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Morse Theory and Bott Periodicity

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In this article, we will go through the basics of Morse theory, which Bott calls “Baby Morse Theory”. It gives us a way to recover the homotopy type of a manifold. After the proof of Morse theory, as an application to compact Lie groups, we will prove Bott periodicity theorem, which calculates the homotopy groups of a unitary group in arbitrary dimension.

We will follow the methods in J. Milnor's *Morse theory* (§1∼3, §23, and part of §20,§22). However, there is a more elementary proof of Bott periodicity theorem, which does not involve Morse theory. It can be found in M. Atiyah and R. Bott, *On the Periodicity Theorem for Complex Vector Boundle*, Acta Methematica, Vol. 112 (1964), pp. 229∼247.

1 Basic Morse Theory

The fundamental concept in Morse theory is: a ”good fuction” $f : M \to \mathbb{R}$ encodes a lot of information about M. To be more specific, Morse theory studies the critical points of good functions on M, and gives a nice way to recover the homotopy type of M.

Consider a smooth function $f : M \to \mathbb{R}$. At each point p on M, f induces a map $f_* : T_pM \to T_{f(p)}\mathbb{R}$ between the tangent spaces of M at p and of $\mathbb{R}$ at $f(p)$.

Definition 1.1. $p \in M$ is a *critical point* of f if the induced map f_* is zero. More specifically, with local coordinate system $(x^1, \cdots, x^n)$, p satisfies

$$\frac{\partial f}{\partial x^1}(p) = \frac{\partial f}{\partial x^2}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0.$$

Definition 1.2. The *Hessian* $H_f(p)$ (or f_{**}) of a function $f : M \to \mathbb{R}$ at p is the $n \times n$ symmetric matrix whose ij-th entry is $\frac{\partial^2 f}{\partial x^i \partial x^j}$, where $(x^1, \cdots, x^n)$ is the local coordinate system at p. We said a critical point p is *nondegenerate* if the matrix $H_f(p)$ is nonsingular.

Definition 1.3. The *index* of a billinear form H is the maximal dimension of sunspace of V on which H is negative definite.

The point p is a nondegenerate critical point of f if and only if $H_f(p)$ has nullity equal to 0. The index of $H_f(p)$ on T_pM will be refered to simply as the index of f at p.

As mention above, we are going to study a "good function" on M. The notion of a "good function" is formalised to mean a Morse function.

Definition 1.4. A map $f : M \to \mathbb{R}$ is a *Morse function* if all the critical points of f are nondegenerate. That is, if $H_f(p)$ at each critical point p is non-singular.

To reach out goal of studying critical point p, we need a nice coordinate system to work near p. This important tool is Morse lemma. Furthermore, we only need information about the index at p to apply this proposition.

Proposition 1.5 (Morse lemma)**.** *If p is a nondegenerate critical point of f and the index of f at p is λ, then there exists local coordinate $(y_1, y_2, \cdots, y_n)$ in a neighborhood U of p with $y^i(p) = 0$ for all i and such that the identity*

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 \cdots + (y^n)^2$$

holds throughout U.

Before we prove Morse lemma, we firstly show the following:

Lemma 1.6. *Let f be a smooth function in a convex neighborhood V of 0 in $\mathbb{R}^n$, with $f(0) = 0$. Then*

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n)$$

for some suitable smooth finction g_i defined in V, with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof.

$$f(x_1, \cdots x_n) = \int_0^1 \frac{df(tx_1, \cdots, tx_n)}{dt}\, dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n) x_i\, dt.$$

Just define $g_i(x_1, \cdots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n)dt$, then we get the result. □

Now, we return to the proof of Morse lemma.

Proof of the Morse lemma. By linear algebra, we can easily show that for any such expression for f, λ must be the index of f at p.

It remains to show that there is such suitable coordinate system $(y^1, \cdots, y^n)$ exists. Without loss of generalization, we may assume that p is the origin of $\mathbb{R}^n$ and that $f(p) = f(0) = 0$. By the previous lemma, we can write

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n)$$

for $(x_1, \cdots x_n)$ on some neighborhood of 0. Since 0 is assumed to be a critical point,

$$g_i(0) = \frac{\partial f}{\partial x^j}(0) = 0.$$

By applying the lemma to g_i's, we get

$$g_i(x_1, \cdots x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, \cdots x_n),$$

for some smooth h_{ij} with

$$h_{ij}(0) = \frac{\partial g_i}{\partial x^j}(0) = \int_0^1 \frac{\partial^2 f}{\partial x^i \partial x^j}(tx_1, \cdots, tx_n) \cdot t \; dt\Big|_{\mathbb{X}=0} = \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(0).$$

It follows that

$$f(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \cdots, x_n).$$

Let $\bar{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, and then we have $\bar{h}_{ij} = \bar{h}_{ji}$ and $f = \sum_{i,j} x_i x_j \bar{h}_{ij}$. Moreover, the matrix $(\bar{h}_{ij}(0))$ is equal to $(\frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(0))$ and hence is nonsingular.

This give us the desired expression for f, in perhaps smaller neighborhood of 0. To complete the proof, we just imitate the prove of usual diagonalization for quadratic forms. The key step is as follows:

It will be proved by induction. Suppose that there is a coordinate $(u_1, \cdots, u_n)$ in a neighborhood U_1 of 0 such that:

$$f = \pm(u_1)^2 \pm \cdots \pm (u_{r-1})^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \cdots u_n)$$

throughout U_1, where the matrix (H_{ij}) are symmetric. We may assume that $H_{rr}(0) \neq 0$.

Now, let $g(u_1, \cdots, u_n)$ be the square root of $|H_{rr}(u_1, \cdots u_n)|$. Then g will be a smooth, non-zero function throughout some smaller neighborhood $U_2 \subset U_1$ of 0. Next, we introduce new coordinates $(v_1, \cdots v_n)$ by $v_i = u_i$, for $i \neq r$.

$$v_r = g \cdot \left[u_r + \frac{1}{H_{rr}} \sum_{i>r} u_i H_{ir} \right]$$

By the inverse function theorem, $(v_1, \cdots v_n)$ can serve as a coordinate function within a sufficiently smaller neighborhood U_3 of 0. So f can be expressed as

$$f = \sum_{i \leq r} \pm (v_i)^2 + \sum_{i,j>r} v_i v_j H'_{ij}$$

throughout U_3. This complete the induction and the proof of Morse lemma. □

Corollary 1.7. *Nondegenerate critical points are isolated.*

Another important tool for us to prove Morse theory is "*1-parameter group of diffeomorphism*". It gives us a way to construct the deformation we need in the proof of Morse theory.

Definition 1.8. A *1-parameter group of diffeomorphism* of a manifold M is a smooth map $\varphi : \mathbb{R} \times M \to M$ such that

1. for each $t \in \mathbb{R}$ the map $\varphi_t : M \to M$ defined by $\varphi_t(q) = \varphi(t, q)$ is a diffeomorphism of M onto itself,
2. for all $t, s \in \mathbb{R}$ we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

Definition 1.9. Given a 1-parameter group φ of diffeomorphisms of M we define a vector field X on M as follows. For every smooth real valued function f let

$$X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$$

This vector field X is said to *generate the group* φ.

Lemma 1.10. *A smooth vector field on M which vanishes outside of a compact set $K \subset M$ generates a unique 1- parameter group of diffeomorphisms of M.*

Proof. Given any smooth curve $t \mapsto C(t) \in M$, we can define the *velocity vector* $\frac{dc}{dt} \in T_{c(t)}M$ by the idenity $\frac{dc}{dt}(f) = \lim_{h\to 0} \frac{f(c(t+h))-f(c(t))}{h}$. Now, let φ be a 1-parameter group of diffeomorphisms, generated by the vector field X. Then for each fixed q the curve $t \mapsto \varphi_t(q)$ satisfies the differential equation $\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}$, with initial condition $\varphi_0(q) = q$. This is true since

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}(f) = \lim_{h\to 0} \frac{f(\varphi_{t+h}(q)) - f(\varphi_t(q))}{h} = \lim_{h\to 0} \frac{f(\varphi_h(p)) - f(p)}{h} = X_p(f),$$

where $p = \varphi_t(q)$. Also, such a differential equation locally, has a unique solution which depends smoothly on the initial condition.

Thus, for each point of M there exists a neighborhood U and a number $\epsilon > 0$ so that this differential equation has a unique solution for $q \in U$, and $|t| < \epsilon$.

By the compactness of K, we can cover it by finite number of such neighborhood U. Now, let $\epsilon_0 > 0$ be the smalllest of the corresponding ϵ. We setting $\varphi_t(q) = q$ for $q \notin K$, then this differential equation has a unique solution $\varphi_t(q)$ for $|t| < \epsilon_0$ and for all $q \in M$. Also, this solution is smooth as a function of both variables. Moreover, it is clear that $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for $|t|, |s|, |t+s| < \epsilon_0$, and each φ is a diffeomorphism.

It remains to defined φ_t for $|t| > \epsilon_0$. Any t can be expressed as

$$t = k(\frac{\epsilon_0}{2}) + r$$

with $k \in \mathbb{Z}$ and $|r| < \frac{\epsilon_0}{2}$. If $K \geq 0$, then set

$$\varphi_t = \varphi_{\frac{\epsilon_0}{2}} \circ \varphi_{\frac{\epsilon_0}{2}} \circ \cdots \circ \varphi_{\frac{\epsilon_0}{2}} \circ \varphi_r,$$

where $\varphi_{\frac{\epsilon_0}{2}}$ is iterated k times. If $k < 0$, we just replace $\varphi_{\frac{\epsilon_0}{2}}$ by $\varphi_{-\frac{\epsilon_0}{2}}$. It is easy to see φ_t is well-defined, smooth and satisfies $\varphi_{t+s} = \varphi_t \circ \varphi_s$. □

Remark. The hypothesis that X vanishes outside a compact set is important. For example, let M be the open interval $(0,1) \subset \mathbb{R}$, and that $X = \frac{d}{dt}$ be standard vector field on M. Then X does NOT generate any 1- parameter group of diffeomorphisms of M.

We now give the proof of Morse theory, and it will be cover by the two main theorems.

For each $a \in \mathbb{R}$, we denote the set $\{p \in M | f(p) \leq a\}$ by M^a. The following theorem is significant as it shows us the homotopy type of M^a can only change if a moves past a critical point, we will investigate the effects of when a moves past a critical point after this theorem.

Theorem 1.11. *If f is a smooth real valued function on M, $a \leq b$ and $f^{-1}[a,b]$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b. In fact, M^a is a deformation retract of M^b.*

Proof. The idea of the proof is to push M^b down to M^a along the orthogonal trajectories of the hypersurfaces $f =$ constant.

Notice that the vector field grad f can be characterized by the identity

$$< X, \operatorname{grad} f >= X(f),$$

for any vector field X. grad f vanishes precisely at the critical points of f. Also, for a curve $c : \mathbb{R} \to M$ with velocity vector $\frac{dc}{dt}$, we have

$$< \frac{dc}{dt}, \operatorname{grad} f >= \frac{dc}{dt}(f) = \frac{d(c \circ f)}{dt}.$$

Let $\rho : M \to \mathbb{R}$ be a smooth function which equal to $\frac{1}{<\operatorname{grad} f, \operatorname{grad} f>}$ throughout the compact set $f^{-1}[a,b]$; and which vanishes outside of a compact neighborhood of this set. Define X by

$$X_q = \rho(q)(\operatorname{grad} f)_q.$$

Then X satisfies the condition of Lemma 1.10, so X generated a 1-parameter group of diffeomorphism φ_t.

For each $q \in M$, consider the function $g_q(t) = f(\varphi_t(q))$. If $\varphi_t(q) \in f^{-1}[a,b]$, then

$$\frac{dg_q(t)}{dt} = \frac{df(\varphi_t(q))}{dt} =< \frac{d\varphi_t(q)}{dt}, \operatorname{grad} f >=< X, \operatorname{grad} f >= +1.$$

Therefore, $g_q(t)$ is linear with derivative 1 as long as $\varphi_t(q) \in f^{-1}[a,b]$. So $f(\varphi_t(q)) = t + f(q)$, whenever $f(\varphi_t(q)) \in [a,b]$. Thus, $\varphi_{b-a} : M \to M$ is a diffeomorphism carries M^a to M^b.

To see M^a is a deformation retract of M^b, define a parameter family of maps $r_t : M^b \to M^b$ by

$$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_{t(a-f(q))}(q) & \text{if } a \leq f(q) \leq b. \end{cases}$$

It is easy to see r_0 is identity, and r_1 is a retraction from M^b to M^a. This complete the proof. ∎

With the next theorem, we will have completely characterized the homotopy type of M based on a Morse function f defined on it.

Theorem 1.12. *Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a nondegenerate critical point of f with index λ. if $f(p) = c$, Suppose for some $\epsilon > 0$, $f^{-1}[c-\epsilon, c+\epsilon]$ is compact. It also contains no critical points of f other then p. Then for all sufficiently small ϵ, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ cell attached.*

Proof. We first modify f to a new function, F, that agrees with f except for in a small neighborhood of p. Then, when we look at those q such that $F(q) \leq c - \epsilon$, there will be an extra portion that $M^{c-\epsilon}$ will not have. Studying this extra portion will allow us to prove the theorem. By Morse lemma, we may write f as

$$f = c - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots (x^n)^2,$$

where $(x^1, \cdots, x^n)$ is a local coordinate in a neighborhood U of p such that

$$x^1(p) = \cdots = x^n(p) = 0.$$

Next, choose $\epsilon > 0$ sufficiently small so that the image of U under the diffeomorphism imbedding $(x^1, \cdots, x^n) : U \to \mathbb{R}^n$ contains the closed ball $\{(x^1, \cdots, x^n) | \sum (x^i)^2 \leq 2\epsilon\}$, and $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no critical points other that p.

We now let g be a smooth function such that:

1. $g(0) > \epsilon$;
2. $g(r) = 0$ for $r > 2\epsilon$;
3. $-1 < g'(r) \leq 0$ for all r.

