NTU 2021 SPRING: DIFFERENTIAL GEOMETRY II
MIDTERM EXAM
A COURSE BY CHIN-LUNG WANG

1. (a) Describe Weiertstrass representation for immersed minimal surface by a meromorphic function g and a holomorphic 1-form W on an open set $M \subset \mathbb{C}$.
(b) Find the parametrization for Ennerper surface for $M = \mathbb{C}$, $g = z$, and $W = dz$.

2. (a) Let $M \hookrightarrow (N, J, \omega)$ be an oriented $2m$-dimensional real submanifold inside a complex n-dimensional Kähler manifold, with dV the induced volume form on M. Show that
$$\frac{\omega^m}{m!}\Big|_{T_pM} \le dV_p$$
for any $p \in M$, with equality holds if and only if T_pM is a J-invariant subspace.
(b) If M is indeed a complex submanifold of a compact Kähler manifold N, show that M minimizes volume in its homology class $[M] \in H_{2m}(N, \mathbb{Z})$. Also any other minimizer must also be a complex Kähler submanifold.

3. (a) Let $(E, \nabla) \to M$ be a vector bundle with connection ∇. For $\Phi \in A^k(\mathrm{End}(E))$. Show that $\nabla\Phi = d\Phi + (-1)^k[\Phi, \omega]$ and $\nabla^2\Phi = [\Phi, \Omega]$.
(b) Prove the second Bianchi identity. Use it to show that $dc_i(E, \nabla) = 0$.

4. (a) Show that the signature is a $\mathbb{Z}$-genus on the real cobordism ring Ω. (You may apply the Poincaré–Lefschetz duality for manifolds with boundary.)
(b) Deduce the Hirzebruch signature formula from Thom's cobordism theorem.

5. (a) Construct the complex spinor module S with $C(V)_{\mathbb{C}} \cong \mathrm{End}\, S$ and show that any complex Clifford module E is of the form $E = S \otimes_{\mathbb{C}} W$ for some vector space W.
(b) Let $n = \dim V = 2p$ and $\epsilon = i^p e_1 \ldots e_n \in C(V)$, where $\{e_1, \ldots, e_n\}$ is an ONB oriented by $\{e_1, Je_1, \ldots, e_p, Je_p\}$. Show that $S^{\pm} = \{v : \epsilon v = \pm v\}$ and for $a \in \mathrm{End}\, S \cong C(V)_{\mathbb{C}}$,
$$\mathrm{str}(a) = (-2i)^p T\sigma(a),$$
where the Berezin integral T picks up the coefficient of the volume form.

6. (a) Let $E = S \otimes W$ be a Clifford module on (M, g) with Clifford connection ∇. Let $e_i \in T_{x_0}M$ be an ONB with dual basis e^i and $c^i := c(e^i)$. Show that
$$F^E_{ij} = -\tfrac{1}{4}\textstyle\sum_{k,l} R_{klij} c^k c^l + F^W_{ij}.$$
(b) Prove the Lichnerowicz formula $D^2 = -\mathrm{tr}\,\nabla^2 + c(F^W) + \frac{1}{4}s_M$ for the Dirac operator D.
(c) Let $p_t(x, y)$ be the heat kernel of D^2 and $\tau(x_0, x)$ be the canonical trivialization of E over $U = B_{x_0}(R)$. Denote $k(t, \mathbf{x}) = \tau(x_0, x)p_t(x, x_0)$ for $x = \exp_{x_0} \mathbf{x}$ and $r(u, t, \mathbf{x}) := \sqrt{u}^n(\delta_u k)(t, \mathbf{x})$. Find $L(u)$ such that $(\partial_t + L(u))r(u, t, \mathbf{x}) = 0$, where δ_u on i-form is
$$(\delta_u \alpha)(t, \mathbf{x}) := \sqrt{u}^{-i}\alpha(ut, \sqrt{u}\mathbf{x}).$$
(d) Compute $\lim_{u \to 0^+} L(u)$.

Date: pm 6:00 – 9:30, 4/23, 2021. You may work on each part independently. Each problem deserves 20 points. Bonus will be assigned to very good answer to (part of) **6**.