{p+2}}dy \le \int_{|y|\le 2} |y|^{-\frac{2p}{p+2}}dy = 2\pi\int_D^2 r^{\frac{2-p}{2+p}}dr = 2^{\frac{2-p}{2+p}}\cdot\pi\cdot(p+2)$$

$$\text{hence } \int_D |v(x)|^p dx \le \frac{1}{(2\pi)^p}\left(2^{\frac{2-p}{2+p}}\cdot\pi\cdot(p+2)\right)^{\frac{p}{2}}\|\nabla v\|_2^{\frac{p-2}{4}}\cdot\int_D\int_D |\nabla v(y)|^2|x-y|^{-\frac{2p}{p+2}}dy\,dx$$

$$\le \frac{1}{(2\pi)^p}\left(2^{\frac{2-p}{2+p}}\pi\cdot(p+2)\right)^{\frac{p}{2}}\cdot\left(2^{\frac{2-p}{2+p}}\pi\cdot(p+2)\right)\|\nabla v\|_2^{\frac{p-2}{4}+\frac{1}{2}} \le C_1 (p+2)^{\frac{p+2}{2}}\|\nabla v\|_2^{\frac{p}{4}}$$

hence $\|v\|_p \le C_2\sqrt{p}\,\|\nabla v\|_2$, where C_1, C_2 are constants independent of p

(ii) $\because C_0^1(D)$ is dense in $H^1(D)$, $\therefore \|v\|_p \le C_2\sqrt{p}\,\|\nabla v\|_2$. $\forall p\ge 2$ #

(3) now $\|u\|_p \le \sum_{i=1}^k \|u_i\|_p \le C_2\sqrt{p}\,\|\nabla u_i\|_2 \le C_3\sqrt{p}\,(\|\nabla u\|_2 + \|u\|_2) \le C_4\sqrt{p}\,\|\nabla u\|_2$

$$\left(\begin{array}{l} u_i = \phi_i u \Rightarrow \nabla u_i = \phi_i\nabla u + u\nabla\phi_i \\ \therefore |\nabla u_i|^2 \le 2\left(\phi_i^2|\nabla u|^2 + u^2|\nabla\phi_i|^2\right) \\ \Rightarrow \sum_i |\nabla u_i|^2 \le \underbrace{\quad}_{\text{depends on } (\phi_i)_{1\le i\le k}}\left(|\nabla u|^2 + |u|^2\right) \end{array}\right)$$

Poincaré ($\int_M u\,d\mu = 0$)

since $\|\nabla u\|_2 \le 1$, $\int_M e^{\beta u^2}d\mu = \int_M \sum_{k=0}^\infty \frac{(\beta u^2)^k}{k!}d\mu \underset{\text{(convergence)}}{\overset{\text{(dominant)}}{=\!=}} \sum_{k=1}^\infty \frac{\beta^k}{k!}\|u\|_{2k}^{2k} + |M|$

$\le M + \sum_{k=1}^\infty \frac{\beta^k}{k!}\left(C_4\sqrt{2k}\right)^{2k}$, which converges if $2\beta C_4^2 e \ll 1$ #

<pf> (of <Cor. 5>)

(1) (i) if $u \equiv c_0$, then $\int_M e^u d\mu = |M| \cdot e^{c_0}$, $e^\wedge \{ \eta \|\nabla u\|_2^2 + (u)_M \} = e^{c_0}$; choose $C \geq |M|$

(ii) otherwise, let $u_0 := u - (u)_M \leq \beta \left(\frac{u_0}{\|\nabla u_0\|_2} \right)^2 + \frac{1}{4\beta} \|\nabla u_0\|_2^2$

then $\frac{u_0}{\|\nabla u_0\|_2}$ satisfies the assumption in <Lemma 4>, hence

$$\int_M e^u d\mu = e^{(u)_M} \int_M e^{u_0} d\mu \leq e^{(u)_M} \cdot e^{\frac{1}{4\beta}\|\nabla u_0\|_2^2} \int_M e^{\beta \left(\frac{u_0}{\|\nabla u_0\|_2} \right)^2} d\mu$$

$$\leq C e^{\eta \|\nabla u\|_2^2 + (u)_M}, \text{ where } \eta = \frac{1}{4\beta} \quad \#$$

(2) replace u by $pu \in H^1(M)$, obtain $e^u \in L^p(M)$, $\forall p \geq 1$ #

(3) $\because \dim M = 2$, $\therefore H^1(M) \xhookrightarrow{(\text{compact})} L^q(M)$, $\forall 1 \leq q < \infty$ $\left(\frac{1}{p} - \frac{1}{n} = \frac{1}{2} - \frac{1}{2} = 0 \right)$

hence $u_i \longrightarrow u$ in $L^p(M)$, $\forall p \geq 1$, now

$$\int_M \tilde{K} (e^{u_i} - e^u) d\mu = \int_M \tilde{K} \int_0^1 \frac{d}{dt} \left(e^{u + t(u_i - u)} \right) dt \, d\mu$$

$$= \int_0^1 \underbrace{\left(\int_M \tilde{K} e^{u + t(u_i - u)} (u_i - u) d\mu \right)}_{} dt \xrightarrow{i \to \infty} 0 \quad \#$$

$$|\cdot| \leq \underbrace{\|\tilde{K}\|_{L^\infty(M)}}_{\text{bounded}} \underbrace{\left(\int_M e^{2u} e^{2|u_i - u|} d\mu \right)^{\frac{1}{2}}}_{\text{bounded by (1)}} \underbrace{\left(\int_M |u_i - u|^2 d\mu \right)^{\frac{1}{2}}}_{\|u_i - u\|_{L^2(M)} \xrightarrow{i \to \infty} 0}$$

Case III : $\chi(M) > 0$

in this case, either M is the sphere S^2 ($\chi(S^2) = 2$), or the real projective space $\mathbb{RP}^2$ ($\chi(\mathbb{RP}^2) = 1$) :

first consider sphere S^2 with standard metric g_0. then $K \equiv 1$.

so $(*)$ becomes $\underline{\underline{\Delta u - 1 + \tilde{K} e^{2u} = 0}}$; subject to $\int_{S^2} \tilde{K} e^{2u} d\mu = 4\pi$

unlike case I ($\chi(M) < 0$), which asserts that $\tilde{K} \le 0 \Rightarrow (*)$ can be solved, even if $\tilde{K} > 0$ here, $(*)$ may NOT be solved.

in fact, for $\tilde{K} = 1 + \varepsilon\phi$, where $\Delta\phi + 2\phi = 0$ on sphere

choose $\varepsilon \ll 1$ s.t. $\tilde{K} > 0$ on S^2. it can be shown that

$(*)$ can NOT be solved for this $\tilde{K}$

however, with some symmetry condition imposed on $\tilde{K}$, $(*)$ can be solved :

but before proving that, we'll need a lemma on the best constant

η in <Cor. 5> in the case of

<Lemma 8> $\exists C > 0$ s.t. $u \in H^1(S^2) \Rightarrow \int_{S^2} e^u d\mu \le C e^{\wedge}\left\{ \frac{1}{16\pi} \|\nabla u\|_2^2 + (u)_{S^2} \right\}$

if in addition, $u(-x) = u(x)$, $\forall x \in S^2$, then

$\int_{S^2} e^u d\mu \le C e^{\wedge}\left\{ \frac{1}{32\pi} \|\nabla u\|_2^2 + (u)_{S^2} \right\}$ (i.e., best constants $\eta = \frac{1}{16\pi}, \frac{1}{32\pi}$)

< Thm. 9 > if $\tilde{K} \in C^\infty(S^2)$ satisfies $\max_{S^2} \tilde{K} > 0$ and $\tilde{K}(-x) = \tilde{K}(x)$, $\forall x \in S^2$,

then $(*)$ can be solved by some $u \in C^\infty(S^2)$ satisfying $u(-x) = u(x)$, $\forall x \in S^2$

<pf> (1) let $S := \{ u \in H^1(M) \mid \int_M u \, d\mu = 0,\ u(-x) = u(x) \text{ a.e.} \}$

and $S^* := \{ u \in S \mid \int_{S^2} \tilde{K} e^{2u} d\mu > 0 \}$ ($\because \tilde{K} > 0$ somewhere, $\therefore S^* \neq \phi$)

(2) by <Lemma 8>, $\exists C > 0$ s.t. $\int_{S^2} e^{2u} d\mu \le C e^{\wedge} \{ \frac{1}{8\pi} \|\nabla u\|_2^2 \}$, $\forall u \in S$;

hence $J(u) = \frac{1}{2} \|\nabla u\|_2^2 - 2\pi \log \int_{S^2} \tilde{K} e^{2u} d\mu$ $(u \in S^*)$ has

$$J(u) \ge \frac{1}{2} \|\nabla u\|_2^2 - 2\pi \left(\log \left(\max_{S^2} |\tilde{K}| \right) + \log C + \frac{1}{8\pi} \|\nabla u\|_2^2 \right) = \frac{1}{4} \|\nabla u\|_2^2 - C_1,$$

let $(u_i)_{i \ge 1}$ be a minimizing sequence s.t. $J(u_i) \searrow C^* := \inf_{u \in S^*} J(u)$ $(> -\infty)$,

$\because (\|\nabla u_i\|_2^2)_{i \ge 1}$ is bounded, by Poincare, $(u_i)_{i \ge 1}$ is bounded in $H^1(S^2)$,

assume $u_i \rightharpoonup u_0 \in H^1(S^2)$, then $\because J(u)$ is weakly L.S.C.

