Differential Geometry (I) (Fall 2012) Quiz

September 19, 2012, pm 3:30 – 5:00

Dept.________________ ID No.________________ Name:________________

A. (20 points) Consider $M = \mathbb{R}$ with one chart given by $(\mathbb{R}, \phi)$ where $\phi(t) = t^3$. Show that this defines a C^∞ structure on M. Is M diffeomorphic to $\mathbb{R}$ with the standard C^∞ structure $(\mathbb{R}, \mathrm{id})$?

B. Let M be a locally compact topological space which is Hausdorff and second countable.

(a) (20 points) Show that M is σ compact. Namely, there is a countable sequence of increasing open sets $\{G_i\}_{i\in\mathbb{N}}$ with $\overline{G}_i$ compact, $\overline{G}_i \subset G_{i+1}$ and $M = \bigcup_{i=1}^\infty G_i$. (Hint: Show first that the sub collection $\{U_i\}$ in the countable basis with $\overline{U}_i$ compact is still a basis.)

(b) (20 points) Show that every open cover $\{U_\alpha\}_{\alpha\in A}$ of M has a countable locally finite refinement $\{V_j\}_{j\in\mathbb{N}}$ with $\overline{V}_j$ being compact.

C. (a) (20 points) Given $f : M \to N$ a C^∞ map, show that in charts $(U, \mathbf{x})$ at $p \in M$ and $(V, \mathbf{y})$ at $f(p) \in N$, df_p is represented by $d\tilde{f}_{\mathbf{x}(p)}$ with Jacobian matrix $\left[\dfrac{\partial \tilde{f}^j}{\partial x^i}\right]$ where $\tilde{f} = \mathbf{y} \circ f \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap f^{-1}(V)) \to \mathbb{R}^{\dim N}$. Namely

$$df_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_j \frac{\partial \tilde{f}^j}{\partial x^i}(\mathbf{x}(p))\frac{\partial}{\partial y^j}\Big|_p.$$

(b) (20 points) Let M be a C^k manifold, $1 \le k \le \infty$, use the theorem that $D_pM \cong (m_p/m_p^2)^*$ to prove

$$\dim D_pM = \begin{cases} \dim M & \text{if} \quad k = \infty, \\ \infty & \text{if} \quad k < \infty. \end{cases}$$

(Hint: For $k = 1$, study functions $f(x) = (x^1)^a$ for $1 < a < 2$.)