DIFFERENTIAL GEOMETRY
MIDTERM EXAM
PM 3:30 – 6:30, 11/07, 2012
A COURSE BY CHIN-LUNG WANG

1. (a) If $f : U \to \mathbb{R}^d$ is C^1 and $A \subset U$ has measure 0 then $f(A)$ also has measure 0.
(b) If $f : M^m \to N^n$ is C^1 and $m < n$, then $f(M)$ has measure 0.
(c) If $f : M \to N$ is C^∞, 1-1, onto and everywhere non-singular, then f is a diffeomorphism.
(d) If M is compact, $\dim M = n$ and $f : M \to \mathbb{R}^n$ is C^∞, then f is singular somewhere.

2. (a) What is the definition of an "immersion"? Show that $f : \mathbb{R}P^2 \to \mathbb{R}^3$ defined by $[x, y, z] \mapsto (xy, yz, zx)$ is an immersion, where $\mathbb{R}P^2 := S^2/\{\pm 1\}$.
(b) Construct an embedding $\mathbb{R}P^2 \to \mathbb{R}^4$. Is it possible to embed $\mathbb{R}P^2$ in $\mathbb{R}^3$?

3. (a) What is "Lie derivative"? For two C^∞ vector fields V, W with compact support on a manifold M, show that $L_V W = [V, W]$.
(b) Prove the Leibniz rule for L_V on tensors and ι_V on forms. If $\alpha \in A^k(M)$, show that $L_V \alpha = (\iota_V d + d\iota_V)\alpha$.

4. (a) For $\omega \in A^k(M)$, state and prove Cartan's "intrinsic formula" for $d\omega$. (If you use the fundamental theorem of tensor calculus then you need to prove it.)
(b) Let $\omega \in A^2(M)$. Define $E_p := \{v \in T_pM \mid \omega(v, w) = 0, \forall w \in T_pM\}$. Show that the subspace distribution $\{E_p\}_{p \in M}$ is integrable if $d\omega = 0$.

5. (a) State and prove the Mayer-Vietoris long exact sequence for $M = U \cup V$.
(b) Show that $H^k_{dR}(S^n) \cong \mathbb{R}$ for $k = 0, n$ and vanishes otherwise.

6. Bonus problem. State and prove *one and only one* of the following theorems:
(1) Sard's theorem,
(2) Whitney imbedding theorem $M^n \hookrightarrow \mathbb{R}^{2n+1}$ for compact M,
(3) C^∞ approximation of a continuous map $f : M \to N$ between compact manifolds.
(4) Smooth dependence of ODE on its initial conditions.
(5) Stokes' theorem on $(M, \partial M)$ and divergence theorem on (M, g),
(6) Homotopy invariance of de Rham cohomology and singular homology.