($\because \int_{S^2} \tilde{K} e^{2u_0} d\mu \ge \limsup_{i \to \infty} \int_{S^2} \tilde{K} e^{2u_i} d\mu$ by <Cor 5.>(3))

$\therefore J(u_0) \le C^*$, and by definition of S^*, $u_0 \in S^*$. hence $J(u_0) = C^*$

(3) now $\forall v \in S$, $u_0 + tv \in S^*$ for all t small, hence

$$\int_{S^2} \nabla u_0 \cdot \nabla v \, d\mu - 2\pi \frac{\int_{S^2} \tilde{K} e^{2u_0} (2v) \, d\mu}{\int_{S^2} \tilde{K} e^{2u_0} d\mu} = 0, \quad \forall v \in S$$

$$\Rightarrow \Delta u_0 + \frac{4\pi \tilde{K} e^{2u_0}}{\int_{S^2} \tilde{K} e^{2u_0} d\mu} = \lambda \text{ for some constant } \lambda \text{; (weak solution)}$$

upon integrating over S^2, obtain $\lambda = 1$.

(4) now $u := u_0 - \frac{1}{2} \log \left(\frac{1}{4\pi} \int_{S^2} \tilde{K} e^{2u_0} d\mu \right)$ has

$$\Delta u - 1 + \tilde{K} e^{2u} = \Delta u_0 - 1 + \tilde{K} e^{2u_0} \cdot \left(\frac{1}{4\pi} \int_{S^2} \tilde{K} e^{2u_0} d\mu \right)^{-1} = 0,$$

i.e. $u \in H^1(M)$ is a weak solution to $(*)$: by

elliptic regularity theorem, $u \in C^\infty(M)$; with $u(-x) = u(x)$, $\forall x \in S^2$ #

upon lifting any $\tilde{K} \in C^\infty(\mathbb{RP}^2)$ to S^2 with $\tilde{K}(-x) = \tilde{K}(x)$,

we immediately obtain the following :

<Cor. 10> on $\mathbb{RP}^2$ with its standard metric, $\tilde{K} \in C^\infty(\mathbb{RP}^2)$ is the Gaussian curvature w.r.t. some Riemannian metric pointwise conformal to the standard one iff $\tilde{K}$ is positive somewhere.

<Remark> $\mathbb{RP}^2$ (with $\chi(M) = 1$) is the only case where the necessary condition (given by Gauss-Bonnett) is also sufficient

A REPORT ON THE BOTT

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Abstract

This report follows the line of John Milnor's book [1]. The Bott periodicity theorem, first proved by Raoul Bott, is a foundational result of K-theory. Bott's original proof made use of Morse theory. Later in [2], the the periodicity theorem was related to the periodicity of Clifford algebra. In this report we briefly review the proof in [1] and use Clifford algebra to interpret the theorem, along the line of [2].

1 Preliminaries

We state without proof the necessary ingredients for the Bott periodicity theorem here. Proofs can be found in [1]. Henceforth we denote by M a C^∞ riemannian manifold.

1.1 Path Spaces

Let M be a C^∞ connected riemannian manifold and denote the distance on M by $\rho(p,q)$ for $p, q \in M$.

Definition 1. *The path space from $p \in M$ to $q \in M$, denoted*

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by $\Omega = \Omega(M;p,q)$ *consists of all piecewise* C^∞ *paths from* p *to* q *parametrised by the unit interval, equipped with the topology defined by the metric, for* $\omega, \omega' \in \Omega(M;p,q)$

$$d(\omega,\omega') = \sup_{0\le t\le 1} \rho(\omega(t),\omega(t')) + \left[\int_0^1 \left(\left|\frac{d\omega}{dt}\right| - \left|\frac{d\omega'}{dt}\right|\right)^2 dt\right]^{1/2}.$$

Definition 2. *An energy functional* E_a^b *on* Ω *is defined by* $\int_a^b \left|\frac{d\omega}{dt}\right|^2 dt$, *for* $0 \le a \le b \le 1$.

It is not difficult to see the path space Ω is a Banach manifold, with tangent space $T_\omega\Omega$ all piecewise C^∞ along ω vanishing at 0 and 1.

Proposition 3. *The critical points of* $E = E_0^1$ *are precisely the geodesics from* p *to* q.

1.2 Morse Theory

Theorem 4. *The set of Morse functions is dense in* $C^\infty(M)$.

The foundation of Morse theory is the following.

Theorem 5. *Let* f *be a* C^∞ *function without degenerate critical points. Then* M *is homotopy equivalent to a CW-complex, each cell of* *which corresponding to a critical point of f, with its dimension equal to the index of the hessian of f at that critical point.*

The following result provide a formula to compute the index of a geodesic.

Theorem 6. *Let $\gamma \in \Omega$ be a geodesic. The index λ of the hessian* $\mathrm{Hess}_\gamma E_0^1$ *has finite index, which equals to the number of* $0 < t < 1$ *such that* $\gamma(t)$ *is conjugate to* $\gamma(0)$*, counted with multiplicity.*

2 Reduction of the path space

The key portion of the proof of Bott periodicity theorem is to study the loop space $\Omega\mathrm{SU}(n)$. There reduction of $\Omega\mathrm{SU}(n)$ is divided into steps that are elaborated in this section.

2.1 Approximation by piecewise C^∞ paths

The metric d makes the energy functional E continuous. Let Ω^* be the (C^0-) path space from p to q with compact-open topology. We have a natural continuous map $\Omega \to \Omega^*$, which is injective.

Lemma 7. *The map* $\Omega \to \Omega^*$ *is a homotopy equivalence.*

We may henceforth identify the two spaces Ω, Ω^* in the homotopy category.

2.2 Change of end points

Lemma 8. *Let* $p, q, p', q' \in M$*. The spaces* $\Omega(M; p, q)$ *is homotopy equivalent to* $\Omega(M; p', q')$*.*

Proof. The proof is straightforward. □

2.3 Approximation by piecewise geodesics

Given a subdivision $\mathbf{t} = (t_0 = 0, t_1, \ldots, t_k = 1)$ of the unit interval $[0, 1]$

Definition 9. *The geodesic path space from $p \in M$ to $q \in M$ with respect to* $\mathbf{t}$, *denoted by $\Omega(\mathbf{t})$ consists of all piecewise geodesic C^∞ paths with respect to* $\mathbf{t}$, *from p to q, parametrised by the unit interval, with the subspace topology from $\Omega(M; p, q)$.*

In order to make it a finite dimensional manifold, we obtain subspaces by limiting the energy E_0^1

Definition 10. *For $c > 0$, let Ω^c be the subspace of Ω consisting of paths $\omega \in \Omega$ with $E_0^1(\omega) \leq c$ and let* $\operatorname{Int}\Omega$ *be the subspace consisting of paths $\omega \in \Omega$ with $E_0^1(\omega) < c$. Denote $\Omega^c(\mathbf{t}) = \Omega^c \cap \Omega(\mathbf{t})$ and* $\operatorname{Int}\Omega^c(\mathbf{t}) = \operatorname{Int}\Omega^c \cap \Omega(\mathbf{t})$.

Proposition 11. *Fix $c > 0$. For all subdivision $\mathbf{t} = (t_0, \ldots, t_k)$ fine enough, the space $\Omega^c(\mathbf{t})$ is a C^∞ manifold of dimension $|\mathbf{t}| - 1 = k - 1$.*

Proof. Take a open covering consisting of geodesically strictly convex open sets (i.e. each pair p, q in the open set is joined by a unique minimal geodesic inside the open set) and let λ be the Lebesgue constant of the covering. If $\mathbf{t}$ is a subdivion of mesh $< \lambda^2/c$, then each path $\omega \in \Omega^c(\mathbf{t})$ is determined uniquely by the points $\omega(t_j)$, $j = 0, \ldots, k$. □

Proposition 12. *For $a < c$, the set $\Omega^a(\mathbf{t})$ is a deformation retract of Ω^a.*

2.4 Approximation by minimal geodesics

Let $d = \rho(p, q)$.

Lemma 13. *Suppose $\Omega^d(M; p, q)$ is a C^0 manifold and that all of the non-minimal geodesics from p to q have indices $\geq \lambda_0$. Then $\pi_i\left(\Omega(M; p, q), \Omega^d(M; p, q)\right) = 0$ for $0 \leq i < \lambda_0$.*

3 Bott periodicity theorem for the unitary group

3.1 Outline of the proof

Equip $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ with the riemannian metric $\langle A, B\rangle = \mathfrak{Re}\,\mathrm{tr}\, AB^*$ on their Lie algebras. Consider the space of geodesics from $I \in \mathrm{SU}(2m)$ to $-I \in \mathrm{SU}(2m)$. The geodesics on $\mathrm{SU}(2m)$ are given by $\exp tA$ for $t \in [0, 1]$ and $A \in \mathfrak{su}(2m)$. In order that $\exp A = -I$, it is necessary and sufficient that each eigenvalue of A be an odd integral multiplicity of $i\pi$. Moreover, in order that $\exp tA$ is a minimal geodesic, it is necessary and sufficient that each eigenvalue of A be $\pm i\pi$. Such matrices are determined by the splits of $(\pm i\pi)$-eigenspaces, both of which are of dimension m since $\mathrm{tr}\, A = 0$. Thus the minimal geodesics from I to $-I$ are in one-to-one correspondence to m-dimensional linear subspaces of $\mathbb{C}^{2m}$. Thus

Lemma 14. *The space of minimal geodesics from I to $-I$ in $\mathrm{SU}(2m)$ is homeomorphic to the complex grassmannian manifold $G_m\left(\mathbb{C}^{2m}\right)$.*

The following lemma exhibit that the non-minimal geodesics are somehow redundant.