We defined F to be a function in U by

$$F = f - g((x^1)^2 + \cdots (x^\lambda)^2 + 2((x^{\lambda+1})^2 + \cdots (x^n)^2).$$

It is convenient to define two functions $\xi, \eta : U \to [0, \infty)$ by

$$\xi = (x^1)^2 + \cdots (x^\lambda)^2 \qquad \text{and} \qquad \eta = (x^{\lambda+1})^2 + \cdots (x^n)^2.$$

Then $f = c - \xi + \eta$ and $F = c - \xi + \eta - g(\xi + 2\eta)$. By the construction of g, $g \geq 0$ for all $r \geq 0$. Moreover, $g(r) = 0$ when $r > 2\epsilon$. So we find that $F \leq f$ when $\xi + 2\eta \geq 2\epsilon$, and $F = f$ when $\xi + 2\eta > 2\epsilon$.

Claim 1. *The region $F^{-1}(-\infty, c+\epsilon]$ coincides with the region $M^{c+\epsilon}$.*

Proof of the claim. If $\xi + 2\eta \geq 2\epsilon$, then $F = f$.

If $\xi + 2\eta > 2\epsilon$, we have

$$F \leq f = x - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c + \epsilon.$$

So the region $\xi + 2\eta \geq 2\epsilon$ lies in both $F^{-1}(-\infty, c+\epsilon]$ and $M^{c+\epsilon}$. □

Claim 2. *The critical points of F in U are the same as those of f in U.*

Proof of the claim. Notice that

$$\frac{\partial F}{\partial \xi} = -1 - g'(\xi + 2\eta) < 0$$
$$\frac{\partial F}{\partial \eta} = 1 - 2g'(\xi + 2\eta) > 1,$$

and

$$dF = \frac{\partial F}{\partial \xi}\, d\xi + \frac{\partial F}{\partial \eta}\, d\eta.$$

So $dF = 0$ in the region $\xi + 2\eta \leq 2\epsilon$ if and only if $d\xi$ and $d\eta$ are both 0. Then F has no critical points in U other than the origin, which was the only critical point of f in U. □

Claim 3. *The region $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $M^{c+\epsilon}$.*

Proof of the claim. Consider $F^{-1}[c-\epsilon, c+\epsilon]$. By Claim 1 and $F \leq f$, we get

$$F^{-1}[c-\epsilon, c+\epsilon] \subset f^{-1}[c-\epsilon, c+\epsilon].$$

Therefore, $F^{-1}[c-\epsilon, c+\epsilon]$ is compact. Also,

$$F(p) = c - g(0) < c - \epsilon,$$

so the only possible critical point p of F is not in $F^{-1}[c-\epsilon, c+\epsilon]$. Thus, we can apply Theorem 1.11 which gives the desired result. □

For convenient, we denote this region $F^{-1}(-\infty, c-\epsilon]$ by $M^{c-\epsilon} \cup H$, where H denotes the closure of $F^{-1}(-\infty, c-\epsilon] \setminus M^{c-\epsilon}$.

Now consider the cell e^λ consisting of all points q with $\xi(q) \leq \epsilon$, $\eta(q) = 0$. Note that e^λ is contained in the "handle" H, since $\frac{\partial F}{\partial \xi} < 0$, we have

$$F(q) \leq F(p) < c - \epsilon,$$

but $f(q) = c - \xi(q) + \eta(q) \geq c - \epsilon$ for $q \in e^\lambda$. So $e^\lambda \subset F^{-1}(-\infty, c-\epsilon] \setminus M^{c-\epsilon} \subset H$.

Claim 4. *$M^{c-\epsilon} \cup e^\lambda$ is a deformation restract of $M^{c-\epsilon} \cup H$.*

Proof of the claim. Define r_t be the identity ontside U, and define r_t within U as follows.

Case 1 On the region $\xi \leq \epsilon$, define r_t by

$$(x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^\lambda, t\lambda^{\lambda+1}, \cdots tx^n).$$

It is easy to cheak r_t maps $F^{-1}(-\infty, c-\epsilon]$ into itself since $\frac{\partial F}{\partial \eta} > 0$. Also, r_1 is the identity and r_0 maps this region into e^λ.

Case 2 Within the region $\epsilon < \xi \leq \eta + \epsilon$, define r_t by

$$(x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^\lambda, s_t x^{\lambda+1}, \cdots, s_t x^n),$$

where the number $s_t \in [0, 1]$ is defined by

$$s_t = t + (1-t)\sqrt{\frac{\xi - \epsilon}{\eta}}.$$

Thus, r_1 is again the identity, and r_0 maps this region into $f^{-1}(c-\epsilon)$. Notice that r_t is continous as $\xi \to \epsilon$, $\eta \to 0$, and this definition coincides with that of the Case 1 when $\xi = \epsilon$.

Case 3 On the region $\eta + \epsilon < \xi$ (i.e. on $M^{c-\epsilon}$). Let r_t be the identity. This coincide the Case 2 when $\xi = \eta + \epsilon$.

Hence, we get the desired maps r_t. □

Together all these four claims, we complete the proof of the therorem. □

It is amazing that a study of local behavior of a Morse fuction f can determine the homotopy type of M.

2 Conjugate Points and Path Space

To prove the Bott periodicity theorem, we need several tool. Some of them are applications of Morse theory.

Let M be a smooth manifold and let p and q be two (not necessarily distinct) points of M. A *piecewise smooth path from* p *to* q will be meant a map $\omega : [0,1] \to M$ such that

1. there exists a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ so that each $\omega|_{[t_{i-1},t_i]}$ is smooth;

2. $\omega(0) = p$ and $\omega(1) = q$.

We denote *the set of all piecewise smooth paths from* p *to* q in M by $\Omega(M; p, q)$, or briefly by $\Omega(M)$ or Ω.

Suppose now that$M$ is a Riemannian manifold. The length of a vector $v \in T_pM$ will be denoted by $\|v\| =< v, v >^{\frac{1}{2}}$. For $\omega \in \Omega$, define *the energy of* ω *from* a *to* b (where $0 \leq a < b \leq 1$) as

$$E_a^b(\omega) = \int_a^b \left\| \frac{d\omega}{dt} \right\|^2 \, dt.$$

We denote E_0^1 by E.

Definition 2.1. Let $p = \gamma(a)$ and $q = \gamma(b)$ be two points on the geodesic γ with $a \neq b$. p and q are said to be *conjugate along* $\gamma(t)$ if there is a non-zero Jacobi field J along $\gamma(t)$ which vanishes at $t = a$ and $t = b$. The *multiplicity of the conjugate points* is the dimension of the vector space of all the Jacobi fields satisfying this condition.

Notice that the *index* of the Hessian of E

$$E_{**} : T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$$

is defined to be the maximum dimension of a subspace of $T_\gamma\Omega$ on on which E_{**} is negative definite.

To compute the index of a geodesic, We will state the following theorem without proof. This theorem allow us compute the index by counting the multiplicity of all the conjugate points.

Theorem 2.2 (Morse). *The index* λ *of* E_{**} *is equal to the number of points* $\gamma(t)$*, with* $0 < t < 1$*, such that* $\gamma(t)$ *is conjugate to* $\gamma(0)$ *along* γ*; each such conjugate point being counted with its multiplicity. This index* λ *is always finite.*

Now, we are going to introduce a useful tool which connect the multiplicity and the eigenvalue of a special linear transformation on T_pM.

Theorem 2.3. *Let $\gamma : \mathbb{R} \to M$ be a geodesic in a locally symmetric manifold. Let $V = \frac{d\gamma}{dt}(0)$ be the velocity vector at $p = \gamma(0)$. Define a linear transformation $K_V : T_pM \to T_pM$ by $K_V(W) = R(V, W)V$. Let $e_1, \cdots, e_n$ denote the eigenvalues of K_V. The conjugate points to p along γ are the points $\gamma(\pi k/\sqrt{e_i})$ where k is any non-zero integer, and e_i is any positive eigenvalue of K_V. The multiplicity of $\gamma(t)$ as a conjugate point is equal to the number of e_i such that t is a multiple of $\pi/\sqrt{e_i}$.*

Proof. First observe that K_V is self-adjoint:

$$< K_V(W), W' >=< W, K_V(W') > .$$

This follows immediately from the symmetry relation

$$< R(X, Y)Z, W >=< R(Z, W)X, Y > .$$

Therefore we may choose an orthonormal basis $U_1, \cdots, U_n$ for M_p so that

$$K_V(U_i) = e_iU_i,$$

where $e_1, \cdots, e_n$ are the eigenvalues. Extend the U_i to vector fields along γ by parallel translation. Then since M is locally symmetric, the condition

$$R(V, U_i)V = e_iU_i$$

remains ture along γ. Any vector field W along γ may be expressed uniquely as

$$W(t) = W_1(t)U_1(t) + \cdots W_n(t)U_n(t).$$

Then the Jacobi equation$\frac{D^2W}{dt^2} + K_V(W) = 0$ takes the form $\sum \frac{D^2W}{dt^2}U_i + \sum e_iW_iU_i = 0$. Since the U_i's are everywhere linearly independent, this is equivalent to the system of n equations

$$\frac{d^2W_i}{dt^2} + e_iW_i = 0.$$

If $e_i > 0$ then

$$w_i(t) = c_i \sin(\sqrt{e_i}t),$$

for some constant c_i. Then the zeros of $W_i(t)$ are at the the multiples of $t = \pi/\sqrt{e_i}$.

If $e_i = 0$, then $W_(t) = c_i t$ and if $e_i < 0$, then $W_i(t) = c_i \sinh(\sqrt{|e_i|}t)$, for some constant c_i. Thus, if $e_i \leq 0$, $W_i(t)$ vanish only at $t = 0$. This complete the proof. $\square$

Let $\sqrt{d}$ be the length of minimal geodesic from p to q, and denote $E^{-1}[0, d]$ by Ω^d. The next theorem shows what conditions make the relative homotopy group $\pi_i(\Omega, \Omega^d) = 0$. These conditions have something to do with the index of geodesic.

Theorem 2.4. *If the space of minimal geodesics from p to q is a topological manifold, and if every non-minimal geodesic from p to q has index at least λ_0, then the relative homotopy group $\pi_i(\Omega, \Omega^d)$ is zero for $0 \leq i \leq \lambda_0$.*

The proof will be based on the following lemmas:

Let K be a compact subset of $\mathbb{R}^n$, and U be a neighborhood of K, and let $f : U \to \mathbb{R}$ be a smooth function such that all critical points of f in K have index $\geq \lambda$.

Lemma 2.5. *If $g : U \to \mathbb{R}$ is a smooth function such that*

$$\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \epsilon \text{ and } \left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \leq \epsilon,$$

for all i,j uniformly throughout K, for some small ϵ, then all the critical points of g have index $\geq \lambda$.

Proof. let

$$K_g = \sum_i \left|\frac{\partial g}{\partial x_i}\right| > 0.$$

Let $e_g^1(x) \leq \cdots \leq e_g^n(x)$ be the n eigenvalues of the matrix that has ij-th entry $(\frac{\partial g}{\partial X_i \partial x_j})$. So we see a critical point of x is of index at least λ if and only if $e_g^l ambda(x)$ is negative. Note that these functions are continuous as the eigenvalues of a matrix depend continuously on the entries of the matrix.

Now, consider $m_g(x) = \max\{K_g(x), -(e_g)^{\lambda_0}\}$ and define $m_f(x)$ similarly. As the critical points of f have index at least λ_0, we must have $-e_f^{\lambda_0}(x) > 0$ whenever $K_f(x) = 0$. So $m_f(x) > 0$, for all $x \in K$. Now, let δ be the minimum of m_f. Suppose g is so "closed" to f so that

$$|K_g(x) - K_f(x)| < \delta \text{ and } |e_g^{\lambda_0}(x) - e_f^{\lambda_0}(x)| < \delta. \qquad (\star)$$

Then m_g is always positive; hence, every critical point of g will have index at least λ_0.

Finally, it is easy to show that $(\star)$ will be satisfied providing that

$$\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \epsilon \text{ and } \left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \leq \epsilon,$$

for sufficiebtly small ϵ. This proves the lemma. □

We can now show a special case of the desired theorem.

Lemma 2.6. *Let $f : M \to \mathbb{R}$ be a smooth function with minimum 0, such that each $M^c = f^{-1}[0, c]$ is compact. If M^0 is a manifold, and the critical points of $M \setminus M^0$ has index at least λ_0, then $\pi_r(M, M^0) = 0$ for $0 \leq r \leq \lambda_0$.*

Proof. Choose a neighborhood around each point of M^0 so that M^0 is a retract of the open set which is unions of the neighborhoodẆe may assume that each point of U is joined to the point of M^0 of which it is in a neighborhood of (we can shrink the neighborhood so that each neighborhood contains only one point of M^0 if necessary).

Let I^r be the unit cube of dimension $r < \lambda_0$. Consider a function

$$h : (I^r, S^r) \to (M, M^0).$$

We are going to show that h is homotopic to a map h' where $f'(I^r) \subset M^0$.

Firstly, We choose a g that approximates f on M^c, where c is the maximum of f on $h(I^r)$. By the previous lemma, we can choose g such that it has no degenerate critical points and each critical point has index at least λ.

Let δ be the minimum of f on $M \setminus U$, then $g^{-1}(M^c)$ has the homotopy type of the union of $g^{-1}(-\infty, \delta]$ and cells of dimenstion λ. Then consider $h : (I^r, S^r) \to (M^c, M^0) \subset (g^{-1}(-\infty, c + \epsilon), M^0)$.

