DIFFERENTIAL GEOMETRY II
MIDTERM EXAM
10:10 – 13:10, APRIL 19, 2013
A COURSE BY CHIN-LUNG WANG

1. (a) Show that a hermitian metric $g_{i\bar{j}}$ on a complex manifold is Kähler if and only if $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$ for all i, j, k. Under the complex coordinates, show that all the non-trivial Christoffel symbols for the Levi-Civita connection are of pure type: $\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}$ and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \bar{\Gamma}^k_{ij}$.
(b) Derive the formulas for Riemannian curvature tensor and Ricci tensor. In particular, show that $R_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\det(g_{k\bar{l}})$.

2. (a) Let $M \hookrightarrow (N, J, \omega)$ be an oriented $2m$-dimensional real submanifold inside a complex n-dimensional Kähler manifold, with dV the induced volume form on M. Show that

$$\frac{\omega^m}{m!}\Big|_{T_pM} \leq dV_p$$

for any $p \in M$, with equality holds if and only if T_pM is a J-invariant subspace.
(b) If M is indeed a complex submanifold of a compact Kähler manifold N, show that M minimizes volume in its homology class $[M] \in H_{2m}(N, \mathbb{Z})$. Also any other minimizer must also be a complex Kähler submanifold.

3. (a) Let M be an immersed minimal surface in $\mathbb{R}^3$. Show that $p \circ N$ induces an isothermal coordinate system near each non-umbilical point $p \in M$. Here $p : S^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection and N is the Gauss map.
(b) Describe the Weierstrass representation of immersed minimal surface in $\mathbb{R}^3$ by a meromorphic function g and holomorphic one form $\omega = f\,dz$. In particular show that the conformal factor λ in $ds^2 = \lambda(dx^2 + dy^2)$ is given by $\lambda = |f|^2(1 + |g|^2)^2$.

4. (a) Let $(E, \nabla) \to M$ be a vector bundle with connection ∇. Prove the Bianchi identity $d\Omega = [\omega, \Omega]$ and show that $dc_i(E, \nabla) = 0$.
(b) Show that $[c_i(E, \nabla)] \in H^{2i}_{dR}(M)$ is independent of the choices of connections.

5. (a) Let $P : \Gamma(E) \to \Gamma(E)$ be a positive definite self-adjoint elliptic operator on a compact manifold (M, g) with eigenfunctions $P\phi_n = \lambda_n\phi_n$, $n \in \mathbb{N}$. Show that ϕ_n's can be chosen to be orthonormal and there exist some $C, \delta > 0$ such that $\lambda_n \geq Cn^\delta$ for all n.
(b) What is the definition of heat kernel? Show that the heat kernel of P is given by

$$H(x, y, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x) \otimes \phi_n(y).$$

6. (a) Construct the complex spinor module S with $C(V)_{\mathbb{C}} \cong \operatorname{End} S$ and show that any complex Clifford module E is of the form $E = S \otimes_{\mathbb{C}} W$ for some vector space W.
(b) Let $n = \dim V = 2p$ and $\epsilon = i^p e_1 \cdots e_n \in C(V)$ where $\{e_1, \cdots, e_n\}$ is an ONB. Show that $S^\pm = \{v \mid \epsilon v = \pm v\}$ and $\operatorname{str}(a) = (-2i)^{n/2} T\sigma(a)$ for $a \in \operatorname{End} S \cong C(V)_{\mathbb{C}}$.