Lemma 15. *Let γ be a non-minimal geodesic from I to $-I$ in $\mathrm{SU}(2m)$.*

Then the hessian $\mathrm{Hess}_\gamma(E_0^1)$ *has index* $\geq 2m+2$. *In other words, there are* $\geq 2m+2$ *points along* γ *conjugate to* I

Combining Lemma 13, Lemma 14, and Lemma 15.

Lemma 16. *The inclusion* $G_m\left(\mathbb{C}^{2m}\right) \hookrightarrow \Omega(M;p,q)$ *induces isomorphisms*

$$\pi_i\left(G_m\left(\mathbb{C}^{2m}\right)\right) \cong \pi_i\left(\Omega(\mathrm{SU}(2m);I,-I)\right) \cong \pi_{i+1}\left(\mathrm{SU}(2m)\right),$$

for $i \leq 2m$.

The second isomorphism in the lemma follows from the fact that $\Omega\left(\mathrm{SU}(2m);I,-I\right)$ is homotopy equivalent to the loop space $\Omega\mathrm{SU}(2m)$ and that $\pi_i\left(\Omega\mathrm{SU}(2m)\right) \cong \pi_{i+1}\left(\mathrm{SU}(2m)\right)$.

Using the fibration

$$\mathrm{U}(m) \xrightarrow{\text{upper-left block}} \mathrm{U}(m+1) \to \mathrm{U}(m+1)/\mathrm{U}(m) \approx S^{2m+1},$$

we see $\pi_i\left(\mathrm{U}(m)\right) \cong \pi_i\left(\mathrm{U}(m+1)\right)$ for $i \leq 2m$ through the inclusion. Hence by induction, $\pi_i(U(m) \cong \pi_i(U(M))$ for $i \leq 2m$ and $M \geq m$ through the inclusion into the upper-left block. Using the fibration

$$\mathrm{U}(m) \xrightarrow{\text{upper-left block}} \mathrm{U}(2m) \to \mathrm{U}(2m)/\mathrm{U}(m),$$

we see that $\pi_i\left(\mathrm{U}(2m)/\mathrm{U}(m)\right) = 0$ for $i \leq 2m-1$. Using the fibration

$$\mathrm{U}(m) \xrightarrow{\text{lower-right block}} \mathrm{U}(2m)/\mathrm{U}(m) \to G_m\left(\mathbb{C}^{2m}\right)$$

we then see that $\pi_{i+1}\left(G_m\left(\mathbb{C}^{2m}\right)\right) \cong \pi_i\left(\mathrm{U}(m)\right)$ for $i \leq 2m-1$. Finally, using the fibration

$$\mathrm{SU}(m) \to \mathrm{U}(m) \xrightarrow{\det} S^1$$

we see that $\pi_i\left(\mathrm{SU}(m)\right) \cong \pi_i\left(\mathrm{U}(m)\right)$ for $i \neq 1$.

Put the isomorphisms and Lemma 16 altogether,

Theorem 17 (Bott periodicity theorem). *Let* $\mathrm{U} = \bigcup_{m=1}^{\infty} \mathrm{U}(m)$. *Then*

$$\pi_{i-1}(\mathrm{U}) \cong \pi_{i+1}(\mathrm{U}),$$

for $i \geq 1$.

Remark. *The periodicity theorem can be reformulated as*

$$\pi_i(\mathrm{U}) \cong \pi_i\left(\Omega^2 \mathrm{U}\right) \quad \textit{for } i \geq 0.$$

Hence by Whitehead's theorem, which states that every weak homotopy equivalence between CW-complexes is a homotopy equivalence, we have a homotopy equivalence $\mathrm{U} \sim \Omega^2 U$.

4 Proofs of auxiliary results

4.1 Proof of Proposition 12

Proof. Denote by λ the Lebesgue constant of the geodesically strictly convex covering. Let $\omega \in \Omega^a$. A retraction r can be defined by taking $r(\omega)\big|_{[t_{i-1},t_i]}$ as the unique minimal geodesic from $\omega(t_{i-1})$ to $\omega(t_i)$. We may extend this retraction to a homotopy r_u from the identity map to r. For $t_{i-1} \leq u \leq t_i$, set

$$\begin{cases} r_u(\omega)\Big|_{[0,t_{i-1}]} = r(\omega)\Big|_{[0,t_{i-1}]}, \\ r_u(\omega)\Big|_{[0,t_{i-1}]} = \text{minimal geodesic from } \omega(t_{i-1}) \text{ to } \omega(u), \\ r_u(\omega)\Big|_{[u,1]} = \omega\Big|_{[u,1]}. \end{cases}$$

$\square$

4.2 Proof of Lemma 7

Proof. Let $\imath\colon \Omega \to \Omega^*$ be the natural injective map. We cover M by geodesically strictly convex open sets and denote by $\mathcal{U}$ this covering and by λ the Lebesgue constant of $\mathcal{U}$. For $k \in \mathbb{Z}_{>0}$, let $t_k = \left(0, 1/2^k, \ldots, \jmath/2^k, \ldots, 1\right)$ a partition of I. Let Ω_k and Ω_k^* be the subset of Ω and Ω^* respectively, consisting of paths ω such that $\omega\left[(\jmath-1)/2^k, \jmath/2^k\right]$ is contained in some $U \in \mathcal{U}$, for $\jmath = 1, \ldots, 2^k$.

Define a function $h\colon \Omega_k^* \to \Omega_k$ by setting $h(\omega)\Big|_{[(\jmath-1)/2^k, \jmath/2^k]}$ to be the unique minimal geodesic from $\omega\left((\jmath-1)/2^k\right)$ to $\omega\left(\jmath/2^k\right)$. Then argue as in the proof of Proposition 12 to show h is a homotopy inverse to $\imath\Big|_{\Omega_k}$. Passing to direct homotopy limit $k \to \infty$, we obtain a homotopy inverse $h\colon \Omega^* \to \Omega$ to $\imath$. $\square$

4.3 Proof of Lemma 13

The virtue of the lemma is to apply Morse theory to the energy functional E, to show that Ω^d is the $(\lambda_0 - 1)$-skeleton of Ω. The following result concerning the index of smooth functions provide a base for a smooth function to be approximated by a Morse function.

Lemma 18. *Let $K \subseteq M$ be a compact subset and let f be a C^∞ function defined on M whose critical points in K have index $\geq \lambda_0$. Then for all C^∞ function g sufficiently closed to f in C^2, the critical points of g in K have index $\geq \lambda_0$.*

The proof is straightforward.

Let f be a C^∞ function on a manifold with minimum 0, and suppose that for each $c \geq 0$, $M^c = f^{-1}[0, c]$ is compact.

Lemma 19. *If the set M^0 is a C^0 manifold, and if every critical point in $M \setminus M^0$ has index $\geq \lambda_0$, then $\pi_r(M, M^0) = 0$ for $0 \leq r < \lambda_0$.*

Remark. *Actually, the assumption on the range of f can be dropped.*

Proof. Firstly observe that M^0 is a neighbourhood retract (actually M^0 is an ANR), say $M^0 \subseteq U \subseteq M$. We may assume U is small enough such that U can be deformed to the corresponding point in M^0 via geodesics within M. Let $h\colon \left(I^r, \dot{I}^r\right) \to (M, M^0)$. We must show that h is homotopic rel $\dot{I}^r$ to a map $h''\colon \left(I^r, \dot{I}^r\right) \to (M^0, M^0)$. Let $c = \sup_{h(I^r)} f$. Let $3\delta > 0$ be the minimum of f on $M \setminus U$.

Approximate f by a Morse function g on $M^{c+2\delta}$ in C^2. We shall choose g closed enough to f such that $|f - g| < \delta$ on $M^{c+2\delta}$, and that the critical points of g in the compact set $f^{-1}[\delta, c + 2\delta]$ has index $\geq \lambda_0$. Thus $g^{-1}[2\delta, c + \delta] \subseteq f^{-1}[\delta, c + 2\delta]$, and by Theorem 5, $g^{-1}]\infty, c + \delta]$ has the homotopy type of $g^{-1}]\infty, 2\delta]$ with cells of dimension $\geq \lambda_0$ attached. Via cellular approximation, h is homotopic to a map $h'\colon \left(I^r, \dot{I}^r\right) \to (g^{-1}]\infty, 2\delta], M^0)$. As $g^{-1}]\infty, 2\delta] \subseteq U$ and U can be deformed to M^0, h' is homotopic to a map $h''\colon \left(I^r, \dot{I}^r\right) \to (M^0, M^0)$. □

Proof of Lemma 13. Recall $d = \rho(p, q)$ is the length of minimal geodesic between p, q, so that M^d is the space of minimal geodesics. It suffices to prove $\pi_i\left(\operatorname{Int}\Omega^c, \Omega^d\right) = 0$ for $c \gg 0$. Moreover, according to Proposition 12, we actually need to show $\pi_i\left(\operatorname{Int}\Omega^c(\mathrm{t}), \Omega^d\right) = 0$, where t is a partition of $I = [0, 1]$ such that $\operatorname{Int}\Omega^c(\mathrm{t})$ is a C^∞ manifold, but this follows from Lemma 19. □