Since $r < \lambda$, then h is homotopic to some h' that maps into $(g^{-1}(-\infty, \delta), M^0)$. This is true because all the critical points of g have index $> \lambda$. However, $g^{-1}(-\infty, 2\epsilon]$ is contained in U and U can be deformed into M^0, so we have $\pi_r(M, M^0) = 0$. □

proof of the Theorem 2.4. We use the energy function restricted to $Int\Omega(t_0, \cdots, t_k)$ to relate the previous theorem to geodesics. Note that the energy function satisfies all the hypotheses of the previous theorem except that it does not range over $[0, \infty)$. We can fix this by just applying some diffeomorphism that takes the range of E into $[0, \infty)$. Call such a diffeomorphism f, then applying the previous lemma to the function $f \star E$ gives $\pi_i(Int\Omega(t_0, \cdots, t)k), \Omega^d) = 0$ as desired. □

There is a more useful form of Theorem 2.4, and in fact, this is what we use to prove the Bott periodicity theorme.

Corollary 2.7. *If the space of minimal geodesics is a topological manifold, and if every non minimal geodesic has index at least* λ_0 *then* $\pi_i(\Omega^d)$ *is isomorphic to* $\pi_{i+1}(M)$ *for* i *at most* $\lambda_0 - 2$.

Proof. $\pi_i(\Omega^d)$ is isomorphic to $\pi_i(\Omega)$ for i less than $\lambda_0 - 1$ because the relative homotopy group is 0, and $\pi_i(\Omega)$ is isomorphic to $\pi_{i+1}(M)$. □

3 Bott Periodicity Theorem

Let $\mathbb{C}^n$ be the space of n-tuples of complex numbers, equipped the standard Hermitian inner product. The unitary group $U(n)$ is the group of all linear transformations $S : \mathbb{C}^n \to \mathbb{C}^n$ which preserve this inner product. Equivalently, using the matrix representation, $U(n)$ is the group of all $n \times n$ complex matrices S such that $SS^* = I$; where S^* denotes the conjugate transpose of S.

For any $n \times n$ complex matrix A the *exponential of* A is defined by the convergent power series expansion:

$$\exp A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

The following properties are easily verified:

1. $\exp(A^*) = (\exp A)^*$; $\exp(TAT^{-1}) = T(\exp A)T^{-1}$;
2. If A and B commute then $\exp(A+B) = (\exp A)(\exp B)$. In particular, $(\exp A)(\exp -A) = I$.
3. The function exp maps a neighborhood of 0 in the space of $n \times n$ matrices diffeomorphically onto a neighborhood of I.

It follows from the above that A is skew-Hermitian (i.e. if $A + A^* = 0$) if and only if $\exp A$ is unitary. It is easy to see:

4. $U(n)$ is a smooth submanifold of the space of $n \times n$ matrices;
5. the tangent space of $T_I U(n)$ can be identified with the space of $n \times n$ skew-Hermitian matrices.

Thus, the Lie algebra $\mathfrak{g}$ of $U(n)$ can be identified with the space of skew-Hermitian matrices. For any tangent vector at I extends uniquely to a *left invariant vector field* on $U(n)$. A directly computation shows that the Lie bracket of left invariant vector field is as same as the Lie bracket of matrices, $[A, B] = AB - BA$.

Since $U(n)$ is compact, it processes a left and right invariant Riemannian metric. Notice that the map $\exp : T_I U(n) \to U(n)$ defined by exponentiation of matrices coincides with the exponential map defined by geodesics on the resulting Riemannian manifold. In fact, for each skew-Hermitian matrix A, the map $t \mapsto \exp(tA)$ defines a 1-parameter subgroup of $U(n)$ and hence defines a geodesic.

We now define an inner product by

$$< A, B >= \mathrm{Re}(\mathrm{trace}(AB^*)) = \mathrm{Re} \sum_{i,j} A_{ij} \bar{B}_{ij}.$$

It is clearly that this is positive definite, $0 \Leftrightarrow A$ or $B = 0$, conjugate symmetric and linear on $\mathfrak{g}$. This inner product on $\mathfrak{g}$ induced a left invariant Riemannian metric on $U(n)$. To check that the resulting metric is also right invariant, we check it is invariant under the adjoint action $U(n)$ on $\mathfrak{g}$.

Definition 3.1. An *adjoint action* is: each $S \in U(n)$ determines an automorphism $X \mapsto SXS^{-1} = (L_S R_S^{-1})X$. The induced mapping $(L_S R^{-1})_*$ is denoted Ad_S. As $\exp(TAT^{-1}) = T \exp(A) T^{-1}$, we then have $\mathrm{Ad}_S A = SAS^{-1}$.

We see that the inner product is invariant under Ad_S by direct computation:

$$\begin{aligned} < \mathrm{Ad}_S A, \mathrm{Ad}_S B > &= \mathrm{Re}(\mathrm{trace}((\mathrm{Ad}_S A)(\mathrm{Ad}_S B))) \\ &= \mathrm{Re}(\mathrm{trace}(SAS^{-1}(SBS^{-1})^*)) \\ &= \mathrm{Re}(\mathrm{trace}(SAS^{-1}(S^{-1})^* B^* S^*)) \\ &= \mathrm{Re}(\mathrm{trace}(SAB^*S^*)) \qquad (\because S \in U(n)) \\ &= \mathrm{Re}(\mathrm{trace}(AB^*)) =< A, B > \end{aligned}$$

It follows that the corresponding left invariant metric on $U(n)$ is also right invariant.

Given $A \in \mathfrak{g}$ we know that there exists $T \in U(n)$ such that TAT^{-1} is in diagonal form:

$$TAT^{-1} = \begin{pmatrix} ia_1 & & & \\ & ia_2 & & \\ & & \ddots & \\ & & & ia_n \end{pmatrix},$$

where the a_i's are real. Also, for any $S \in U(n)$, there exists $T \in U(n)$ such that:

$$TST^{-1} = \begin{pmatrix} e^{ia_1} & & & \\ & e^{ia_2} & & \\ & & \ddots & \\ & & & e^{ia_n} \end{pmatrix},$$

where the a_i's are real again. Hence, $\exp : \mathfrak{g} \to U(n)$ is surjective.

We may treat the special unitary group $SU(n)$ in the same way. $SU(n)$ is defined as the subgroup of $U(n)$ consisting of matrices of determinant 1. It is easy to show that for $T \in U(n)$ such that TAT^{-1} is diagonal, then

$$\det(\exp A) = \det(T(\exp A)T^{-1}) = \det(\exp(TAT^{-1})) = e^{\text{trace}(TAT^{-1})} = e^{\text{trace}\, A}.$$

This shows that the Lie algebra of $SU(n)$, $\mathfrak{g}'$, is the set of all matrices A such that $A + A^* = 0$ and $\text{trace}\, A = 0$.

To apply Mores theory, we need to consider the geodesics from I to $-I$. In other words, we consider all $A \in \mathfrak{g} = T_I U(n)$ such that $\exp A = -I$. Suppose A is such matrix. If it is not of diagonal form, let $T \in U(n)$ so that TAT^{-1} is diagonal. Then we have:

$$\exp(TAT^{-1}) = T(\exp)AT^{-1} = T(-I)T^{-1} = -I.$$

Thus, we may assume that A is diagonal:

$$A = \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix}.$$

Then

$$\exp A = \begin{pmatrix} e^{ia_1} & & \\ & \ddots & \\ & & e^{ia_n} \end{pmatrix}.$$

So in this case, $\exp A = -I$ if and only if A is of the form

$$\begin{pmatrix} ik_1\pi & & & \\ & ik_2\pi & & \\ & & \ddots & \\ & & & ik_n\pi \end{pmatrix},$$

for some odd integers $k_1, \cdots, k_n$.

It is clearly that the length of geodesic $t \mapsto \exp tA$ from $t = 0$ to $t = 1$ is

$$|A| = \sqrt{\mathrm{Re}(\mathrm{trace}\, AA^*))} = \sqrt{\mathrm{trace}(AA^*)},$$

so the length of the geodesic is determined by

$$\pi\sqrt{k_1^2 + \cdots + k_n^2}.$$

Thus, A determines a minimal geodesic if and only if each $k_i = \pm 1$, and in this case the length is $\pi\sqrt{n}$.

Now, treat A as a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$, then A is complete determined by $\mathrm{Eigen}(i\pi)$, the eigenspace with respect to eigenvalue $i\pi$, and $\mathrm{Eigen}(-i\pi)$, the eigenspace with respect to eigenvalue $-i\pi$. In fact, since $\mathbb{C}^n$ splits as

$$\mathrm{Eigen}(i\pi) \oplus \mathrm{Eigen}(-i\pi),$$

it can only determined by $\mathrm{Eigen}(i\pi)$, which can be an arbitrary subspace of $\mathbb{C}^n$. Hence, the space of all minimal geodesic in $U(n)$ from I to $-I$ can be identified with the space of all sub-vector-space of $\mathbb{C}^n$.

Unfortunately, this space is inconvenient to use since its element has varying dimensions. This difficulty can be removed by replacing $U(n)$ by $SU(n)$ and setting $n = 2m$. In this case, all the discussion above remain valid. But the additional condition that $k_1 + \cdots k_{2m} = 0$ with $k_i = \pm 1$ restricts $\mathrm{Eigen}(i\pi)$ to being an arbitrary m-dimensional sub-vector-space of $\mathbb{C}^{2m}$. This proves the following:

Lemma 3.2. *The space of minimal geodesics from I to $-I$ in $SU(2m)$ is homeomorphic to the complex Grassmann manifold $G_m(\mathbb{C}^{2m})$, consisting of all m-dimensional sub-vector-spaces of c^{2m}.*

Lemma 3.2 shows the minimal geodesics from I to $-I$ in $SU(2m)$ is a manifold. To apply Corollary 2.7, we also need the information about index of non-minimal geodesics.

Lemma 3.3. *Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.*

Proof. To compute the index of non-minimal geodesic from I to $-I$ on $SU(2m)$, let $A \in \mathfrak{g}'$ be a matrix corresponds to a geodesic from I to $-I$ (i.e. the eigenvalues of A have the form $ik_1\pi, \cdots, ik_n\pi$ where k_i's are odd integers with sum zero).

We need to find the conjugate points to I along the geodesic $t \mapsto \exp(tA)$. According to Theorem 2.3, these will be determined by the positive eigenvalues of the linear transformation

$$K_A : \mathfrak{g}' \to \mathfrak{g}',$$

where

$$K_A(W) = R(A, W)A = \frac{1}{4}[[A, W], A].$$

We may assume A is diagonal matrix:

$$\begin{pmatrix} ik_1\pi & & \\ & \ddots & \\ & & ik_n\pi \end{pmatrix}$$

with $k_1 \geq k_2 \geq \cdots \geq k_n$. If $W = (w_{j\ell})$, then a direct computation shows that

$$[A, W] = i\pi(k_j - k_\ell)w_{j\ell},$$

and hence

$$[A, [A, W]] = -\pi^2(k_j - k_\ell)^2 w_{j\ell}.$$

So,

$$K_A(W) = \frac{\pi^2}{4}(k_j - k_\ell)^2 w_{j\ell}.$$

Now we find a basis for $\mathfrak{g}'$ consisting of eigenvectors of K_A, as follows:

1. For each $j < \ell$, let $E_{j\ell}$ be the matrix with $+1$ in the $j\ell$-th entry, -1 in the $j\ell$-th place and zeros elsewhere. It is in $\mathfrak{g}'$ and is an eigenvector corresponding to the eigenvalue $\frac{\pi^2}{4}(k_j - k_\ell)^2$.

2. Similarly for each $j < \ell$, let $E'_{j\ell}$ be the matrix with $+i$ in the $j\ell$-th entry, $-i$ in the $j\ell$-th place and zeros elsewhere. It is in $\mathfrak{g}'$ and is an eigenvector corresponding to the eigenvalue $\frac{\pi^2}{4}(k_j - k_\ell)^2$

3. Each diagonal matrix in $\mathfrak{g}'$ is an eigenvector with eigenvalue 0.

Thus, the non-zero eigenvalues of K_A are the numbers $\frac{\pi^2}{4}(k_j - k_\ell)^2$ with $k_j > k_\ell$. Each such eigenvalue is to be counted twice.

Now consider the geodesic $\gamma(t) = \exp(tA)$. Each eigenvalue $e = \frac{\pi^2}{4}(k_j - k_\ell)^2 > 0$ give rise to a series of conjugate points along γ corresponding to the values

$$t = \frac{\pi}{\sqrt{e}}, \frac{2\pi}{\sqrt{e}}, \frac{3\pi}{\sqrt{e}}, \cdots .$$

This gives

$$t = \frac{2}{(k_j - k_\ell)}, \frac{4}{(k_j - k_\ell)}, \frac{6}{(k_j - k_\ell)}, \cdots .$$

The numder of such value of t in $(0, 1)$ is equal to $\frac{k_j - k_\ell}{2} - 1$ (We need to minus one since the value $t = 1$ is not included).

Now let us apply the Index Theorem. For each j, ℓ with $k_j > k_\ell$, we obtain two copies of the eigenvalue $\frac{\pi^2}{4}(k_j - k_\ell)^2$, and hence a contribution of $2(\frac{k_j - k_\ell}{2} - 1)$ to the index. Sum over all j, ℓ, this gives

$$\lambda = \sum_{k_j > k_\ell} (k_j - k_\ell - 2)$$

for the index of the geodesic γ.

Now, we divided it into three cases:

Case 1 At least $m+1$ of the k_i's are negative. In this case at least one of the positive k_i must be ≥ 3, and we have

$$\lambda \geq \sum_{1}^{m+1} (3 - (-1) - 2) = 2(m + 1).$$

Case 2 At least $m+1$ of the k_i's are positive. In this case at least one of the negative ≤ -3, so

$$\lambda \geq \sum_{1}^{m+1} (1 - (-3) - 2) = 2(m + 1).$$

Case 3 m of the k_i are positive and m are negative but not all are ± 1 (since we assume that γ is non-minimal). Then one is ≥ 3 and one is ≤ -3, so

$$\lambda \geq \sum_{1}^{m-1} (3 - (-1) - 2) + \sum_{1}^{m+1} (1 - (-3) - 2) + (3 - (-3) - 2) = 4m \geq 2(m + 1).$$

Thus, in either case we have $\lambda \geq 2m+2$. This proves Lemma.

