DIFFERENTIAL GEOMETRY
FINAL EXAM
AM 8:30 – 12:30, 1/04, 2013
A COURSE BY CHIN-LUNG WANG

1. (a) For any $p \in M$, show that for the Riemann normal coordinate system $(U, \mathbf{x})$ we have $\Gamma^k_{ij}(p) = 0$ and $\partial_k g_{ij}(p) = 0$ for all i, j, k.
(b) Prove the second Bianchi identity $R_{ij[kl;m]} = 0$.
(c) Let $n \geq 3$. If $R_{ij}(x) = \lambda(x) g_{ij}(x)$, show that $\lambda = R/n$ must be a constant.

2. (a) Derive the first variation for piece-wise C^1 curves. Use it to show that if $\gamma : [0, \ell] \to M$ is a closed piece-wise C^1 curve which is a critical point of the length functional then γ is a C^∞ closed geodesic.
(b) Derive the second variation formula for geodesics. Use it to prove Synge's theorem: An even dimensional compact oriented manifold with $K > 0$ must be simply connected.

3. (a) For a normal variation of closed immersed minimal hypersurface $M^m \to \overline{M}^{m+1}$ with variation field $\eta = f\vec{n}$, it is known that

$$A''(0) = \int_M |\nabla f|^2 - (\overline{\mathrm{Ric}}(\vec{n}, \vec{n}) + \|B\|^2) f^2.$$

Use it to show that if $\overline{M}^3$ has $\bar{R} > 0$, there is no stable minimal immersion of orientable surface M with $g(M) \geq 1$. (Hint: Show that $\overline{\mathrm{Ric}}(\vec{n}, \vec{n}) = \frac{1}{2}\bar{R} - K_M - \frac{1}{2}\|B\|^2$.)
(b) Construct an isometrically and minimally embedded flat tori T in S^3. Why is this not a counterexample to (a)?

4. (a) Prove the Bochner formula: Let $e_i \in T_pM$ be an ONB and $\eta^i \in T^*_pM$ be its dual basis. Then for $\omega \in A^k(M)$,

$$(\triangle\omega)(p) = -\mathrm{tr}\, \nabla^2 \omega - \sum_{i,j} \eta^i \wedge \iota_{e_j} R(e_i, e_j)\omega.$$

(b) If M is compact with $R_{ij} \geq 0$, show that $h^1(M) \leq \dim M$. Moreover if $\mathrm{Ric}(p) > 0$ for some $p \in M$ then $h^1(M) = 0$.

5. (a) Let G be a compact Lie group. Construct a bi-invariant metric $\langle\,,\rangle$ on G.
(b) Compute $\nabla_X Y$ and $R(X, Y)Z$ for $X, Y, Z \in \mathfrak{g}$, and prove $K \geq 0$. When does G have $\mathrm{Ric} > 0$? $K > 0$? Apply your answers to $U(n)$ and $SU(n)$.

The Bonus problem is on the next page:

6. Give the details of one and only one problem in the following list:

(i) Prove the Gauss Lemma, the local minimality of arc length along geodesics, and the existence of convex neighborhoods.

(ii) State and prove the Hopf-Rinow Theorem.

(iii) Define Jacobi fields along a geodesic. For M a complete Riemannian manifold and $p \in M$, show that $d\exp_p$ is singular at v if and only if there is a Jacobi field $J \not\equiv 0$ along $\gamma(t) = \exp_p(tv)$ with $J(0) = 0 = J(1)$. Use it to prove the Cartan–Hadamard Theorem.

(iv) Derive the second variation formula for immersed minimal submanifods and then deduce from it the formula for $A''(0)$ in **3.** (a).

(v) Define Riemannian submersion. Derive the formula for the Levi-Civita connection and O'Neil's curvature formula. Apply it to compute the sectional curvature of $\mathbb{C}P^n$.

(vi) Define L^2 Sobolev spaces $H_s(\mathbb{R}^m)$, $s \in \mathbb{R}$. Prove the Sobolev Lemma and the Rellich Lemma. Extend all these to a vector bundle over a compact manifold.

(vii) Use Garding's inequality to prove the Hodge decomposition theorem on $A^k(M)$.