4.4 Proof of Lemma 15

In order to prove Lemma 15, we invoke a lemma that counts the index of a geodesic in a *locally symmetric space.*

Definition 20. *A riemannian manifold M is called a locally symmetric space if for every geodesic γ and every parallel vector fields U, V and W along γ, $R(U, V)W$ is also parallel along γ.*

For example, a Lie group with an invariant metric is a locally symmetric space. Now let $\gamma\colon \mathbb{R} \to M$ be a geodesic on a locally symmetric space. Denote $\gamma(0) = p$ and $V = \gamma'(0)$. Define a linear transformation $K_V\colon T_pM \to T_pM$ by $K_V(W) = R(W, V)V$. We have

Lemma 21. *The conjugate points to p along γ are the points $\gamma\left(\pi k/\sqrt{e_i}\right)$ where k is any non-zero integer, and e_i is any positive eigenvalue of K_V. The multiplicity of $\gamma(t)$ as a conjugate point is equal to the number of e_i such that t is a multiple of $\pi/\sqrt{e_i}$.*

Remark. *In view of Theorem 6, the number is exactly the index of γ.*

Proof. First note that K_V is self-adjoint due to the symmetry of Riemann curvature tensor. Thus there is an orthonormal basis $\{U_i\}_{i=1}^n$ for T_pM such that $K_V(U_i) = e_iU_i$, where $e_1, \dots, e_n$ are eigenvalues. We may extend U_i to a vector field along γ by parallel translation. Then $R(V, U_i)V = e_iU_i$ along γ by the local symmetry. Let $W(t) = \sum_i w_iV_i$ be a vector field along γ. Then W is a Jacobi field if and only if

$$\nabla_t\nabla_t W + K_V(W) = \sum_i \frac{d^2w_i}{dt^2}U_i + \sum_i e_iw_iU_i = 0.$$

By the linear independence of $\{U_i\}$, $\frac{d^2w_i}{dt^2} + e_iw_i = 0$. For $e_i > 0$, the equation has solution $c\sin\sqrt{e_i}t$ with a constant $c \in \mathbb{R}$. For $e_i = 0$, the solution is ct with a constant c. For e_i, the solution is $c\sinh\sqrt{-e_i}$ with a constant $c \in \mathbb{R}$. The lemma is now clear from the solution space of the set of equations. $\square$

Proof of Lemma 15. Now return to $\mathrm{SU}(n)$ where $n = 2m$, denoting $\mathfrak{g} = \mathfrak{su}(n)$. Let $\gamma(t) = \exp tA$ be a geodesic from I to $-I$, the index of γ is determined by $K_A : \mathfrak{g} \to \mathfrak{g}$, where $K_A(W) = R(W, A)A = \frac{1}{4}\left[[A, W], A\right]$. A is conjugate to a matrix $\operatorname{diag}(i\pi k_1, \ldots, i\pi k_n)$ where where $k_1 \geq \cdots \geq k_n$ are odd integers. Then for $W = (w_{jl})_{j,l}$,

$$K_A(W) = \left(\frac{\pi^2}{4}\left(k_j - k_l\right)^2 w_{jl}\right)_{j,l}.$$

The differences of matrix basis elements $E_{ij} - E_{ji}$ and $i\left(E_{jl} - E_{lj}\right)$ are eigenvectors of eigenvalue $\frac{\pi^2}{4}\left(k_j - k_l\right)^2$, each counted twice. For each positive eigenvalue $e = \frac{\pi^2}{4}\left(k_j - k_l\right)^2 > 0$, invoking Lemma 21, we see the conjugate points along γ are

$$t = \pi/\sqrt{e}, 2\pi/\sqrt{e}, 3\pi/\sqrt{e}, \cdots = \frac{2}{k_j - k_l}, \frac{4}{k_j - k_l}, \frac{6}{k_j - k_l}, \ldots.$$

Hence the number of conjugate points lying strictly between I and $-I$ along γ is equal to $k_j - k_l - 2$, counted with multiplicity ($= 2$). Thus, summing over j, l, we obtain the index λ of γ

$$\lambda = \sum_{k_j > k_l} \left(k_j - k_l - 2\right).$$

If γ is minimal, then $k_1 = \cdots = k_m = -k_{m+1} = \cdots = -k_{2m} = 1$, so $\lambda = 0$. If γ is non-minimal, by an elementary argument one see that $\lambda \geq 2m + 2$. $\square$

5 Bott periodicity theorem for the orthogonal group

The strategry of the proof is roughly the same as for U. We study the space of minimal geodesics, and claim that the non-minimal geodesics only contribute cells of high dimensions. ALso, construct a fibration over S^1 as in the complex case $\mathrm{SU}(n) \to \mathrm{U}(n) \to S$. However, the proof is rather technical, so we skip many details, and instead, draw a conceptual proof.

5.1 Clifford algebra

We briefly state some results of the Clifford algebra $C_k = C(\mathbb{R}^k)$. For proofs and details, see [2]. Let F denote either of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and let $F(n)$ denote the $n \times n$ matrix algebra over F.

Proposition 22. *The structures of Clifford algebras C_k are given by:*

- $C_0 \cong \mathbb{R}$,
- $C_1 \cong \mathbb{C}$,
- $C_2 \cong \mathbb{H}$,
- $C_3 \cong \mathbb{H} \oplus \mathbb{H}$,
- $C_4 \cong \mathbb{H}(2)$,
- $C_5 \cong \mathbb{C}(4)$,
- $C_6 \cong \mathbb{R}(8)$,
- $C_7 \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$,
- $C_{k+8} \cong C_k \otimes_{\mathbb{R}} \mathbb{R}(16)$ *for* $k \geq 0$.

In particular, C_k is semisimple artinian for $k \geq 0$ and is simple for $k \not\equiv 3 \pmod 4$.

Therefore, all finite dimensional modules are completely reducible. Let $M(C_k)$ be the Grothendieck group of finite dimensional C_k-modules. We have

Proposition 23.

- $M(C_0) \cong \mathbb{Z}$,
- $M(C_1) \cong \mathbb{Z}$,
- $M(C_2) \cong \mathbb{Z}$,

- *$M(C_3) \cong \mathbb{Z} \oplus \mathbb{Z}$,*
- *$M(C_4) \cong \mathbb{Z}$,*
- *$M(C_5) \cong \mathbb{Z}$,*
- *$M(C_6) \cong \mathbb{Z}$,*
- *$M(C_7) \cong \mathbb{Z} \oplus \mathbb{Z}$,*
- *$M(C_{k+8}) \cong M(C_k)$ for $k \geq 0$.*

Let γ be a class of irreducible C_8-module. Multiplication by γ gives an isomorphism $M(C_k) \cong M(C_{k+8})$.

The inclusion $i\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ gives rise to $i\colon C_k \hookrightarrow C_{k+1}$ and $i^*\colon M(C_{k+1}) \to M(C_k)$ in a natural way. Denote by A_k the cokernel of $i^*\colon M(C_{k+1}) \to M(C_k)$. Then

Proposition 24.

- *$A(C_0) \cong \mathbb{Z}_2$,*
- *$A(C_1) \cong \mathbb{Z}_2$,*
- *$A(C_2) \cong 0$,*
- *$A(C_3) \cong \mathbb{Z}$,*
- *$A(C_4) \cong 0$,*
- *$A(C_5) \cong 0$,*
- *$A(C_6) \cong 0$,*
- *$A(C_7) \cong \mathbb{Z}$,*
- *$A(C_{k+8}) \cong A(C_k)$ for $k \geq 0$.*

Let γ be a class of irreducible C_8-module. Multiplication by γ gives an isomorphism $A(C_k) \cong A(C_{k+8})$.

Let V be a real vector bundle over M and denoted by $\Pi(V)$ the Thom space of V. Let $M(V)$ be the Grothendieck group of graded $C(V)$-modules, and let $A(V)$ be the cokernel of the natural homomorphism $M(V \oplus 1) \to M(V)$. Also, denoted by $KO(M)$ the Grothendieck group of real bundles and by $\widetilde{KO}(M)$ the cokernel of $KO(\{\mathrm{pt}\}) \to KO(M)$.

Theorem 25. *There is a group homomorphism $A(V) \to \widetilde{KO}(\Pi(V))$.*

Another form of Bott periodicity states $A_k \cong \widetilde{KO}\,(\Pi(V))$.

Definition 26. *Let V be a finite dimensional C_k-module on which C_k acts isometrically. Define* $\mathrm{Aut}_{C_k} \doteq \varinjlim_V \mathrm{Aut}_{C_k}(V)$.

By the classification Proposition 22 and Morita equivalence between C_k and $C_k \otimes_{\mathbb{R}} \mathbb{R}(n)$, we have

Proposition 27.

- $\mathrm{Aut}_{C_0} \cong \mathrm{O}$,
- $\mathrm{Aut}_{C_1} \cong \mathrm{U}$,
- $\mathrm{Aut}_{C_2} \cong \mathrm{Sp}$,
- $\mathrm{Aut}_{C_3} \cong \mathrm{Sp} \times \mathrm{Sp}$,
- $\mathrm{Aut}_{C_4} \cong \mathrm{Sp}$,
- $\mathrm{Aut}_{C_5} \cong \mathrm{U}$,
- $\mathrm{Aut}_{C_6} \cong \mathrm{O}$,
- $\mathrm{Aut}_{C_7} \cong \mathrm{O} \times \mathrm{O}$,
- $\mathrm{Aut}_{C_{k+8}} \cong \mathrm{Aut}_{C_k}$ *for $k \geq 0$.*

5.2 Idea of the proof

We begin by a definition of isometry groups over Clifford algebras.