□

Then now we can prove the following:

Theorem 3.4. *The inclusion map* $G_m(\mathbb{C}^{2m}) \to \Omega(SU(2m); I, -I)$ *induces isomorphisms of homotopy groups in dimensions* $\leq 2m$. *Hence,*

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1} SU(2m),$$

for $i \leq 2m$.

Proof. By Lemma 3.2, $\Omega(SU(2m); I, -I)$ is a topological manifold, since it is isomorphic to $G_m(\mathbb{C}^{2m})$. Also, every non-minimal geodesic has index at least $\lambda_0 = 2m+2$, then by Corollary 2.7,

$$\pi_i G_m(\mathbb{C}^{2m}) = \pi_i(\Omega^{\pi^2 n}) \cong \pi_{i+1}(SU(2m)),$$

for $i \leq \lambda_0 - 2 = 2m$. □

We are now going to establish the relation between of homotopy groups of $U(m)$ and those of $SU(m)$.

Lemma 3.5. *The group* $\pi_i G_m(\mathbb{C}^{2m})$ *is isomorphic to* $\pi_{i-1} U(m)$ *for* $i \leq 2m$. *Moreover,*

$$\pi_{i-1} U(m) \cong \pi_{i-1} U(m+k)$$

for $i \leq 2m$, $k \in \mathbb{N}$; *and*

$$\pi_j(U(m)) \cong \pi_j(SU(m)),$$

for $j \neq 1$.

Proof. We can choose fibrations

$$U(m) \to U(m+1) \to \mathbb{S}^{2m+1}$$

and

$$U(m) \to U(2m) \to U(2m)/U(m).$$

From the first one, we get

$$\cdots \to \pi_i \mathbb{S}^{2m+1} \to \pi_{i-1} U(m) \to \pi_{i-1} U(m+1) \to \pi_{i-1} \mathbb{S}^{2m+1} \to \cdots,$$

and this becomes

$$0 \to \pi_{i-1}U(m) \to \pi_{i-1}U(m+1) \to 0,$$

when $i \leq 2m$.

Also, the second fibration gives

$$\cdots \to \pi_i U(2m)/U(m) \to \pi_{i-1}U(m) \to \pi_{i-1}U(2m) \to \pi_{i-1}U(2m)/U(m) \to \cdots,$$

which implies $\pi_i(U(2m)/U(m)) = 0$, for $i \leq 2\pi$.

Notice that the complex Grassmann manifold $G_m(\mathbb{C}^{2m})$ can be identified with $U(2m)/(U(m) \times U(m))$, so we have a fibration:

$$U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m}).$$

Using this fibration and $\pi_i(U(m)/U(2m)) = 0$, for $i \leq 2m$, we now get:

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i-1}U(m),$$

for $i \leq 2m$.

Finally, from the fibration

$$SU(m) \to U(m) \to \mathbb{S}^1,$$

we obtain that

$$\pi_j SU(m) \cong \pi_j U(m),$$

for $j \neq 1$. This proves the lemma. □

From now on, we use $\pi_i U$ to denote the i-th *stable homotopy group of the unitary group.*

So, we see that:

$$\pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U.$$

The first and the third isomorphism follows from Lemma 3.5, and the second isomorphism comes from Theorem 3.4. This proves the famous Bott Periodicity Theorem.

Theorem 3.6 (Bott Periodicity Theorem)**.** *For $i \geq 1$,*

$$\pi_{i-1}U \cong \pi_{i+1}U.$$

Final Report on Atiyah–Singer Index Theorem via Twisted Signature Formula

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Contents

1 Introduction

Let $E, F \to M$ be two real or complex vector bundle over a compact manifold M. Let $P : C^\infty(M, E) \to C^\infty(M, F)$ be an elliptic operator. The **index** of P is defined by

$$\begin{aligned} \operatorname{ind} P &:= \dim \ker P - \dim \operatorname{coker} P \\ &= \dim \ker P^* P - \dim \ker P P^*. \end{aligned}$$

We had already studied that the index of an elliptic operator is expressible as an integral on M via the heat kernel. This is known as McKean-Singular formula

$$\operatorname{ind} P = \int_M \left(\operatorname{tr}_{E_x} H_{P^*P}(x, x, t) - \operatorname{tr}_{F_x} H_{PP^*}(x, x, t) \right) \operatorname{dvol}_M(x).$$

Also, in the lecture, we had introduced the Dirac operator D on a Clifford module E. In order to prove the local index theorem, we need to use the Lichnerowicz formula and the Getzler scaling method.

In this report, we will give a different approach to Atiyah–Singer index theorem via twisted signature formula due to Gilkey [Gil95]. Precisely, we will fiest study the index of twisted Dirac operator for the vector bundle E:

$$D : C^\infty \left(\bigwedge^\pm (T^*M) \otimes E \right) \to C^\infty \left(\bigwedge^\mp (T^*M) \otimes E \right).$$

Then, we will prove the twisted signature formula:

$$\operatorname{ind} D = \int_M L(M) \operatorname{ch}(E)$$

using the invariance theory. Finally, based on the twisted signature formula, we will prove the Atiyah–Singer index theorem for general elliptic operator P by interpreting the index as a function on K-ring under K-theory language.

In the second section, we will introduce the important invariant for an elliptic operator. These invariants are highly related to the heat kernel $H(x, y, t)$ of the elliptic operator P. In the third section, we will study the invariance theory on manifolds and vector bundles. Explicitly, we will prove that all the invariants in terms of derivatives of metric and connection one form are all linearly span of wedge products of Pontryagin classes of TM and Chern classes of E. This will help us in proving the twisted signature formula. In the fourth section, we give the proof of twisted signature formula via the invariants. After that, in the fifth section, we will prove the Atiyah–Singer Index Theorem for general elliptic operators. To achieve this goal, we need to interpret the index of an elliptic operator as a function in K-theory language:

$$\operatorname{ind} : K(\Sigma(T^*M); \mathbb{C}) / K(M; \mathbb{C}) \to \mathbb{C},$$

where $\Sigma(T^*M)$ is the fiberwise suspension of the unit sphere bundle $S(T^*M)$. Then, we will prove that the group $K(\Sigma(T^*M); \mathbb{C}) / K(M; \mathbb{C})$ is generated by the special bundles $\left\{ \Pi_+ (\Sigma_{\sigma_{P_E}}) \right\}_{E \in \operatorname{Vect}(M)}$. Then, the Atiyah–Singer Index Theorem reduce to these special case that we may simply apply the twisted signature formula.

As a supplement, in the last section, we will give a sketch of original proof of Atiyah–Singer Index Theorem [AS63]. There are three key points in the proof. First, we introduce the group of elliptic symbols $\operatorname{Ell}(M)$. By analyzing the group, we reduced the Atiyah–Singer Index Theorem to the twisted signature formula. Next, we introduce the cobordism ring A on the pair (M, E), where E is a complex vector bundle on M. Using the knowledge of singular integral operators on manifolds, we can prove that the null-cobordant elements in A satisfy the theorem. Therefore, the concept of index is well-defined on the cobordism ring A. Finally, generalizing the Thom isomorphism theorem, we see that $A \otimes \mathbb{Q}$ is generated by $\left(\mathbb{CP}^{2i}, 1\right)$ and (S^{2j}, V_j) as a polynomial algebra. Eventually, to achieve the theorem is to verify the case $\mathbb{CP}^{2i}$ and S^{2j} as the generators of $A \otimes \mathbb{Q}$.

2 Local Formula for the Index

Theorem 1. Let P be a self-adjoint elliptic operator of order $d > 0$ on a vector bundle E over compact manifold M^m such that the symbol $\sigma_P(x, \xi)$ of P is positive definite for $\xi \neq 0$. Then,

1. If we choose a coordinate system for M near a point $x \in M$ and choose a local frame for E, we can define $e_n(x)$ depending on the symbol $\sigma_P(x, \xi)$ such that if $H(t, x, y)$ is the heat kernel of e^{-tP} then

$$H(t, x, x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} e_n(x) \quad \text{as } t \to 0^+$$

i.e., given any integer k, there exists $n(k)$ such that:

$$\left| H(t, x, x) - \sum_{n \leq n(k)} t^{\frac{n-m}{d}} e_n(x) \right|_{\infty,k} < C_k t^k \quad \text{for } 0 < t < 1.$$

2. Moreover, $e_n(x) \in \mathrm{END}(E, E)$ is invariantly defined independent of the coordinate system and local frame for E.

Theorem 2.

(a) Let $P_i : C^\infty(E_i) \to C^\infty(E_i)$ be elliptic self-adjoint differential operators of order $d > 0$ with positive definite symbol. We set $P = P_1 \oplus P_2 : C^\infty(E_1 \oplus E_2) \to C^\infty(E_1 \oplus E_2)$. Then P is an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite symbol and $e_n(x, P_1 \oplus P_2) = e_n(x, P_1) \oplus e_n(x, P_2)$.

(b) Let $P_i : C^\infty(E_i) \to C^\infty(E_i)$ be elliptic self-adjoint partial differential operators of order $d > 0$ with positive definite symbol defined over compact manifolds M_i. We let

$$P = P_1 \otimes 1 + 1 \otimes P_2 : C^\infty(E_1 \otimes E_2) \to C^\infty(E_1 \otimes E_2)$$

over $M = M_1 \times M_2$. Then, P is an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite symbol over M and

$$e_n(x, P) = \sum_{p+q=n} e_p\left(x_1, P_1\right) \otimes e_q\left(x_2, P_2\right)$$

Proof. These follow from the identities:

$$\begin{aligned} e^{-t(P_1 \oplus P_2)} &= e^{-tP_1} \oplus e^{-tP_2} \\ e^{-t(P_1 \otimes 1 + 1 \otimes P_2)} &= e^{-tP_1} \otimes e^{-tP_2} \end{aligned}$$

so the heat kernels satisfy the identities:

$$\begin{aligned} H\left(t, x, x, P_1 \oplus P_2\right) &= H\left(t, x, x, P_1\right) \oplus H\left(t, x, x, P_2\right) \\ H\left(t, x, x, P_1 \otimes 1 + 1 \otimes P_2\right) &= H\left(t, x_1, x_1, P_1\right) \otimes H\left(t, x_2, x_2, P_2\right). \end{aligned}$$

We equate equal powers of t in the asymptotic series:

$$\begin{aligned} &\sum t^{\frac{n-m}{d}} e_n(x,\, P_1 \oplus P_2) \\ &\qquad \sim \sum t^{\frac{n-m}{d}} e_n\left(x, P_1\right) \oplus \sum t^{\frac{n-m}{d}} e_n\left(x, P_2\right) \\ &\sum t^{\frac{n-m}{d}} e_n(x,\, P_1 \otimes 1 + 1 \otimes P_2) \\ &\qquad \sim \left\{\sum t^{\frac{p-m_1}{d}} e_p\left(x_1, P_1\right)\right\} \otimes \left\{\sum t^{\frac{q-m_2}{d}} e_q\left(x_2, P_2\right)\right\} \end{aligned}$$

Hence, the proof is complete. □

We define the scalar invariant

$$a_n(x, P) = \operatorname{Tr} e_n(x, P),$$

where the trace is the fiber trace in E over the point x. These scalar invariants $a_n(x, P)$ gives

$$\begin{aligned} \operatorname{Tr} e^{-tP} &= \int_M \operatorname{Tr}_{E_x} H(t, x, x)\, \mathrm{dvol}_M(x) \\ &\sim \sum_{n=0}^{\infty} t^{\frac{m-n}{d}} \int_M a_n(x, P)\, \mathrm{dvol}_M(x). \end{aligned}$$

This is a spectral invariant of P which can be computed from local information about the symbol of P.

Let P be an elliptic operators on the vector bundle E and let Δ_i be the associated Laplacians. We define:

$$a_n(x, P) = \sum_i (-1)^i \operatorname{Tr} e_n(x, \Delta_i)$$

then McKean-Singer formula gives

$$\operatorname{ind}(P) = \sum_i (-1)^i \operatorname{Tr} e^{-t\Delta_i} \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M a_n(x, P) \operatorname{dvol}_M(x)$$

Let $t \to 0^+$, we get the following theorem.

Theorem 3. Let P be an elliptic differential operators on the vector bundle E over compact manifold M^m.

(a) $a_n(x, P)$ can be computed in any coordinate system and relative to any local frames depending on the symbol of P and of P^*.

(b)

$$\int_M a_n(x, P) \operatorname{dvol}_M(x) = \begin{cases} \operatorname{ind}(P) & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

3 Invariance Theory

We let $\mathcal{P}_m$ denote the ring of all invariant polynomials in $\{g_{ij}, g_{ij;k}, g_{ij;k\ell}, \ldots\}$, the derivatives of the metric, for a manifold M of dimension m. We defined $\operatorname{ord}(g_{ij;\alpha}) = |\alpha|$; let $\mathcal{P}_{m,n}$ be the subspace of invariant polynomials which are homogeneous of order n. Then, we have the following useful coordinate free characterization:

Lemma 4. Let $P \in \mathcal{P}_m$, then $P \in \mathcal{P}_{m,n}$ if and only if $P(c^2 g)(x_0) = c^{-n} P(g)(x_0)$ for every $c \neq 0$.

