NTU 2020 FALL: DIFFERENTIAL GEOMETRY
MIDTERM EXAM
A COURSE BY CHIN-LUNG WANG

1. Let M be a C^∞ manifold with open cover $\bigcup_{\alpha \in A} U_\alpha$. Construct a C^∞ partition of unity $\{\psi_i\}_{i \in \mathbb{N}}$ subordinate to it with supp ψ_i being compact.

2. Show that if $f \in C^1(U, \mathbb{R}^d)$ $A \subset U$ has measure 0 then $f(A)$ also has measure 0.

3. (a) Show that $f : \mathbb{R}P^2 := S^2/\pm 1 \to \mathbb{R}^3$ defined by

$$[x, y, z] \mapsto (xy, yz, zx)$$

is an immersion.

(b) Construct an embedding $\mathbb{R}P^2 \to \mathbb{R}^4$. Is it possible to embed $\mathbb{R}P^2$ in $\mathbb{R}^3$?

4. Prove Cartan's homotopy formula: if $\alpha \in A^k(M)$ then

$$L_V \alpha = (\iota_V d + d\iota_V)\alpha.$$

5. For $\omega \in A^k(M)$, state and prove Cartan's intrinsic formula for $d\omega$. (If you use the fundamental theorem of tensor calculus then you have to prove it too.)

6. (a) Compute $H^k_{dR}(S^n)$ and $H^k_{dR}(\mathbb{T}^2)$ for all $k \geq 0$.

(b) Let $B = \overline{B_0(1)}$ in $\mathbb{R}^n$. Show that every $f \in C^0(B, B)$ has a fixed point.

For **7** and **8**, (M, g) is a Riemannian manifold with Levi–Civita connection.

7. (a) Prove Gauss' Lemma and the local minimality of geodesics.

(b) Prove the existence of convex neighborhood at any $p \in M$.

8. (a) Prove the second Bianchi identity $R_{ij[kl;m]} = 0$.

(b) Let $n = \dim M \geq 3$. If

$$R_{ij} = \lambda g_{ij},$$

show that $\lambda = s/n$ and λ is a constant.

- You may replace ONE problem above by stating and proving a substantial result you have well prepared but not shown above.

Date: pm 6:00 – 9:15, 11/11, 2020. Show your works in details. Each problem deserves 15 points.