Definition 28. *Let V be a finite dimensional C_k-module on which C_k acts isometrically. Define* $\mathrm{Aut}_{C_k} = \varinjlim_V \mathrm{Aut}_{C_k}(V)$.

Since we have $C_k \hookrightarrow C_{k+1}$, there is an inclusion $\mathrm{Aut}_{C_{k+1}}(V) \hookrightarrow \mathrm{Aut}_{C_k}(V)$, passing to colimit $\mathrm{Aut}_{C_{k+1}} \hookrightarrow \mathrm{Aut}_{C_k}$. Let $\Xi_k = \mathrm{Aut}_{C_k} / \mathrm{Aut}_{C_{k+1}}$ be the quotient homogeneous space of orbits. We see from Proposition 22 that $\mathrm{Aut}_{C_{k+8}} / \mathrm{Aut}_{C_{k+9}} \approx \mathrm{Aut}_{C_k} / \mathrm{Aut}_{C_{k+1}}$ by Morita equivalence. The key lemma is the following:

Theorem 29. *(For proof, see [1], §24) For $k \geq 0$, there is a homotopy equivalence*

$$\begin{cases} \Xi_{k+1} \sim \Omega\Xi_k, & k \equiv 0,3 \pmod 4; \\ \Xi_{k+1} \times \mathbb{Z} \sim \Omega\Xi_k, & k \equiv 2 \pmod 4; \\ \Xi_{k+1} \sim \Omega\Xi_k \times \mathbb{Z}, & k \equiv 1 \pmod 4). \end{cases}$$

Applying the theorem, we obtain

$$\begin{aligned} \pi_k(\mathrm{O}) &\cong \pi_k\left(\mathrm{Aut}_{C_7} / \mathrm{Aut}_{C_8}\right) = \pi_k(\Xi_7) \\ &\cong \pi_{k-8}(\Xi_{15}) \cong \pi_{k-8}(\Xi_7) \cong \pi_{k-8}(\mathrm{O}). \end{aligned}$$

This concludes

Theorem 30 (Bott periodicity theorem, real case).

$$\pi_i(\mathrm{O}) \cong \pi_{i+8}(\mathrm{O}),$$

for $i \geq 0$.

Remark. *Actually, the space* Aut_{C_k} *corresponds to the Grothendieck group M_k, as* Aut_{C_k} *is the structure group of C_k-bundles. On the other hand, the space* $\mathrm{Aut}_{C_k} / \mathrm{Aut}_{C_{k+1}}$ *corresponds to the cokernel A_k.*

5.3 Sketch of the proof of Theorem 29

As written at the beginning of this section, we shall estimate the index of non-minimal geodesics, and construct fibrations in special cases.

Firstly, when $k \not\equiv 2 \pmod 4$, the space Ξ_k is simply connected, and thus $\Omega\Xi_k$ is connected. As we have done in the complex case, we split the space V into irreducible components (as we diagonalised A), construct eigenvectors of K_A, and then apply Lemma 21 to obtain a formula that counts the index. Since we are interested in non-minimal geodesics $\exp tA$, we assume some block in the block matrix A has entry $k_j \geq 3$, and then conclude that the index is $\geq n/m_k - 1$, where m_k is the dimension of irreducible C_k-modules.

The troublesome case is $k \equiv 2 \pmod 4$. In this case Ξ_k has its fundamental group $\mathbb{Z}$. The way to overcome such difficulties is to construct a fibration $\Xi_k \to S$, in analogy to the fibration $\mathrm{SU}(n) \to \mathrm{U}(n) \to S^1$. In this way we can impose a condition on the summation $\sum_j k_j$ of matrix entries. The remaining affairs are then quite similar to those of the previous cases, that we shall not repeat here.

References

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[2] Michael. F. Atiyah, Raoul Bott, Arnold Shapiro, *Clifford Modules*, Topology 3, Suppl 1, 1964, 3-38.

Final Report:
On the Complex Cobordism Ring

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Abstract

We will give a quick review of complex orientation, some basic algebraic topology, and then prove the structure theorem for the complex cobordism ring.

Contents

Notations

By a manifold I mean a (connected) C^∞ manifolds. By a theorem of Whitney, it can be embedded into some Euclidean space $\mathbb{R}^N$. A morphism between manifolds is a C^∞ map of manifolds.

1 Introduction

The complex cobordism theory was established by J. Milnor [Mil60] and R. Thom [Tho54]. There are several ways to define the complex cobordism ring. One is using the "complex orientation", as

I will do later. Instead of this, one can also use the concept of "stable tangent (normal) bundles". For the second, see [Mil60] for the detail. To understand what I am doing, I will review some facts about "real cobordism theory".

We say that two n-dimensional manifolds, say M and N, are **cobordant**, if there exists an $(n+1)$-dimensional manifold W so that $\partial W = M - N$. Let Ω_n be the set of all cobordant classes of n-dimensional manifolds. The Cartesian product operation gives an associative bilinear product operation

$$\Omega_n \times \Omega_m \to \Omega_{m+n}.$$

And use the operation "disjoint union" as the addition operation. Then

$$\Omega = \bigcup_{n \geq 0} \Omega_n$$

will be a ring. This is the (oriented) cobordism ring. The main theorem of this part is that

Theorem 1.1. *Ω_n is finite if $n \neq 0 \pmod 4$, and Ω_n is a finitely generated group with rank $p(r)$, the number of partition of r, when $n = 4r$.*

The main reference here is [MS74].

Indeed, Let $B := Gr(k, p+k)$ be the grassmannian variety and E be its universal k-plane bundle. Pick any bundle metric g on E and let $T(E)$ be the associated Thom space. We may define a map F from $\pi_{k+p}(T(E), t_0) \otimes \mathbb{Q}$ to $\mathbb{Q} \otimes \Omega_p$. Here t_0 is the base point. By a theorem of Thom, F is actually an isomorphism.

To compute the oriented cobordism ring (tensoring with $\mathbb{Q}$), it suffices to compute the group $\pi_{k+p}(T(E), t_0) \otimes \mathbb{Q}$. Now by a theorem of Hurewicz type, we have an isomorphism $H_{k+p}(T(E), t_0) \otimes \mathbb{Q} \cong H_p(B, \mathbb{Q})$.

We have established the isomorphism between $\mathbb{Q} \otimes \Omega_n$ and $H_p(B, \mathbb{Q})$. The structure theorem follows from an explicit computation of $H_p(B, \mathbb{Q})$.

An immediate corollary is that

Corollary 1.2. *$\Omega \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ with independent generators $\mathbb{C}P^N$, $N = 2, 4, 6, \cdots$.*

I will prove a similar result in complex setting. But the first obstruction is that we can't define the "cobordism" for complex manifold since the dimension is not even. Instead of this, we define a new concept "stably almost complex structure", which is defined by allowing a direct sum. In the first section, I will define a relative version of "stably almost complex structure", called the "complex orientation". And then prove a structure theorem for the complex setting.

2 Complex Orientation and the Complex Cobordism Ring

Given a morphism $f : Z \to X$ between manifolds. Assume that the dimension of f, i.e., $\dim_z Z - \dim_{f(z)} X$ for any $z \in Z$, is even. A **complex orientation of** f is a factorization $Z \to E \to X$ with $i : Z \to E$ an embedding endowed with a complex structure on its normal bundle ν_i and $p : E \to X$ a complex vector vector bundle over X. We denote such a factorization by (E, i, p) when f is specific.

We say that two factorizations, say (E,i,p) and (E',i',p'), are equivalent if E and E' can be embedded into as sub-vector bundles of an E'', such that in E'', i and i' are isotopic compatible with the normal complex structure. In other words, the isotopy is given by an embedding $i'' : Z \times I \to E'' \times I$ over I endowed with a complex structure on its normal bundle which matches to that of i and i' in E'' at end points.

A complex orientation for a morphism of odd dimension $f : Z \to X$ is defined by the orientation of the composition $Z \to X \to X \times \mathbb{R}$, or equivalently, the factorization of the form $Z \to E \times \mathbb{R} \to X$ with $E \to X$ a complex vector bundle and the normal bundle of $E \to Z \times \mathbb{R}$ is a complex vector bundle.

If $f : Z \to X$ is a complex-oriented morphism and $g : Y \to X$ is any morphism which is transversal to f, then we can define the "pull-back" of f by g as the complex orientation of the morphism $Y \times_X Z \to Y$.

Two proper complex-oriented morphisms $f_i : Z_i \to X$ are said to be **cobordant** if there exist a proper complex-oriented morphism $b : W \to X \times \mathbb{R}$ and morphisms $\epsilon_i : X \to X \times \mathbb{R}$, $\epsilon_i(x) = (x,i)$, is transversal to b and the pull-back of b by ϵ_i, with the induced complex orientation, is isomorphic to f_i.

Definition 2.1. The collection of cobordism classes of proper complex-oriented morphisms over X of dimension $-q$ is called the q-th complex cobordism ring over X and is denoted by $U^q(X)$. Write $U^*(X) = \bigcup_{q \in \mathbb{Z}} U^q(X)$.

Remark 2.2. The original definition of the complex cobordism ring is not the same as we defined above. For another definition, see [Mil60] or [Sto68].

It's quite easy to define sum and product on $U^*(X)$ to make it a ring.