Proof. Fix $c = 0$ and let X be a normalized coordinate system for the metric g at the point x_0. Suppose that $x_0 = (0, \ldots, 0)$ is the center of the coordinate system X. Let $Y = cX$ be a new coordinate system, then we have

$$\frac{\partial}{\partial y_i} = c^{-1} \frac{\partial}{\partial x_i} \quad c^2 g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$
$$d_y^\alpha = c^{-|\alpha|} d_x^\alpha \qquad g_{ij;\alpha}\left(Y, c^2 g\right) = c^{-|\alpha|} g_{ij;\alpha}(X, g).$$

This implies that if A is any monomial of P that:

$$A\left(Y, c^2 g\right)(x_0) = c^{-\operatorname{ord}(A)} A(X, g)(x_0)$$

Since Y is normalized coordinate system for the metric $c^2 g$, $P(c^2 g)(x_0) = P(Y, c^2 g)(x_0)$ and $P(g)(x_0) = P(X, g)(x_0)$. This proves the Lemma. $\square$

If $P \in \mathcal{P}_m$, we can always decompose $P = P_0 + \cdots + P_n$ into homogeneous polynomials. Above lemma implies the P_j are all invariant separately. Therefore, $\mathcal{P}_m$ has a direct sum decomposition $\mathcal{P}_m = \mathcal{P}_{m,0} \oplus \mathcal{P}_{m,1} \oplus \cdots \oplus \mathcal{P}_{m,n} \oplus \cdots$ and has the structure of a graded algebra. Using Gauss lemma and Taylor's theorem, we can always find a metric with the $g_{ij;\alpha}(x_0) = c_{ij,\alpha}$ arbitrary constants for $|\alpha| \geq 2$ and $g_{ij}(x_0) = \delta_{ij}$, $g_{ij;k}(x_0) = 0$. Consequently, if $P \in \mathcal{P}_m$ is non-zero as a polynomial, then we can always find g so $P(g)(x_0) \neq 0$ so P is non-zero as a formula.

Finally, note that $\mathcal{P}_{m,n}$ is zero if n is odd since we may take $c = -1$ in the above lemma.

Lemma 5. $a_n(x, \Delta_p)$ defines an element of $\mathcal{P}_{m,n}$, where Δ_p is the Laplacian on p-forms.

Proof. First, the Laplacian is defined to be $\Delta = dd^* + d^*d = \pm d * d * \pm * d * d$. In the flat metric metric, Laplacian is given by $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ which is smooth in the metric g. For general metric, we parametrize the metric and we can differentiate the matrix representation of Hodge star $*$ and each derivative applied to $*$ reduces the order of differentiation by 1 and increase the order of $g_{ij;\alpha}$ by 1. Thus, the linear term and the constant term of Laplacian is also smooth in g_{ij}, $g_{ij;k}$, and $g_{ij;k\ell}$.

Note that $e_n(x, \Delta)$ for the second order elliptic operator Δ is defined as

$$H(t,x,x) \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} e_n(x,P),$$

where $H(t,x,x)$ be the heat kernel of e^{-tP}. Then, $a_n(x,\Delta) = \operatorname{Tr} e_n(x,\Delta)$ is the fiber trace and $a_n(x,\Delta)$ is homogeneous of order n in $\mathcal{P}_{m,n}$. □

Weyl's theorem (See theorem 10 at the end of this section) on the invariants of the orthogonal group gives a spanning set for the spaces $\mathcal{P}_{m,n}$:

Lemma 6. We introduce formal variables $R_{i_1 i_2 i_3 i_4; i_5 \ldots i_k}$ for the multiple covariant derivatives of the curvature tensor. The order of such a variable is $k+2$. We consider the polynomials in these variables and contract on pairs of indices. Then, all possible such expressions generate $\mathcal{P}_m$. In particular,

(1) $\{1\}$ spans $\mathcal{P}_{m,0}$.

(2) $\{R_{ijij}\}$ spans $\mathcal{P}_{m,2}$.

(3) $\{R_{ijij;kk}, R_{ijij}R_{klkl}, R_{ijik}R_{ljlk}, R_{ijkl}R_{ijkl}\}$ spans $\mathcal{P}_{m,4}$. This particular spanning set for $\mathcal{P}_{m,4}$ is linearly independent and forms a basis if $m \geq 4$.

If $I = \{1 \le i_1 \le \cdots \le i_p \le m\}$, let $|I| = p$ and $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. A p-form valued polynomial is a collection $\{P_I\} = P$ for $|I| = p$ of polynomials P_I in $\{g_{ij}, g_{ij;k}, \cdots\}$. We write $P = \sum_{|I|=p} P_I dx^I$ as a formal sum to represent P. If all the $\{P_I\}$ are homogeneous of order n, we say P is homogeneous of order n. We define:

$$P(X, g)(x_0) = \sum_I P_I(X, g) dx^I \in \bigwedge^p (T^*M)$$

to be the evaluation of such a polynomial. We say P is invariant if $P(X, g)(x_0) = P(Y, g)(x_0)$ for every normalized coordinate systems X and Y. Similar to above lemma, we have

Lemma 7. Let P be p-form valued and invariant. Then, P is homogeneous of order n if and only if $P(c^2 g)(x_0) = c^{p-n} P(g)(x_0)$ for every $c \neq 0$.

Proof. Fix $c = 0$ and let X be a normalized coordinate system for the metric g at the point x_0. Suppose that $x_0 = (0, \ldots, 0)$ is the center of the coordinate system X. Let $Y = cX$ be a new coordinate system, then we have

$$\frac{\partial}{\partial y_i} = c^{-1} \frac{\partial}{\partial x_i} \quad c^2 g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$
$$d_y^\alpha = c^{-|\alpha|} d_x^\alpha \qquad g_{ij;\alpha}\left(Y, c^2 g\right) = c^{-|\alpha|} g_{ij;\alpha}(X, g)$$

$$dy^1 \wedge \cdots \wedge dy^p = c^p dx^1 \wedge \cdots \wedge dx^p.$$

This implies that if A is any monomial of P that:

$$A\left(Y, c^2 g\right)(x_0) = c^{p-\mathrm{ord}(A)} A(X, g)(x_0)$$

Since Y is normalized coordinate system for the metric $c^2 g$, $P(c^2 g)(x_0) = P(Y, c^2 g)(x_0)$ and $P(g)(x_0) = P(X, g)(x_0)$. This proves the Lemma. □

Let $\mathcal{P}_{m,n,p}$ be the space of p-form valued invariants which are homogeneous of order n.

Let $P_j(g) = p_j(TM)$ be the j-th Pontryagin class computed relative to the curvature tensor of the Levi-Civita connection. If we expand p_j in terms of the curvature tensor, then p_j is homogeneous of order $2j$ in the $\{R_{ijkl}\}$ tensor so p_j is homogeneous of order $4j$ and invariant in $\mathcal{P}_{m,4j,4j}$. If ρ is a partition of $k = i_1 + \cdots + i_j$, we define $p_\rho = p_{i_1} \ldots p_{i_j} \in \mathcal{P}_{m,4k,4k}$. The $\{p_\rho\}$ form a basis of the Pontryagin $4k$ forms. Also, by considering products of these manifolds with flat tori T^{m-4k} we see that the $\{p_\rho\}$ are linearly independent in $\mathcal{P}_{m,4k,4k}$ if $4k \le m$.

Lemma 8. $\mathcal{P}_{m,n,n}$ is spanned by the Pontryagin classes, i.e.,

(1) $\mathcal{P}_{m,n,n} = 0$ if n is not divisible by $4k$.

(2) $\mathcal{P}_{m,4k,4k} = \text{span}\{p_\rho\}$ for $4k \leq m$ has dimension $\pi(k)$, where $\pi(k)$ is the integral partition number of k.

Proof. There is a natural restriction map $r : \mathcal{P}_{m,n,p} \to \mathcal{P}_{m-1,n,p}$. Note that $r : \mathcal{P}_{m,n,n} \to \mathcal{P}_{m-1,n,n}$ is injective for $n < m$ since $r(Adx^1 \wedge \cdots \wedge dx^p) = Adx^1 \wedge \cdots \wedge dx^p$ appears in $r(P)$. The Pontryagin classes have dimension $\pi(k)$ for $n = 4k$. By induction, $r^{m-n} : \mathcal{P}_{m,n,n} \to \mathcal{P}_{n,n,n}$ is injective so $\dim \mathcal{P}_{m,n,n} \leq \dim \mathcal{P}_{n,n,n}$. Apply this lemma for the special case $n = m$. If n is not divisible by $4k$, then $\dim \mathcal{P}_{n,n,n} = 0$ which implies $\dim \mathcal{P}_{m,n,n} = 0$. If $n = 4k$, then $\dim \mathcal{P}_{n,n,n} = \pi(k)$ implies that $\pi(k) \leq \dim \mathcal{P}_{m,n,n} \leq \dim \mathcal{P}_{n,n,n} \leq \pi(k)$ so $\dim \mathcal{P}_{m,n,n} = \pi(k)$. Lastly, since the Pontryagin classes span a subspace of exactly dimension $\pi(k)$ in $\mathcal{P}_{m,n,n}$, this completes the proof. □

Finally, we discuss the invariance on vector bundles. Let E be a complex vector bundle. Suppose that E is equipped with a Hermitian fiber metric and let ∇ be a Hermitian connection on E. Let $\vec{s} = (s_1, \ldots, s_a, \ldots, s_v)$ be a local orthonormal frame for E and introduce variables ω_{abi} for the connection 1-form;

$$\nabla(s_a) = \omega_{abi} dx^i \otimes s_b, \qquad \text{i.e., } \nabla\vec{s} = \omega \otimes \vec{s}.$$

We introduce variables $\omega_{abi;\alpha} = d_x^\alpha(\omega_{abi})$ for the partial derivatives of the connection 1-form. We shall also use the notation $\omega_{abi;jk\ldots}$. We use indices $1 \leq a, b, \cdots \leq v$ to index the frame for E and indices $1 \leq i, j, k \leq m$ for the tangent space variables. We define:

$$\text{ord}(\omega_{abi;\alpha}) = 1 + |\alpha| \qquad \text{and} \qquad \deg_k(\omega_{abi;\alpha}) = \delta_{i,k} + \alpha(k).$$

Let $\mathcal{Q}$ be the collection of polynomials in the $\{\omega_{abi;\alpha}\}$ variables for $|\alpha| \geq 1$. If $Q \in \mathcal{Q}$, we define the evaluation $Q(X, \vec{s}, \nabla)(x_0)$. We normalize the choice of frame $\vec{s}$ by requiring $\nabla(\vec{s})(x_0) = 0$. We also normalize the coordinate system X as before so $X(x_0) = 0$, $g_{ij}(X,g)(x_0) = \delta_{ij}$, and $g_{ij;k}(X,g)(x_0) = 0$. We say Q is invariant if $Q(X, \vec{s}, \nabla)(x_0) = Q(Y, \vec{s}', \nabla)(x_0)$ for any normalized frames $\vec{s}, \vec{s}'$ and normalized coordinate systems X, Y; we denote this common value by $Q(\nabla)$. Let $\mathcal{Q}_{m,p,v}$ denote the space of all invariant p-form valued polynomials in the $\{\omega_{abi;\alpha}\}$ variables for $|\alpha| \geq 1$ defined on a manifold M of dimension m and for a vector bundle of complex fiber dimension v. Let $\mathcal{Q}_{m,n,p,v}$ denote the subspace of invariant polynomials homogeneous of order n in the variables $\{\omega_{abi;\alpha}\}$ of the connection forms. Similarly as was done for the $\mathcal{P}$, we can show there is a direct sum decomposition

$$\mathcal{Q}_{m,p,v} = \bigoplus_n \mathcal{Q}_{m,n,p,v} \qquad \text{and} \qquad \mathcal{Q}_{m,n,p,v} = 0 \quad \text{for } n-p \text{ odd}.$$

Let P be invariant and let A be a monomial. We let $c(A,P)$ be the coefficient of A in P. We say A is a monomial of P if $c(A,P) \neq 0$. Let T_j be the linear transformation:

$$T_j\left(x_k\right) = \begin{cases} -x_j & \text{if } k = j \\ x_k & \text{if } k \neq j \end{cases}$$

This is reflection in the hyperplane defined by $x_j = 0$. Then

$$T_j^*(A) = (-1)^{\deg_j(A)} A$$

for any monomial A. Since

$$T_j^* P = \sum (-1)^{\deg_j(A)} c(A,P) A = P = \sum c(A,P) A.$$

we conclude $\deg_j(A)$ must be even for any monomial A of P. If A has the form:

$$A = g_{i_1 j_1;\alpha_1} \ldots g_{i_r j_r;\alpha_r}$$

we define the length of A to be:

$$\ell(A) = r.$$

It is clear $2\ell(A) + \operatorname{ord}(A) = \sum_j \deg_j(A)$ so $\operatorname{ord}(A)$ is necessarily even if A is a monomial of P. This also provides another proof $\mathcal{P}_{m,n} = 0$ if n is odd.

In addition, we let $\mathcal{R}_{m,n,p,v}$ denote the space of p-form valued invariants which are homogeneous of order n in the $\{g_{ij;\alpha}, \omega_{abk;\beta}\}$ variables for $|\alpha| \geq 2$ and $|\beta| \geq 1$. The spaces $\mathcal{P}_{m,n,p}$ and $\mathcal{Q}_{m,n,p,v}$ are both subspaces of $\mathcal{R}_{m,n,p,v}$. Furthermore, wedge product gives a natural map $\mathcal{P}_{m,n,p} \otimes \mathcal{Q}_{m,n',p',v} \to \mathcal{R}_{m,n+n',p+p',v}$. We say that $R \in \mathcal{R}_{m,p,p,v}$ is a characteristic form if it is in the linear span of wedge products of Pontryagin classes of $T(M)$ and Chern classes of E. The characteristic forms are characterized abstractly by the following Theorem.

Theorem 9.

1. $\mathcal{R}_{m,n,p,v} = 0$ if $n < p$ or if $n = p$ and n is odd.

