NTU 2020 DIFFERENTIAL GEOMETRY
FINAL EXAM
A COURSE BY CHIN-LUNG WANG

1. (a) Let (M, g) be a complete Riemannian manifold. Show that any non-trivial $[\gamma] \in \pi_1(M)$ is represented by a C^∞ closed curve γ_0 with shortest length.
(b) Derive the second variation formula for geodesics. Use it to prove that an even dimensional compact oriented manifold with $K > 0$ is simply connected.

2. (a) State and prove the Cartan–Ambrose theorem on local isometry $(M, p) \to (N, q)$ via a linear isometry $\Phi : T_pM \cong T_qN$ and identification on geodesics.
(b) Apply it to show that a space is locally symmetric if and only $\nabla R = 0$.

3. (a) Define L^2 Sobolev spaces $H_s(\mathbb{R}^m)$, $s \in \mathbb{R}$. State and prove the Sobolev Lemma.
(b) Extend (a) to the setting of a vector bundle over a compact manifold.

4. (a) Prove the Bochner formula: let $e_i \in T_pM$ be an ONB and $\eta^i \in T_p^*M$ be its dual basis. Then for $\omega \in A^k(M)$,

$$(\triangle\omega)(p) = -\mathrm{tr}\,\nabla^2\omega - \sum_{i,j} \eta^i \wedge \iota_{e_j} R(e_i, e_j)\omega.$$

(b) If (M, g) is compact with $\mathrm{Ric} \geq 0$, show that $h^1(M) \leq \dim M$. If moreover $\mathrm{Ric}(p) > 0$ for some $p \in M$ then $h^1(M) = 0$.

5. (a) Let G be a connected Lie group. Show that $Z(G) = \mathrm{Ker}\,\mathrm{Ad}$.
(b) Let G be a simply connected Lie group and H be a connected Lie group. Show that there is a one-to-one correspondence between Lie group homomorphism $G \to H$ and Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$.

6. (a) Construct a bi-invariant metric on a compact Lie group G. For $G = \mathrm{U}(n), \mathrm{O}(n)$ give the bi-invariant metric in explicit formula.
(b) Let G be a Lie group with a left invariant metric $\langle\,,\rangle$, determine ∇^{LC}. If the metric is bi-invariant, compute $R(X, Y)Z$ for $X, Y, Z \in \mathfrak{g}$ and prove $K \geq 0$.

*There are some more harder/technical problems listed in the next page. Nevertheless, if you prefer to, you may replace one and only one problem above by solving one of them.

Date: am 8:30 – 12:30, 1/13, 2021.
Each problem deserves 20 points.

(i) Define Jacobi fields along a geodesic. For a complete Riemannian manifold (M, g) and $p \in M$, show that $d\exp_p$ is singular at v if and only if there is a Jacobi field $J \not\equiv 0$ along $\gamma(t) = \exp_p(tv)$ with $J(0) = 0 = J(1)$. Use it to prove the Cartan–Hadamard Theorem: if (M^m, g) is complete with $K \le 0$ then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^m$.

(ii) Derive the second variation formula for a normal variation of closed immersed minimal hypersurface $M^m \to \overline{M}^{m+1}$ with variation field $\eta = f\vec{n}$:

$$A''(0) = \int_M |\nabla f|^2 - (\overline{\mathrm{Ric}}(\vec{n}, \vec{n}) + \|B\|^2) f^2.$$

Use it to show: if $\overline{M}^3$ has $\bar{s} > 0$ then there is no stable minimal immersion of orientable surface M with $g(M) \ge 1$. Construct an isometrically and minimally embedded flat tori T in S^3 and explain why this is not a counterexample.

(iii) Use Garding's inequality to prove the compactness theorem and the regularity theorem, and then prove the Hodge decomposition theorem on $A^k(M)$.

(iv) Define the Riemannian submersion. Derive the formula for the Levi-Civita connection and prove O'Neil's curvature formula. Apply it to compute the sectional curvature of $\mathbb{C}P^n$ by viewing it as a Riemannian submersion $f : S^{2n+1} \to \mathbb{C}P^n$.

(v) Let $M = G/H$ be a symmetric space with compact G. Show that

$$H^*(M, \mathbb{R}) \cong A^*_{\mathrm{inv}}(M) = \mathbb{H}^*(M).$$

In particular, for semi-simple G show that $H^3(G; \mathbb{R}) \ne 0$.