The cobordism ring is somehow uniquely determined in the following sense:

Let h be a contravariant functor from the category of differentiable manifolds to the category of sets. For any morphism $g : X \to Y$, let $g^* : h(Y) \to h(X)$ be the corresponding morphism. Suppose furthermore, for each complex-oriented morphism $f : Z \to X$, there is a given morphism $f_* : h(Z) \to h(X)$ such that the following conditions are satisfied:

(i) Assume that

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array}$$

is a fibred product, where g is transversal to f. Then $g^* f_* = f'_* g'^* : h(Z) \to h(Y)$.

(ii) If f_0 is homotopic to f_1, then $f_0^* = f_1^*$.

(iii) If $f : Z \to X$ and $g : X \to Y$ are proper complex-oriented maps, and if gf is endowed with the composite complex orientation, then $(gf)_* = g_* f_*$.

Proposition 2.3. *Given an element $a \in h(pt)$, there exists a unique morphism $\theta : U^* \to h$ of functors commuting with Gysin maps and such that $\theta(1) = a$.*

Proof. Let $x \in U^*(X)$ be a cobordism class represented by a proper complex-oriented morphism $f : Z \to X$. Then $x = f_*\pi_Z^*1$, where $\pi_Z : Z \to pt$ is the canonical map. So we must have $\theta(x) = f_*\pi_Z^*a$. This proves the uniqueness of θ. For the existence, it suffices to show that $f_*\pi_Z^*a$ only depends on x.

So suppose $g : T \to X$ is cobordant to f via $b : W \to X \times \mathbb{R}$. g is the pull-back of b by ϵ_1 and f is the one obtained by ϵ_0. Then

$$\theta(x) = f_*\pi_Z^*a = \epsilon_0^*b_*\pi_W^*1 = \epsilon_1^*b_*\pi_W^*1 = g_*\pi_T^*a = \theta(x).$$

□

We state a well-known result and close this section. For a proof, see [Sto68].

Proposition 2.4. *If X is of the homotopy type of a finite complex, then $U^q(X)$ is finitely generated abelian group.*

3 Some Algebraic Topology

3.1 Formal Group Laws

A power series $F(T_1, T_2)$ with coefficients in a commutative ring R is said to be a one-dimensional (commutative) **formal group law over** R if the following identities hold:

1. $F(0,T) = F(T,0) = 0$.
2. $F(T_1, F(T_2, T_3)) = F(F(T_1, T_2), T_3)$.
3. $F(T_1, T_2) = F(T_2, T_1)$.

Let F be a formal group law over R. If S is a commutative algebra over R, then we can form a group in the following way: Let N be the set of nilpotent elements in S. Define a new multiplicative operation on N via F, that is, for $x, y \in N$, $x \cdot y := F(x, y)$. The point is that this is well-defined since we evaluate on the nilpotent elements. The construction is clear functorial. So we have construct

Proposition 3.1. *For any group law F over a commutative ring R, there exits a functor A from the category of commutative algebras over R to the category of groups.*

There is a universal commutative one-dimensional formal group law over a universal commutative ring defined as follows.

Let $F(x, y) = x + y + \sum_{i,j} c_{i,j} x^i y^j$ for indeterminates $c_{i,j}$.

We define the universal ring R^∞ to be the commutative ring generated by the elements $c_{i,j}$ with the relations given by the associativity and commutativity laws for formal group laws. More or less by definition, the ring R^∞ has the following universal property:

For any commutative ring S, one-dimensional formal group laws over S correspond to ring homomorphisms from R^∞ to S.

In other words, we have

Proposition 3.2. *The functor A is representable, and is represented by R^∞, i.e., there exists a natural transformation of functors $A(-) \to \mathrm{Hom}(R^\infty, -)$.*

In [Laz75], Lazard proved the universal ring R^∞ is just a polynomial ring over $\mathbb{Z}$ on generators of even degrees. The degree of $c_{i,j}$ is $2(i+j-1)$. See also [Qui69].

Proposition 3.3. *There exists a unique series $F(T_1, T_2) = \sum_{i,j} c_{i,j} T_1^i T_2^j$ with $c_{i,j} \in U^{2-2i-2j}(pt)$ such that*

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2))$$

for any two line bundles over the same manifold X.

Proof. By Leray-Hirsch theorem for projective bundles, we have the ring isomorphism

$$U^*(\mathbb{C}P^n \times \mathbb{C}P^n) = U^*(pt)[z_1, z_2]/(z_1^{n+1}, z_2^{n+1}),$$

where z_i is the Euler class of the bundle $pr_i^* \mathcal{O}(1)$. Using the ring isomorphism, we may find unique $c_{i,j}^n$ such that

$$e(pr_1^*\mathcal{O}(1) \otimes pr_2^*\mathcal{O}(1)) = \sum_{i,j \le n} c_{i,j}^n z_1^i z_2^j.$$

Let $n \to \infty$. The coefficients $c_{i,j}^n$ doesn't change. We get a well-defined power series $F(T_1, T_2)$ with coefficients in $U^*(pt)$.

Any line bundle is a pull-back of $\mathcal{O}(1)$ via some map to $\mathbb{C}P^N$ for some N. Let $L_1 = f^*\mathcal{O}(1)$ and $L_2 = g^*\mathcal{O}(1)$ with $f, g : X \to \mathbb{C}P^N$. (We may choose an N such that it works for L_1 and L_2.)

Consider the composite map $h_i = \Delta \circ (f \times g) \circ pr_i : X \to \mathbb{C}P^N$. $h_1 = f$ and $h_2 = g$.

$$L_i = h_i^*\mathcal{O}(1) = \Delta^*(f \times g)^* pr_i^* \mathcal{O}(1).$$

The theorem follows from the functoriality. □

3.2 The Landweber-Novikov Operations

Let t_i, $i \ge 0$, be a sequence of indeterminates of degree $-2i$. For any complex rank n vector bundle $E \to X$, let $i : X \to E$ be the zero section. Then $i^* i_* 1 \in U^{2n}(X)$ is defined to be the **Euler class of** E, denoted by $e(E)$.

We define

$$c_t(E) := \mathrm{Norm}\left(\sum_{j \ge 0} t_j e(\mathcal{O}(1))^j \right)$$

Note that this is well-defined since we have the Leray-Hirsch theorem for complex projective bundles. Sorting by degree, we may write

$$c_t(E) = \sum_\alpha t^\alpha c_\alpha(E),$$

where the sum is taken for all $\alpha = (\alpha_1, \alpha_2, \cdots)$ with all but finitely many $\alpha_i = 0$.

If $f : Z \to X$ is a complex-oriented map of even dimension, whose orientation is given by $Z \to E \to X$, then the difference $\nu_f := f^*E - \nu_i$ can be viewed as an element in $K(Z)$. The **Landweber-Novikov** operation

$$s_t := \sum t^\alpha s_\alpha : U^*(X) \to U^*(X)[t]$$

is defined by $f_*1 \mapsto f_*c_t(\nu_f)$.

Proposition 3.4. *The operator s_t is well-defined.*

Proof. Define a Gysin homomorphism $f_!$ on the functor $F : X \mapsto U^*(X)[t]$ by a twisted pushforward

$$f_!(x) := f_*(c_t(\nu_f) \cdot x)$$

Then by functoriality of the cobordism theory, s_t is well-defined. □

3.3 The Steenrod Operation

Let G be a group acting on the set $\{1,2,\cdots,k\}$ and let h be a G-equivariant theory. The **external Steenrod operation**

$$P_{ext} : U^{-2q}(X) \to h^{-2qk}(X^k)$$

is defined by $P_{ext}(f_*1) = f_*^k 1$. $f : Z \to X$ is a proper complex-oriented map of even dimension $2q$ and $f^k : Z^k \to X^k$ is the k-fold product. It can be though as a G-map. In this case, f^k has a natural equivariant complex orientation since the dimension of f is even.

Again, use the argument in proposition 1.3, we can show that this definition is independent of the choice of the map f. Pull-back by diagonal map $\Delta : X \to X^k$, we obtained the **Steenrod operation**

$$P(f_*1) := \Delta^* f_*^k 1.$$

We have the following proposition

Proposition 3.5. *Suppose G acts transitively on $\{1,\cdots,k\}$, and let ρ denote the corresponding representation of G on the subspace of $(z_1,\cdots,z_k)$ in $\mathbb{C}^k$ such that $\sum z_i = 0$. Let $f : Z \to X$ be a proper complex-oriented map of dimension $2q$ and m is an integer larger than dimension of Z, so that $m\epsilon + \nu_f$ is a vector bundle over Z, well-defined up to isomorphism, where ϵ is the trivial complex line bundle. Then*

$$e(\rho)^m P(f_*1) = f_*(e(\rho \otimes (m\epsilon + \nu_f)))$$

in $h^{2m(k-1)-2qk}(X)$.

The proof is actually using some techniques in equivairant cohomology theory. The proof is elementary. For the detail of the proof, see [Qui71].

Consider $G = \mathbb{Z}_k$, the cyclic group of order k, and η representation of $\mathbb{Z}_k$ on $\mathbb{C}$. Let $F(T_1, T_2)$ be formal group law introduced before. Let C be the subring of $U^{even}(pt)$ generated by the coefficients of F. If i is an integer, we let $[i]_F(T) \in C[[T]]$ be the operation of "multiplication by i" for the formal group. We have the following formula:

1. $[i]_F(T) = F(T, [i-1]_F(T))$.
2. $[1]_F(T) = T$.
3. $[i]_F(T) = iT+$ higher order terms.