2. If $R \in \mathcal{R}_{m,n,n,v}$ then R is a characteristic form.

Proof. Let $0 \neq R \in \mathcal{R}_{m,n,p,v}$. Note that an invariant polynomial is homogeneous of order n if $R\left(c^2 G, \nabla\right) = c^{p-n} R(G, \nabla)$. Also, $n - p$ must be even and that if A is a monomial of R, A is a monomial of exactly one of the R_I. We decompose A in the form:

$$A = g_{i_1 j_1/\alpha_1} \ldots g_{i_q j_q/\alpha_q} \omega_{a_1 b_1 k_1/\beta_1} \ldots \omega_{a_r b_r k_r/\beta_r} = A^g A^\omega$$

and define $\ell(A) = q+r$ to the length of A. We choose A such that $\deg_k(A^g) = 0$ for $k > 2q$. By making a coordinate permutation we can assume that the $k_\nu \leq 2q + r$ for $1 \leq \nu \leq r$. We choose the β_i so that $\beta_1(k) = 0$ for $k > 2q + r + 1$, $\beta_2(k) = 0$ for $k > 2q + r + 2$, …, $\beta_r(k) = 0$ for $k > 2q + 2r$. This choice of A so that $\deg_k(A) = 0$ for $k > 2\ell(A)$. If A is a monomial of R_I for $I = \{1 \leq i_1 < \cdots < i_p \leq m\}$ then $\deg_{i_p}(A)$ is odd. We have the estimate $p \leq i_p \leq 2\ell(A) \leq \sum |\alpha_\nu| + \sum (|\beta_\mu| + 1) = n$ so that $\mathcal{P}_{m,n,p,v} = \{0\}$ if $n < p$ or if $n - p$ is odd. Hence, we prove the first part of this theorem.

In the limiting case, we must have equalities so $|\alpha_\nu| = 2$ and $|\beta_\mu| = 1$. Furthermore, $i_p = p$ so there is some monomial A such that $\deg_k(A) = 0$ for $k > p = n$ and $Adx^1 \wedge \cdots \wedge dx^p$ appears in R. Consider the natural restriction map

$$r : \mathcal{R}_{m,n,p,v} \to \mathcal{R}_{m-1,n,p,v}$$

and the argument above shows $r : \mathcal{R}_{m,n,n,v} \to \mathcal{R}_{m-1,n,n,v}$ is injective for $n < m$. Since the restriction of a characteristic form is again a characteristic form, it suffices to prove the second part for the case $m = n = p$.

Let $0 \neq R \in \mathcal{R}_{n,n,n,v}$, then R is a polynomial in the $\{g_{ij;k\ell}, \omega_{abi;j}\}$ variables. The restriction map r was defined by considering products $M_1\times S^1$. Fix non-negative integers (s,t) so that $n = s + t$. Let M_1 be a Riemannian manifold of dimension s. Let M_2 be the flat torus of dimension t and let E_2 be a vector bundle with connection ∇_2 over M_2 Let $M = M_1 \times M_2$ with the product metric and let E be the natural extension of E_2 to M which is flat in the M_1 variables. Explicitly, if $\pi_2 : M \to M_2$ is a projection on the second factor, then $(E,\nabla) = \pi_2^*(E_2,\nabla_2)$ is the pull back bundle with the pull back connection. We define

$$\pi_{(s,t)}(R)(g_1,\nabla_2) = R(g_1 \times 1, \nabla).$$

Using the fact that $\mathcal{P}_{s,n_1,p_1} = 0$ for $s < p_1$ or $n_1 < p_1$ and the fact $\mathcal{Q}_{t,n_2,p_2,k} = 0$ for $t < p_2$ or $n_2 < p_2$, it follows that $\pi_{(s,t)}$ defines a map

$$\pi_{(s,t)} : \mathcal{R}_{n,n,v} \to \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}.$$

Algebraically, let $A = A^g A^\omega$ be a monomial, then we define:

$$\pi_{(s,t)}(A) = \begin{cases} 0 & \text{if } \deg_k(A^g) > 0 \text{ for } k > s \text{ or } \deg_k(A^\omega) > 0 \text{ for } k \leq s \\ A & \text{otherwise.} \end{cases}$$

In this definition, we set $g_{ij;k\ell} = 0$ if any of these indices exceeds s and set $\omega_{abi;j} = 0$ if either i or j is less than or equal to s.

We will use these projections to reduce the proof to the case in which $R \in \mathcal{Q}_{t,t,t,v}$. Let $0 \neq R \in \mathcal{R}_{n,n,n,v}$ and let $A = A^g A^\omega$ be a monomial of R. Let

$s = 2\ell(A^g) = \mathrm{ord}(A^g)$ and let $t = n - s = 2\ell(A^\omega) = \mathrm{ord}(A^\omega)$. We choose A such that $\deg_k(A^g) = 0$ for $k > s$. Since $\deg_k(A) \geq 1$ must be odd for each index k, we can estimate:

$$t \leq \sum_{k>s} \deg_k(A) = \sum_{k>s} \deg_k(A^\omega) \leq \sum_{k} \deg_k(A^\omega) = \mathrm{ord}(A^\omega) = t.$$

Then, as all these inequalities must be equalities, we conclude $\deg_k(A^\omega) = 0$ for $k \leq s$ and $\deg_k(A^\omega) = 1$ for $k > s$. In particular, this shows that $\pi_{(s,t)}(R) \neq 0$ for some (s,t) so that

$$\bigoplus_{s+t=n} \pi_{(s,t)} : \mathcal{R}_{n,n,n,v} \to \bigoplus_{s+t=n} \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}$$

is injective.

Finally, we prove that $\mathcal{Q}_{t,t,t,v}$ consists of characteristic forms of E. Note that $\mathcal{P}_{s,s,s}$ consists of Pontryagin classes of $T(M)$. The characteristic forms generated by the Pontryagin classes of $T(M)$ and of E are elements of $\mathcal{R}_{n,n,n,v}$ and $\pi_{(s,t)}$ just decomposes such products. Thererfore, π is surjective when restricted to the subspace of characteristic forms. This proves π is bijective and also that $\mathcal{R}_{n,n,n,v}$ is the space of characteristic forms. This will complete the proof.

Now, we have reduced the proof of Theorem to showing $\mathcal{Q}_{t,t,t,v}$ consists of the characteristic forms of E. We noted that $0 \neq Q \in \mathcal{Q}_{t,t,t,v}$ is a polynomial in the $\{\omega_{abi;j}\}$ variables and that if A is a monomial of Q, then $\deg_k(A) = 1$ for $1 \leq k \leq t$. Since $\mathrm{ord}(A) = t$ is even, we conclude $\mathcal{Q}_{t,t,t,v} = 0$ if t is odd. The components of the curvature tensor are given by:

$$\Omega_{abij} = \omega_{abi;j} - \omega_{abj;i} \quad \text{and} \quad \Omega_{ab} = \sum \Omega_{abij} dx_i \wedge dx_j$$

up to a possible sign convention. If A is a monomial of P, we decompose:

$$A = \omega_{a_1 b_1 i_1; i_2} \ldots \omega_{a_u b_u i_{t-1}; i_t},$$

where $2u = t$. All the indices i_ν are distinct. Then, we can express P in terms of the expressions:

$$\begin{aligned}\bar{A} =& \left(\omega_{a_1 b_1 i_1; i_2} - \omega_{a_1 b_1 i_2; i_1}\right) \ldots \left(\omega_{a_u b_u i_{t-1}; i_t} - \omega_{a_u b_u i_t; i_{t-1}}\right) dx_{i_1} \wedge \cdots \wedge dx_{i_t} \\ =& \Omega_{a_1 b_1 i_1 i_2} \ldots \Omega_{a_u b_u i_{t-1} i_t} dx_{i_1} \wedge \cdots \wedge dx_{i_t}\end{aligned}$$

Using the alternating structure of these expression, we can express P in terms of expressions of the form:

$$\Omega_{a_1 b_1} \wedge \cdots \wedge \Omega_{a_u b_u}$$

so that $Q = Q(\Omega)$ is a polynomial in the components Ω_{ab} of the curvature. Since the value of Q is independent of the frame chosen, Q is the invariant. Hence, we see that in fact Q is a characteristic form which completes the proof. $\square$

Weyl's theorem on invariants In the remaining of this section, we will review H. Weyl's theorem briefly. Let V be a real vector space with a fixed inner product. Let $\mathrm{O}(V)$ denote the group of linear maps of $V \to V$ which preserve this inner product. Let $\otimes^k(V) = V \otimes \cdots \otimes V$ denote the k-th tensor product of V. If $g \in \mathrm{O}(V)$, we extend g to act orthogonally on $\otimes^k(V)$. We let $z \mapsto g(z)$ denote this action. Let $f : \otimes^k(V) \to \mathbb{R}$ be a multi-linear map, then we say f is $\mathrm{O}(V)$ invariant if $f(g(z)) = f(z)$ for every $g \in \mathrm{O}(V)$. By letting $g = -1$, we can see there are no $\mathrm{O}(V)$ invariant maps if k is odd. We let $k = 2j$ and construct a map $f_0 : \otimes^k(V) = (V \otimes V) \otimes (V \otimes V) \otimes \cdots \otimes (V \otimes V) \to \mathbb{R}$ using the metric to map $(V \otimes V) \to \mathbb{R}$. More generally, if ρ is any permutation of the integers 1 through k, we define $z \mapsto z_\rho$ as a map from $\otimes^k(V) \to \otimes^k(V)$ and let $f_\rho(z) = f_0(z_\rho)$. This will be $\mathrm{O}(V)$ invariant for any permutation ρ. H. Weyl's theorem states that the maps $\{f_\rho\}$ define a spanning set for the collection of $\mathrm{O}(V)$ invariant maps.

For example, let $k = 4$. Let $\{v_i\}$ be an orthonormal basis for V and express any $z \in \otimes^4(V)$ in the form $a_{ijkl} v_i \otimes v_j \otimes v_k \otimes v_l$ summed over repeated indices. Then, after weeding out duplications, the spanning set is given by:

$$f_0(z) = a_{iijj}, \quad f_1(z) = a_{ijij}, \quad f_2(z) = a_{ijji}$$

where we sum over repeated indices. f_0 corresponds to the identity permutation; f_1 corresponds to the permutation which interchanges the second and third factors; f_2 corresponds to the permutation which interchanges the second and fourth factors. We note that these need not be linearly independent; if $\dim V = 1$ then $\dim(\otimes^4 V) = 1$ and $f_1 = f_2 = f_3$. However, once $\dim V$ is large enough these become linearly independent.

We are interested in p-form valued invariants. We take $\otimes^k(V)$ where $k-p$ is even. Again, there is a natural map we denote by

$$f^p(z) = f_0(z_1) \wedge \Lambda(z_2)$$

where we decompose $\otimes^k(V) = \otimes^{k-p}(V) \otimes \otimes^p(V)$. We let f_0 act on the first $k-p$ factors and then use the natural map $\otimes^p(V) \xrightarrow{\Lambda} \wedge^p(V)$ on the last p factors. If ρ is a permutation, we set $f^p_\rho(z) = f^p(z_\rho)$. These maps are equivariant in the sense that $f^p_\rho(gz) = g f^p_\rho(z)$, where we extend g to act on $\wedge^p(V)$ as well. Again, these are a spanning set for the space of equivariant multi-linear maps from $\otimes^k(V)$ to $\wedge^p(V)$.

If $k = 4$ and $p = 2$, then after eliminating duplications this spanning set becomes:

$$\begin{array}{lll} f_1(z) = a_{iijk} v_j \wedge v_k, & f_2(z) = a_{ijik} v_j \wedge v_k, & f_3(z) = a_{ijki} v_j \wedge v_k \\ f_4(z) = a_{jiki} v_j \wedge v_k, & f_5(z) = a_{jiik} v_i \wedge v_k, & f_6(z) = a_{jkii} v_j \wedge v_k. \end{array}$$

Again, these are linearly independent if $\dim V$ is large, but there are relations if $\dim V$ is small. Generally speaking, to construct a map from $\otimes^k(V) \to \Lambda^p(V)$ we must alternate p indices (the indices j, k in this example) and contract the remaining indices in pairs (there is only one pair i, i here).

Theorem 10 (H. Weyl's Theorem on the invariants of the orthogonal group)**.** The space of maps $\{f_\rho^p\}$ constructed above span the space of equivariant multi-linear maps from $\otimes^k V \to \wedge^p V$.

The theorem becomes an algebra problem and we refer to Weyl's book [Wey39].

4 Twisted Signature Formula

Untwisted Case Let $\bigwedge(T^*M)$ be the space of complex valued forms on M^m and ϵ be the chirality element on $\bigwedge(T^*M)$. ϵ gives an endomorphism $\epsilon : \bigwedge(T^*M) \to \bigwedge(T^*M)$ so that $\epsilon^2 = 1$. Also, $c(\xi)\epsilon = -\epsilon c(\xi)$ for any ξ, so ϵ anti-commutes with the symbol of $(d + d^*)$. If we decompose $\bigwedge(T^*M) = \bigwedge^+(T^*M) \oplus \bigwedge^-(T^*M)$ into the ± 1 eigenvalues of ϵ, then $(d+d^*)$ decomposes to define:

$$(d + d^*)_\pm : C^\infty\left(\bigwedge^\pm(T^*M)\right) \to C^\infty\left(\bigwedge^\mp(T^*M)\right)$$

where the adjoint of $(d + d^*)_+$ is $(d + d^*)_-$.

Proposition 11. Write $D = (d + d^*)_+ : C^\infty\left(\bigwedge^+(T^*M)\right) \to C^\infty\left(\bigwedge^-(T^*M)\right)$. Suppose that $4 \mid m$, then we have

$$\operatorname{ind} D = \sigma(M),$$

where $\sigma(M)$ is the signature of M.

Proof. Note that if ω is a harmonic form in $\bigwedge^q(T^*M)$, then $\omega \pm \epsilon\omega \in \ker D|_{\bigwedge^\pm}$. Then, if $q \neq \frac{m}{2}$, we have an isomorphism

$$D|_{\bigwedge^+(T^*M)\cap(\ker\Delta^q \oplus \ker\Delta^{m-q})} \xrightarrow{\cong} \ker D^*|_{\bigwedge^-(T^*M)\cap(\ker\Delta^q \oplus \ker\Delta^{m-q})}.$$

Then, we obtain that

$$\begin{aligned}\operatorname{ind}(D) &= \operatorname{ind}\left(D|_{\bigwedge^+\cap\bigwedge^{m/2}}\right) = \dim\ker\left(D|_{\bigwedge^+\cap\bigwedge^{m/2}}\right) - \dim\ker\left(D|_{\bigwedge^-\cap\bigwedge^{m/2}}\right)\\ &= \dim\left(\bigwedge^+\cap\ker(d+d^*)\Big|_{\bigwedge^{m/2}}\right) - \dim\left(\bigwedge^-\cap\ker(d+d^*)\Big|_{\bigwedge^{m/2}}\right).\end{aligned}$$

For $\omega\in\bigwedge^+\cap\ker\,(d+d^*)|_{\bigwedge^{m/2}}$, we have $*\omega=\omega$ (as $\epsilon = *$ on $\bigwedge^{m/2}$), so

$$[\omega]\cdot[\omega] = \int_M \omega\wedge\omega = \int_M \omega\wedge *\omega > 0.$$

Similarly, for $\omega\in\bigwedge^-\cap\ker\,(d+d^*)|_{\bigwedge^{m/2}}$, we have $[\omega]\cdot[\omega]<0$. Finally, since

$$\begin{aligned}H^{m/2}_{\mathrm{dR}} &\cong \ker(d+d^*)|_{\bigwedge^{m/2}}\\ &= \left(\bigwedge^+\cap\ker(d+d^*)\Big|_{\bigwedge^{m/2}}\right)\oplus\left(\bigwedge^-\cap\ker(d+d^*)\Big|_{\bigwedge^{m/2}}\right).\end{aligned}$$

It follows that

$$\operatorname{ind}(D) = \sigma(M).$$

□

Moreover, by the Hirzebruch signature formula, we have

$$\operatorname{ind}(D) = \int_M L(M).$$

Twisted Case Let E be a complex vector bundle over a compact manifold M equipped with a Riemannian connection ∇. We take the Levi-Civita connection on $T^*(M)$ and on $\bigwedge(T^*M)$ and let ∇ be the tensor product connection on $\bigwedge(T^*M)\otimes E$. We define an operator $(d+d^*)_E$: $C^\infty(M,\bigwedge(T^*M)\otimes E)\to C^\infty(M,\bigwedge(T^*M)\otimes E)$ by the composition

$$\begin{aligned}(d+d^*)_E : C^\infty\left(\bigwedge(T^*M)\otimes E\right) &\xrightarrow{\nabla} C^\infty\left(T^*M\otimes\bigwedge(T^*M)\otimes E\right)\\ &\xrightarrow{c\otimes 1} C^\infty\left(\bigwedge(T^*M)\otimes E\right).\end{aligned}$$

Note that if $E=1$ is trivial vector bundle with flat connection, then the resulting operator is just $d+d^*$.

We define $\epsilon_E=\epsilon\otimes 1$ on $\bigwedge(T^*M)\otimes E$, then we have $\epsilon_E^2=1$ and ϵ_E anti-commutes with $(d+d^*)_E$. The ±1 eigenspaces of ϵ_E are $\bigwedge^\pm(T^*M)\otimes E$ and the twisted signature operator is defined by the following:

$$(d+d^*)^\pm_E : C^\infty\left(\bigwedge^\pm(T^*M)\otimes E\right)\to C^\infty\left(\bigwedge^\mp(T^*M)\otimes E\right),$$

where as $(d+d^*)_E^-$ is the adjoint of $(d+d^*)_E^+$. Then, we have the following theorem.

Theorem 12 (Twisted Signature Formula)**.**

$$\operatorname{ind}(d+d^*)_E^+ = \int_M L(M)\operatorname{ch}(E).$$

Before we give the proof of this formula, we first construct a non-trivial line bundle L over S^2 such that $\int_{S^2}\operatorname{ch}_1(L) = 1$. This example will help us in proving the formula.

Example 13. Let $e(x)$ be a linear map from $\mathbb{R}^m$ to the set of self-adjoint matrices with $e(x) = |x|^2 I$. If $\{v_0, \ldots, v_m\}$ is any orthonormal basis for $\mathbb{R}^m$, then $\{e(v_0), \ldots, e(v_m)\}$ forms a set of Clifford matrices, i.e., $e(v_i)e(v_j) + e(v_j)e(v_i) = 2\delta_{ij}$.

If $x \in S^m$, we let $\Pi_\pm(x)$ be the image of $\frac{1}{2}(1+e(x)) = \pi_\pm(x)$. This is the span of the $+1$ eigenvectors of $e(x)$. If $e(x)$ is a $2k \times 2k$ matrix, then $\dim \Pi_\pm(x) = k$. Then, we have a decomposition $S^m \times \mathbb{C}^{2k} = \Pi_+ \oplus \Pi_-$. We project the flat connection on $S^m \times \mathbb{C}^{2k}$ to the two subbundles to define connections $\nabla_\pm$ on $\Pi_\pm$. If $e^0_\pm$ is a local frame for $\Pi_\pm(x_0)$, we define $e_\pm(x) = \pi_\pm e^0_\pm$ as a frame in a neighborhood of x_0. We compute

$$\nabla_\pm e_\pm = \pi_\pm d\pi_\pm e^0_\pm, \quad \Omega_\pm e_\pm = \pi_\pm d\pi_\pm d\pi_\pm e^0_\pm.$$

Since $e^0_\pm = e_\pm(x_0)$, this yields the identity:

$$\Omega_\pm(x_0) = \pi_\pm d\pi_\pm d\pi_\pm(x_0)$$

Since Ω is tensorial, this holds for all x.

Let $m = 2j$ be even. We want to compute ch_j. Suppose first $x_0 = (1, 0, \ldots, 0)$ is the north pole of the sphere. Then:

$$\begin{aligned}
\pi_+(x_0) &= \frac{1}{2}(1+e_0) \\
d\pi_+(x_0) &= \frac{1}{2}\sum_{i\geq 1} dx_i e_i \\
\Omega_+(x_0) &= \frac{1}{2}(1+e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^2 \\
\Omega_+(x_0)^j &= \frac{1}{2}(1+e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^{2j} \\
&= 2^{-m-1} m!\,(1+e_0)(e_1 \ldots e_m)(dx_1 \wedge \cdots \wedge dx_m)
\end{aligned}$$

The volume form at x_0 is $dx_1 \wedge \cdots \wedge dx_m$. Since e_1 anti-commutes with the matrix $e_1 \dots e_m$, this matrix has trace 0 so we have computation:

$$\mathrm{ch}_j(\Omega_+)(x_0) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{tr}(e_0 \dots e_m) \operatorname{dvol}(x_0)/j!.$$

A similar computation shows this is true at any point x_0 of S^m so that:

$$\int_{S^m} \mathrm{ch}_j(\Pi_+) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{tr}(e_0 \dots e_m) \operatorname{vol}(s^m)/j!$$

Since the volume of S^m is $j! 2^{m+1} \pi^j / m!$, we conclude:

Lemma 14. Let $e(x)$ be a linear map from $\mathbb{R}^{m+1}$ to the set of self-adjoint matrices. Suppose that $e(x)^2 = |x|^2 I$ and define bundles $\Pi_\pm(x)$ over S^m corresponding to the ± 1 eigenvalues of e. Let $m = 2j$ be even, then:

$$\int_{S^m} ch_j(\Pi_+) = i^j 2^{-j} \operatorname{Tr}(e_0 \dots e_m).$$

In particular, if

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then $\operatorname{tr}(e_0 e_1 e_2) = -2i$ so $\int_{S^2} \mathrm{ch}_1(\Pi_+) = 1$ which shows Π_+ is a non-trivial line bundle L over S^2.

Now, we begin our proof of twisted signature formula.

Proof. Let $\Delta_E^\pm$ be the associated Laplacians to $(d+d^*)_E^\pm$. Then, we have

$$\operatorname{ind}\left((d+d^*)_E^+\right) = \dim\ker\left(\Delta_E^+\right) - \dim\ker\left(\Delta_E^-\right).$$

Then, using local formula for the index, we can write

$$\operatorname{ind}\left((d+d^*)_E^+\right) = \int_M a_n(x, E) = \int_M \left(a_n\left(x, \Delta_E^+\right) - a_n\left(x, \Delta_E^-\right)\right) \operatorname{dvol}_M(x).$$

Note that the leading symbol of $\Delta^\pm$ is $|\xi|^2 I$. The first order symbol is linear in the $g_{ij;k}$ and the connection form on E. The 0-th order symbol is linear in the $g_{ij;k\ell}$ and the connection form and quadratic in the $g_{ij;k}$ and the connection form. Thus, $a_n(x, E) \in \mathcal{R}_{m,n,m,v}$. Theorem 9 implies $a_n = 0$ for $n < m$ while a_m is a characteristic form of $T(M)$ and of E.

If $m = 2$ and $v = 1$, then $\mathcal{R}_{2,2,2,1}$ is one dimensional and is spanned by the first Chern class $c_1(E) = \mathrm{ch}(E) = \frac{i}{2\pi}\Omega$. Consequently, $a_2 = c \cdot c_1$ in this case. Also, by direct computation on above example 13, this normalizing constant is given by $c = 1$.

Lemma 15. Let $m = 2$ and let L be a line bundle over M^2. Then,

$$\operatorname{ind}\left((d + d^*)_E^+\right) = \int_M c_1(L).$$

Now, we know $a_m(x, E)$ is a characteristic form which integrates to $\operatorname{ind}\left((d + d^*)_E^+\right)$, so it suffices to verify the formula

$$\operatorname{ind}(d + d^*)_E^+ = \sum_{4s+2t=m} \int_M L_s(M) \wedge \operatorname{ch}_t(E).$$

If E_1 and E_2 are bundles, we let $E = E_1 \oplus E_2$ with the direct sum connection. Since $\Delta_E^\pm = \Delta_{E_1}^\pm \oplus \Delta_{E_2}^\pm$, we conclude

$$\operatorname{ind}(d + d^*)_E^+ = \operatorname{ind}(d + d^*)_{E_1}^+ + \operatorname{ind}(d + d^*)_{E_2}^+.$$

Since the integrals are additive, the local formulas must be additive. Then, we have

$$a_n(x, E_1 \oplus E_2) = a_n(x, E_1) + a_n(x, E_2).$$

Let $\{P_\rho\}_{|\rho|=s}$ be the basis for $\mathcal{P}_{m,4s,4s}$ and expand

$$a_m(x, E) = \sum_{4|\rho|+2t=m} P_\rho \wedge Q_{m,t,v,\rho}$$

for $Q_{m,t,v,\rho} \in \mathcal{Q}_{m,2t,2t,v}$ a characteristic form of E. Then the additivity under direct sum implies:

$$Q_{m,t,v,\rho}(E_1 \oplus E_2) = Q_{m,t,v_1,\rho}(E_1) + Q_{m,t,v_2,\rho}(E_2).$$

If $v = 1$, then $Q_{m,t,1,\rho}(E) = c \cdot c_1(E)^t$ since $\mathcal{Q}_{m,2t,2t,1}$ is one dimensional. If A is diagonal matrix, then the additivity implies:

$$Q_{m,t,v,\rho}(A) = Q_{m,t,v,\rho}(\lambda) = c \cdot \sum_j \lambda_j^t = c \cdot \operatorname{ch}_t(A).$$

Since Q is determined by its values on diagonal matrices, we conclude:

$$Q_{m,t,v,\rho}(E) = c(m, t, \rho) \operatorname{ch}_t(E)$$

where the normalizing constant does not depend on the dimension v. Therefore, we expand a_m in terms of $\operatorname{ch}_t(E)$ to express

$$a_m(x, E) = \sum_{4s+2t=m} P_{m,s} \wedge \operatorname{ch}_t(E) \quad \text{for } P_{m,s} \in \mathcal{P}_{m,4s,4s}.$$

We complete the proof of the theorem by identifying $P_{m,s} = L_s$ by the Hirzebruch signature formula. Hence, we have reduced the proof of the theorem to the case $v = 1$.

Now, we prove by induction on m; The above lemma establishes this theorem if $m = 2$. Suppose $m \equiv 0 \bmod 4$. If we take E to be the trivial bundle, then if $4k = m$

$$a_m(x, 1) = L_k = P_{m,k}$$

follows from the Hirzebruch signature formula. Then, we may assume $4s < m$ in computing $P_{m,s}$. Let $M = M_1 \times S^2$ and let $E = E_1 \otimes E_2$ where E_1 is a line bundle over M_1 and where E_2 is a line bundle over S^2 such that $\int_{S^2} c_1(E_2) = 1$ constructed in the example 13. We take the product connection on $E_1 \otimes E_2$ and decompose:

$$\bigwedge^+(E) = \bigwedge^+(E_1) \otimes \bigwedge^+(E_2) \oplus \bigwedge^-(E_1) \otimes \bigwedge^-(E_2)$$
$$\bigwedge^-(E) = \bigwedge^-(E_1) \otimes \bigwedge^+(E_2) \oplus \bigwedge^+(E_1) \otimes \bigwedge^-(E_2)$$

The decomposition of the Laplacians yields

$$\begin{aligned} \operatorname{ind}(d + d^*)^+_E &= \operatorname{ind}(d + d^*)^+_{E_1} \operatorname{ind}(d + d^*)^+_{E_2} \\ &= \operatorname{ind}(d + d^*)^+_{E_1}. \end{aligned}$$

Also, since the signatures are multiplicative, the local formulas are multiplicative that

$$a_m(x, E) = \sum_{p+q=m} a_p(x_1, E_1)\, a_q(x_2, E_2)$$

and the fact $a_p = 0$ for $p < m_1$ and $a_q = 0$ for $q < m_2$. Thus we conclude:

$$a_m(x, E) = a_{m_1}(x_1, E_1)\, a_{m_2}(x_2, E_2),$$

where $m_2 = 2$ and $m_1 = m - 2$. Besides, we use the identity:

$$\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \operatorname{ch}(E_2)$$

to conclude

$$\begin{aligned} \operatorname{ind}(d + d^*)^+_{E_1} &= \operatorname{ind}(d + d^*)^+_E \\ &= \left\{ \sum_{4s+2t=m-2} \int_{M_1} P_{m,s} \wedge \operatorname{ch}_t(E_1) \right\} \int_{M_2} \operatorname{ch}_1(E_2) \\ &= \sum_{4s+2t=m-2} \int_{M_1} P_{m,s} \wedge \operatorname{ch}_t(E_1). \end{aligned}$$

Also, $P_{m,s} = P_{m-2,s}$ for $4s \leq m - 2$. Therefore, by induction, $P_{m-2,s} = L_s$ and we complete the proof of the theorem. □

5 Toward the Atiyah–Singer Index Theorem

In this section, we shall discuss the Atiyah-Singer theorem for an elliptic operator by interpreting the index as a map in K-theory. Let $P : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic complex with leading symbol $\sigma_P(x,\xi) : S(T^*M) \to \mathrm{Hom}(E_1, E_2)$. We let $\Sigma(T^*M)$ be the fiberwise suspension of the unit sphere bundle $S(T^*M)$. We form the disk bundles $D_\pm(M)$ over M corresponding to the northern and southern hemispheres of the fiber spheres of $\Sigma(T^*M)$. We define $\Pi_+(\Sigma_{\sigma_P})$ by the bundle $E_1^+ \cup E_2^-$ over the disjoint union $D_+(M) \cup D_-(M)$ attached using the clutching function $\sigma_P(x,\xi)$ over their common boundary $S(T^*M)$. If E_1 is a rank k bundle, then $\Pi_+(\Sigma_{\sigma_P})$ is a rank k bundle on $\Sigma(T^*M)$. Then, the Atiyah–Singer Index Theorem states that

Theorem 16 (Index theorem)**.** Let $P : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic operator. Let $\mathrm{Td}(M) = \mathrm{Td}(TM \otimes \mathbb{C})$ be the Todd class of the complexification of the real tangent bundle. Then,

$$\mathrm{ind}(P) = (-1)^{\dim M} \int_{\Sigma(T^*M)} \mathrm{Td}(M) \wedge \mathrm{ch}\left(\Pi_+(\Sigma_{\sigma_P})\right).$$

Remark that the additional factor of $(-1)^{\dim M}$ could have been avoided if we changed the orientation of $\Sigma(T^*M)$.