Now consider the cohomology theory $h(-) = U^*(Q \times_G -)$. Let L be a line bundle equipped with a trivial G-action. $e(\rho \otimes L)$ will be the Euler class in $U^*(B \times Z)$ of the bundle induced from ρ and the bundle L. Let $v = e(\eta) \in U^2(B)$, we have

$$e(\rho \otimes L) = \prod_{i=1}^{k-1} e(\eta^i \otimes L) = \prod_{i=1}^{k-1} F([i]_F(v), e(L)) = w + \sum_{j \geq 1} a_j(v) e(L)^j.$$

$a_j(T) \in C[[T]]$ and $w = e(\rho) = (k-1)! v^{k-1} + \sum_{j \geq k} b_j v^j$.

For the vector bundle $E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$, we have

$$e(\rho \otimes E) = \prod_{i=1}^{r} e(\rho \otimes L_i) = \sum_{l(\alpha) \leq r} w^{r-l(\alpha)} (a(v))^\alpha c_\alpha(E),$$

where $l(\alpha) = \sum \alpha_j$. Using a standard argument of splitting principle, we obtains the formula for general vector bundle E. See [BT82] for the detail.

Pluging this, we obtain:

Proposition 3.6. *Let $Q \to B$ be a principal $\mathbb{Z}_k$-bundle and let P be the Steenrod k-th operation. Let v be the Euler class of the line bundle over B induced from the character sending the generator to $e^{2\pi i/k}$. Let w be the Euler class of the bundle induced from the reduced regular representation ρ. Then the Steenrod operation is related to the Landweber-Novikov operations by the formula*

$$w^{n+q} P x = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x)$$

where $x \in U^{-2q}(X)$. Here n is a sufficiently large integer, depends on $\dim X$ and q. $a_j(T)$ are power series with coefficients in the subring C of $U^{even}(pt)$ generated by the coefficients of the formal group law F.

4 Structure of $U^*(X)$

4.1 A Key Lemma

For any positive integer k, we define $\Phi(T) := [k]_F(T)/T$. Under power series expansion, we may write

$$\Phi(T) = k + d_1 T + d_2 T^2 + \cdots,$$

where $d_j \in C$. For any fixed manifold Y, we consider the cobordism theory $h^q(X) := U^q(X \times Y)$. Let $\mathbb{Z}_k$ act on $S^{2n-1} \subset \mathbb{C}^n$ with the generator acting via multiplication with $e^{2\pi i/k}$. Let $v_n \in h^2(S^{2n-1}/\mathbb{Z}_k)$ be the Euler class of the line bundle induced from $\eta : \mathbb{Z}_k = \langle \sigma \rangle \to \mathbb{C}^*$, $\sigma \mapsto e^{2\pi i/k}$. Finally let $j_n : S^{2n-1}/\mathbb{Z}_k \to S^{2n+1}/\mathbb{Z}_k$ be induced by the natural inclusion of $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$.

Proposition 4.1. *Let $x \in h^q(S^{2n+1}/\mathbb{Z}_k)$ such that $v_{n+1} \cdot x = 0$. Then there exists a $y \in h^q(pt)$ such that $y \cdot \Phi(v_n) = j_n^* x$ in $h^q(S^{2n-1}/\mathbb{Z}_k)$.*

Proof. Recall that if E is a complex vector bundle of (complex) dimension n over X and $\pi : SE \to X$ be its sphere bundle under any Riemannian metric, then there exists an exact "Gysin" sequence

$$h^{q-2n}(X) \xrightarrow{\cup e} h^q(X) \xrightarrow{\pi^*} h^q(SE) \xrightarrow{\pi_*} h^{q-2n+1}(X).$$

Consider the map $p_n : S^{2n-1} \times_{\mathbb{Z}_k} S^1 \to S^1/\mathbb{Z}_k$, the projection on the second factor. Think it as a sphere bundle of the bundle over $S^1/\mathbb{Z}_k$ induced from the representation $n\eta$.

One obtains the following commutative diagram:

$$\begin{tikzcd} {} \arrow[r, "\cup v_1^{n+1}"] & h^q(S^1/\mathbb{Z}_k) \arrow[r, "p_{n+1}^*"] \arrow[d, "\mathrm{id}"] & h^q(S^{2n+1} \times_{\mathbb{Z}_k} S^1) \arrow[r, "p_{n+1,*}"] \arrow[d, "(j_n')^*"] & h^{q-2n-1}(S^1/\mathbb{Z}_k) \arrow[d, "v_1"] \\ {} \arrow[r, "\cup v_1^n"] & h^q(S^1/\mathbb{Z}_k) \arrow[r, "p_n^*"] & h^q(S^{2n-1} \times_{\mathbb{Z}_k} S^1) \arrow[r, "p_{n,*}"] & h^{q-2n+1}(S^1/\mathbb{Z}_k) \end{tikzcd}$$

The first diagram is commutative. For the second, note that the inclusion j_n' can be thought as an inclusion of sphere bundles. This follows from the following claim:

Claim: Let E, F be two complex vector bundles over X, and let $f : S(E \oplus F) \to X$, $g : SE \to X$ be the associated sphere bundles. If $j : SE \to S(E \oplus F)$ is the inclusion, then

$$g_* j^* z = e(F) \cdot f_* z$$

for any $z \in h^*(S(E \oplus F))$.

Proof of the claim. The projection $p : S(E \oplus F) \to F$ is transversal to the zero section $s : X \to F$ and the pull-back of s by p is isomorphic to j, hence

$$j_* 1 = p^* s_* 1 = f^* s^* s_* 1 = f^* e(F).$$

Here, we need the fact that p and sf are homotopic. Thus,

$$g_* j^* z = f_* j_* j^* z = f_*(j_* 1 \cdot z) = f_*(f^* e(F) \cdot z) = e(F) \cdot f_* z.$$

□

To complete the proof, consider the element $v_1 \in h^2(S^1/\mathbb{Z}_k)$. The element v_1 comes from an element in $U^2(S^1/\mathbb{Z}_k)$. $v_1 = 0$ because of the dimension reason.

Let $\pi_{n+1} : S^{2n+1} \times_{\mathbb{Z}_k} S^1 \to S^{2n+1}/\mathbb{Z}_k$ be the projection to the first factor. Then we may regard π_{n+1} the sphere bundle over $S^{2n+1}/\mathbb{Z}_k$ of the line bundle induced from η. Again, we have the exact Gysin sequence

$$\begin{tikzcd} h^{q+1}(S^{2n+1} \times_{\mathbb{Z}_k} S^1) \arrow[r, "\pi_{n+1,*}"] \arrow[d, "j_n'^*"] & h^q(S^{2n+1}/\mathbb{Z}_k) \arrow[r, "v_{n+1}"] \arrow[d, "j_n^*"] & h^{q+2}(S^{2n+1}/\mathbb{Z}_k) \\ h^{q+1}(S^{2n-1} \times_{\mathbb{Z}_k} S^1) \arrow[r, "\pi_{n*}"] & h^q(S^{2n-1}/\mathbb{Z}^k). & \end{tikzcd}$$

Using a standard exact sequence argument, we know that $x = \pi_{n+1,*}z$ for some z. So $j_n^* x = \pi_{n*}j_n'^* z$. $j_n'^* z = p_n^* z'$ for some $z' \in h^{q+1}(S^1/\mathbb{Z}_k)$. Let $i : pt \to S^1/\mathbb{Z}_k$ be the natural inclusion. Then we have the decomposition

$$z' = y' \cdot 1 + y \cdot i_* 1,$$

where $y' \in h^{q+1}(pt)$ and $y \in h^q(pt)$. Indeed, consider the commutative diagram

$$\begin{tikzcd} h^{q-1}(pt) \arrow[r, "ue"] & h^{q+1}(pt) \arrow[r, "\pi^*"] & h^{q+1}(S^1) \arrow[r, "\pi_*"] & h^q(pt) \\ & h^q(pt) \arrow[r, "i_*"] & h^{q+1}(S^1/\mathbb{Z}_k) \arrow[u] \arrow[r, "\pi_*"] & h^q(pt) \arrow[u, equal] \end{tikzcd}$$

The composition of the lower diagram is the identity map. Using an exact sequence argument, we obtain the decomposition. Now note that $\pi_{n*}p_n^* 1 = 0$ since $1 = p_n^* 1 = \pi_n^* 1$.

On the other hand, consider the morphism $pr_1^* : U^*(-) \to U^*(- \times Y) = h^*(-)$. This morphism commutes with Gysin maps by functoriality. Now $i_* 1$ is the cobordism class of the morphism $i : pt \to S^1/\mathbb{Z}_k$. So $p_n^* i_* 1$ is the cobordism class of the morphism $S^{2n-1} \cong S^{2n-1} \times_{\mathbb{Z}_k} \mathbb{Z}_k \to S^{2n-1} \times_{\mathbb{Z}_k} S^1$. So $\pi_{n*}p_n^* i_* 1$ is the cobordism class of the projection map $S^{2n-1} \to S^{2n-1}/\mathbb{Z}_k$. The proof will be completed if we prove the following:

Proposition 4.2. *Let $f : Q \to B$ be a principal $\mathbb{Z}_k$-bundle with B being compact and let $L := Q \times_{\mathbb{Z}_k} \mathbb{C}$ be the line bundle associated to the character η. Then $f_* 1 = \Phi(e(L))$ in $U^0(B)$.*

Proof of the proposition. Let i be the zero section of L and let $g : L \to B$ be the projection. Then the line bundle $g^* L$ has a tautological section s, which is transversal to zero and vanishing on $i(B)$. The bundle $g^* L$ with the trivialization off $i(B)$ given by s. Hence the line bundle $g^* L$ extends to a line bundler M over the one-point compactification $L \cup \{\infty\}$. Let $i_* : U^q(B) \cong U^{q+2}(L \cup \{\infty\}, \{\infty\})$ be the Thom isomorphism and we have $e(M) = i_* 1$.

A similar trick applies. The bundle $g^* L^k$ with the section s^k extends to the bundle M^k. Let $j : Q \to L$ be the natural inclusion and t be the section defined by

$$t(z, j(q)) = ((z, j(q)), (z^k, j(q)^{\otimes k}))$$

By projection formula, we have

$$j_* 1 = e(M^{\otimes k}) = [k]_F(i_* 1) = i_* 1 \cdot \Phi(i_* 1) = i_* \Phi(e(L)).$$

Finally, note that i_* is an isomorphism and $j_* = i_* f_*$. This completes the proof. □

We have shown that $\pi_{n*}p_n^* 1 = 0$ and $\pi_{n*}p_n^* i_* 1 = \Phi(v_n)$. It follows that

$$j_n^* x = \pi_{n*}p_n^* z' = y \cdot \Phi(v_n)$$

as expected. □

4.2 The Main Theorem

Theorem 4.3. *Let $\tilde{U}^*(X)$ be the ideal consisting of elements in $U^*(X)$ which vanish when restrict to any point of X. If X is of the homotopy type of a finite complex, e.g., X is a manifold, then*

$$U^*(X) = C\cdot\sum_{q\geq 0} U^q(X),\quad \tilde{U}^*(X) = C\cdot\sum_{q>0}\tilde{U}^q(X).$$

Here $C \subset U^{even}(p)$ is the subring (with unity) generated by the coefficients of the formal group law F.

Proof. By suspension isomorphisms,

$$U^{2k-1}(X) \cong \tilde{U}^{2k}(S^1\times X/p\times X), \tag{1}$$

$$U^{2k}(X) \cong \tilde{U}^{2k+2}(S^2\times X/p\times X), \tag{2}$$

$$\tilde{U}^{2k-1}(X) \cong \tilde{U}^{2k}(S^1\times X/p\times X\cup S^1\times\{x_0\}). \tag{3}$$

For the details, see [Spa66] and [Hat02]. It suffices to show that

$$\tilde{U}^{even}(X) = C\cdot\sum_{q>0} U^{2q}(X).$$

Let $R := C\cdot\sum_{q>0}U^{2q}(X)$. By localisation, it suffices to show that $R_{(p)} = \tilde{U}^{even}(X)_{(p)}$ for any prime $p\in\mathbb{Z}$.

Proceeding by descending induction, suppose we have shown that $R^{-2j}_{(p)} = \tilde{U}^{-2j}(X)_{(p)}$ for $j < q$. Notice that the initial case $q = 0$ is trivially true. Pick $x\in\tilde{U}^{-2q}(X)$. By the formula between the Steenrod operation and Landweber-Novikov operations, we may find a sufficiently large n so that

$$w^{n+q}Px = \sum_{l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x) \tag{4}$$

in the cohomology theory $U^{2n-2q}(S^{2m+1}/\mathbb{Z}_p\times X)$ for all m.

Since p is a prime, $(p-1)!$ is a unit in the localisation ring $Z_{(p)}$. Thus, we may write $v^{p-1} = w\cdot\theta(v)$, where $\theta(v)$ is a formal power series with coefficients in $C_{(p)}$. By induction hypothesis, $s_\alpha(x)\in R$. Hence the equation above becomes $v^m(w^qPx - x) = \psi(v)$ in the theory $U^*(S^{2m+1}/\mathbb{Z}_p\times X)_{(p)}$ with $\psi(T)\in R_{(p)}[[T]]$.

Suppose $m\geq 1$ is the smallest integer so that the equation holds. Let $i^* : U^*(S^{2m+1}/\mathbb{Z}_p\times X)\to U^*(X)$ be the ring homomorphism induced from the inclusion of a point in $S^{2m+1}/\mathbb{Z}_p$. We obtain

$$i^*(v^m(w^qPx - x)) = i^*(\psi(v)).$$

Note that when we restrict the equation to X, the class v vanishes by the functoriality of Euler classes. So $\psi(v) = 0$ and we may write $\psi(T) = T\psi_1(T)$ with $\psi_1(T)\in R_{(p)}[[T]]$ and

$$v(v^{m-1}(w^qPx - x) - \psi_1(v)) = 0.$$

By previous lemma, there exists a $y\in U^*(X)_{(p)}$ of degree $2(m-1)-2q$ such that

$$v^{m-1}(w^qPx - x) - \psi_1(v) = y\cdot\Phi(v)$$

in $U^*(S^{2m-1}/\mathbb{Z}_p \times X)_{(p)}$.

Again, one notices that the class x will vanish when restricting on a base point of X and by definition, $\psi_1(T)$ will also vanish since its coefficients lying in $R_{(p)}$. One obtains $y'\Phi(v) = 0$, where y' is the component of y in $U^*(pt)_{(p)}$. Subtracting y from y', we may assume $y \in \tilde{U}^*(X)_{(p)}$.

Now if $m > 1$, then $y \in R_{(p)}$ by induction hypothesis, and $\psi_1(v) + y\Phi(v)$ sits in $R_{(p)}[[v]]$. This contradicts to the minimality of m. Hence $m = 1$.

Apply i^* again, as $q > 0$, one obtains

$$-x = \psi_1(0) + py \quad \text{in } \tilde{U}^{-2q}(X)_{(p)}. \tag{5}$$

Since x is arbitrary, we have $U^{-2q}(X) \subset R^{-2q}_{(p)} + pU^{-2q}(X)_{(p)}$. By Nakayama lemma, since $U^{-2q}(X)$ is finitely generated abelian group, $U^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$.

If $q = 0$, then we have $x^p - x = \psi_1(0) + py$. Note that $\tilde{U}^0(X)$ is nilpotent. So $x = x^p - \psi_1(0) - py$ can be reduce to $-x = \psi_1(0) + py$ for some ψ_1' and y'. In any case we have $U^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$. This completes the induction step.

□

It's easy to see that $U^{even}(pt) = \mathbb{Z}$ and $U^q(pt) = 0$ for $q > 0$. So the theorem implies the following

Corollary 4.4. *$U^{even}(pt) = C$ and $U^{odd}(pt) = 0$.*

4.3 A Special Case: $X = pt$

There is a unique natural transformation from $U^*(X)$ to $H^*(X, \mathbb{Z})$ compatible with Gysin homomorphisms. It is called the Thom homomorphism, denoted by ϵ. Let β be the composition of Landweber-Novikov operator and ϵ,

$$\beta : U^*(X) \to H^*(X)[t].$$

Indeed, it is defined by

$$\beta(f_* z) = f_*(c_t^H(\nu_f) \cdot \beta z) \tag{6}$$

for a proper complex-oriented map $f : Z \to X$. Suppose now X is a point, then the formula shows that βx is the polynomial with coefficients being Chern numbers of x.

Recall that $c_t(L) = \sum_{j \geq 0} t_j e(L)^j$ with $t_0 = 1$. Together with , we have

$$\beta(e^U(L)) = \sum_{j \geq 0} t_j (e^H(L))^{j+1}, \quad t_0 = 1.$$

Replacing L by $L_1 \otimes L_2$, we obtain the formula

$$\beta F(\theta_t(T_1), \theta_t(T_2)) = \theta_t(T_1 + T_2),$$

where $\theta_t(T) = \sum_{j \geq 0} t_j T^{j+1}$.

Therefore, there are ring homomorphisms

$$R^\infty \xrightarrow{\delta} U^*(pt) \xrightarrow{\beta} \mathbb{Z}[t], \quad F_{univ} \mapsto F \mapsto \theta_t^*(T_1 + T_2).$$

Theorem 4.5. *δ is an isomorphism, and β is an injection. Consequently, $U^*(pt)$ is a polynomial ring over $\mathbb{Z}$ with one generator of degree $-2q$ for each $q > 0$, and any element in $U^*(pt)$ is determined by the set of its Chern numbers.*

Proof. By previous corollary, the map δ is onto. Tensoring with $\mathbb{Q}$, we will claim the map $R^\infty \otimes \mathbb{Q} \to \mathbb{Q}[t]$ is an isomorphism.

Indeed, fixed a $\mathbb{Q}$-algebra R, a map $u : \mathbb{Z}[t] \to R$ is completely determined by a power series

$$\theta_u := \sum u(t_i) T^{i+1}.$$

By our notation, the composition $u\beta\delta$ may be identified with the formal group $\theta_u^*(T_1 + T_2)$. By formal Lie theory, any formal group law over R is of the form θ_u. Here we used the fact that R is a $\mathbb{Q}$-algebra.

Thus, for a $\mathbb{Q}$-algebra R, $\beta\delta$ induces a 1-1 correspondence between maps $\mathbb{Z}[t] \to R$ and maps $R^\infty \to R$. So $\mathbb{Q} \otimes \beta\delta$ is an isomorphism. By structure theorem of R^∞, the ring is torsion free. $\beta\delta$ is injective. So δ is an injection, and hence an isomorphism. Consequently, β is an injection. □

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