The first step of the proof is to reduce to the case $\dim M = m$ even and M orientable. If m is odd. We can take $Q : C^\infty(S^1) \to C^\infty(S^1)$ to be an elliptic operator with index $+1$. Then, we form the operator $R = P \otimes Q$ and thus reduce to the even-dimensional case. If M is not orientable, consider M' to be the orientable double cover of M. Then, we can reduce the proof to the orientable manifolds.

Before the next step, we recall the topological K-ring and the Chern character on it. Let X be a compact Hausdorff topological space. The Grothendieck group $K(X)$ is the free abelian group generated by all complex vector bundles on X modulo short exact sequences. Besides, the tensor product of bundles induces a commutative ring structure on $K(X)$. An important cohomological invariant is the Chern character $\mathrm{ch} : K(X) \to H^*(X;\mathbb{Q})$ which is a ring homomorphism.

Next, we interpret the index of an elliptic operator to the $\mathbb{C}$-valued map on the K ring.

Lemma 17. There is a natural map $\mathrm{ind} : K(\Sigma(T^*M);\mathbb{C}) \to \mathbb{C}$ which is linear so that $\mathrm{ind}(P) = \mathrm{ind}(\Pi_+(\Sigma_{\sigma_P}))$ if $P : C^\infty(E_1) \to C^\infty(E_2)$ is an elliptic operator over M with symbol σ_P.

Proof. We simply define ind : $\text{Vect}\left(\Sigma\left(T^*M\right)\right) \to \mathbb{Z}$ so that $\text{ind}\left(\Pi_+(\Sigma_{\sigma_P})\right) = \text{ind}(P)$ if P is an elliptic operator. Also, we have $\Sigma_{\sigma_P \oplus \sigma_Q} = \Sigma_{\sigma_P} \oplus \Sigma_{\sigma_Q}$ and therefore $\Pi_+(\Sigma_{\sigma_P \oplus \sigma_Q}) = \Pi_+(\Sigma_{\sigma_P}) \oplus \Pi_+(\Sigma_{\sigma_Q})$. Moreover, we have $\text{ind}(P \oplus Q) = \text{ind}(P) + \text{ind}(Q)$. Thus, we extend the map to ind : $K\left(\Sigma\left(T^*M\right)\right) \to \mathbb{Z}$ to be $\mathbb{Z}$-linear. Finally, tensoring with $\mathbb{C}$ to extend ind : $K\left(\Sigma\left(T^*M\right); \mathbb{C}\right) \to \mathbb{C}$. □

Lemma 18. Let $\pi : \Sigma\left(T^*M\right) \to M$ be the natural projection map. If $E \in K\left(\Sigma\left(T^*M\right); \mathbb{C}\right)$ can be written as $\pi^* E_1$ for $E_1 \in K(M; \mathbb{C})$, then $\text{ind}(E) = 0$. Thus, ind : $K\left(\Sigma\left(T^*M\right); \mathbb{C}\right) / K(M; \mathbb{C}) \to \mathbb{C}$.

Proof. If $E = \pi^* E_1$, then the clutching function defining E is just the identity map. Consequently, the corresponding elliptic operator P can be taken to be a self-adjoint operator on $C^\infty(E)$ which has index zero. □

These two lemmas show that all the information contained in an elliptic complex from the point of view of computing its index is contained in the corresponding description in K-theory. Since the index map is linear, it suffices to prove the Atiyah–Singer Index Theorem on the generators given by the twisted signature operator due to the following lemma:

Lemma 19. Assume M is orientable and of even dimension m. Let P_E be the operator of the twisted signature operator of E. The bundles $\left\{\Pi_+(\Sigma_{\sigma_{P_E}})\right\}_{E \in \text{Vect}(M)}$ generate $K\left(\Sigma\left(T^*M\right); \mathbb{C}\right) / K(M; \mathbb{C})$ additively.

To prove this lemma, we need following lemmas:

Lemma 20. Let $P : C^\infty\left(\bigwedge^+\right) \to C^\infty\left(\bigwedge^-\right)$ be the operator of the signature operator. Let $\omega = \text{ch}_{m/2}\left(\Pi_+(\Sigma_{\sigma_P})\right) \in H^m\left(\Sigma\left(T^*M\right); \mathbb{C}\right)$. Then, if ω_M is the orientation class of M, we have

1. $\omega_M \wedge \omega$ gives the orientation of $\Sigma\left(T^*M\right)$.
2. If S^m is a fiber sphere of $\Sigma\left(T^*M\right)$, then $\int_{S^m} \omega = 2^{m/2}$.

Proof. Let $(x_1, \ldots, x_m)$ be an oriented local coordinate system on M so that the $\{dx_j\}$ are orthonormal at $x_0 \in M$. If $\xi = (\xi_1, \ldots, \xi_m)$ are the dual fiber coordinates for T^*M, then:

$$\sigma_P(\xi) = \sum_j \sqrt{-1}\xi_j \left(c\left(dx_j\right)\right)$$

gives the symbol of $d + d^*$, where $c(\cdot)$ denotes the Clifford multiplication. We let $e_j = \sqrt{-1}c\left(dx_j\right)$; these are self-adjoint matrices such that $e_j e_k + e_k e_j = 2\delta_{jk}$. The orientation class is defined by:

$$\epsilon = \sqrt{-1}^{m/2} c\left(dx_1\right) \ldots c\left(dx_m\right) = (-\sqrt{-1})^{m/2} e_1 \ldots e_m$$

The bundles $\bigwedge^{\pm}$ are defined as the ± 1 eigenspaces of ϵ. Consequently,

$$\Sigma_{\sigma_P(\xi,t)} = t\epsilon + \sum \xi_j e_j$$

Therefore, when S^m is given its natural orientation. by lemma 14, we have

$$\begin{aligned}\int_{S^m} \mathrm{ch}_{m/2}\, \Pi_+(\Sigma_{\sigma_P}) &= \sqrt{-1}^{m/2} 2^{-m/2}\, \mathrm{tr}\,(\epsilon e_1 \ldots e_m) \\ &= \sqrt{-1}^{m/2} 2^{-m/2}\, \mathrm{tr}\left(\epsilon \sqrt{-1}^{m/2} \epsilon\right) \\ &= (-1)^{m/2} 2^{-m/2}\, \mathrm{tr}(I) = (-1)^{m/2} 2^{-m/2} 2^m \\ &= (-1)^{m/2} 2^{m/2}\end{aligned}$$

Besides, S^m is in fact given the orientation induced from the orientation on $\Sigma(T^*M)$ and on M. At the point $(x, 0, \ldots, 0, 1)$ in $T^*M \oplus \mathbb{R}$ the natural orientations are:

$$\begin{aligned}\text{of } X :& dx_1 \wedge \cdots \wedge dx_m \\ \text{of } \Sigma(T^*M) :& dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_m \wedge d\xi_m \\ =& (-1)^{m/2} dx_1 \wedge \cdots \wedge dx_m \wedge d\xi_1 \wedge \cdots \wedge d\xi_m \\ \text{of } S^m :& (-1)^{m/2} d\xi_1 \wedge \cdots \wedge d\xi_m\end{aligned}$$

Thus, with the induced orientation, the integral becomes $2^{m/2}$ and the lemma is proved. □

Consequently, ω provides a cohomology extension and we conclude the following lemma:

Lemma 21. Let $\rho : \Sigma(T^*M) \to M$ where M is orientable and even dimensional. Then,

1. $\rho^* : H^*(M; \mathbf{C}) \to H^*(\Sigma(T^*M); \mathbf{C})$ is injective.
2. If ω is as defined in Lemma 20, then we can express any $\alpha \in H^*(\Sigma(T^*M); \mathbb{C})$ uniquely as $\alpha = \rho^*\alpha_1 + \rho^*\alpha_2 \wedge \omega$ for some $\alpha_i \in H^*(M; \mathbb{C})$.

Since ρ^* is injective, we shall drop it and regard $H^*(M; \mathbb{C})$ as being a subspace of $H^*(\Sigma(T^*M); \mathbb{C})$.

The Chern character gives an isomorphism $K(M; \mathbb{C}) \simeq H^{\mathrm{even}}(M; \mathbb{C})$. When we interpret Lemma 21 in K-theory, we conclude that we can decompose $K(\Sigma(T^*M); \mathbb{C}) = K(M; \mathbb{C}) \oplus K(M; \mathbb{C}) \otimes \Pi_+(\Sigma_{\sigma_P})$. $\mathrm{ch}(E)$ generates $H^{\mathrm{even}}(M; \mathbb{C})$ as E ranges over $K(M; \mathbb{C})$. Therefore $K(\Sigma(T^*M); \mathbb{C}) / K(M; \mathbb{C})$ is generated as an additive module by the twisted signature operator with coefficients in bundles over M. $\Pi_+(\Sigma_{P_E}) = E \otimes \Pi_+(\Sigma_P)$ if P_E is the symbol of the twisted signature operator on E. This proves the lemma 19 and hence Atiyah–Singer Index Theorem.

6 Original Proof Atiyah–Singer Index Theorem

Setting and statement First, we recall the topological K-ring and the Chern character on it. Let X be a compact Hausdorff topological space. The Grothendieck group $K(X)$ is the free abelian group generated by all complex vector bundles on X modulo short exact sequences. Besides, the tensor product of bundles induces a commutative ring structure on $K(X)$. An important cohomological invariant is the Chern character $\mathrm{ch} : K(X) \to H^*(X; \mathbb{Q})$ which is a ring homomorphism.

For two vector bundles E, F on a topological space Y with an isomorphism σ on a suitable subspace Y_0, we can define a difference element

$$d(E, F, \sigma) \in K(Y/Y_0),$$

where Y/Y_0 is Y with Y_0 pinched to a point by the following way. Let $I = [0, 1]$ and form the subspace

$$A = Y \times 0 \cup Y \times 1 \cup Y_0 \times I$$

of $Y \times I$. On A, we define a complex vector bundle L by putting E on $Y \times 1$, F on $Y \times 0$ and using σ to join them along $Y_0 \times I$. Then, $d(E, F, \sigma)$ is defined to be the image of L in the following composition of maps:

$$K(A) \to K_1(Y \times I/A) \cong K_1(S(Y/Y_0)) \cong K(Y/Y_0),$$

where K_1 group is defined using vector bundles on a suspension of the space and K_0, K_1 groups fits into an exact sequence similar to the relative homology exact sequence.

We denote by $B(M)$ the unit ball bundle of $T^*(M)$. Since an elliptic operator $P : C^\infty(E) \to C^\infty(F)$ gives an isomorphism between the sphere bundles, it defines an element

$$d(p^*E, p^*F, \sigma(D)) \in K(B(M)/S(M)),$$

where $p : B(M) \to M$ is the projection. Hence, P defines an element

$$\mathrm{ch}\, d(E, F, \sigma(P)) \in H^*(B(M)/S(M); \mathbb{Q}).$$

Also, using the Thom isomorphism

$$\phi_* : H^k(M; \mathbb{Q}) \cong H^{m+k}(B(M)/S(M); \mathbb{Q}),$$

we obtain finally the element

$$\phi_*^{-1}\,\mathrm{ch}\,d(E,F,\sigma(P))\in H^*(M;\mathbb{Q})$$

which we shall simply denote by $\mathrm{ch}(P)$. Now, we can state the Atiyah–Singer Index Theorem.

Theorem 22 (Atiyah–Singer Index Theorem)**.** Let $P:C^\infty(M,E)\to C^\infty(M,F)$ be an elliptic operator. Let $\mathrm{Td}(M)=\mathrm{Td}(TM\otimes\mathbb{C})$ be the Todd genus of the complexification of the real tangent bundle. Then,

$$\mathrm{ind}(P)=\int_M \mathrm{Td}(M)\,\mathrm{ch}\,(P)\,.$$

The group of elliptic symbols Let $P:C^\infty(M,E)\to C^\infty(M,F)$ be an elliptic operator with symbol $\sigma(P)$. The index of P $\mathrm{ind}\,(P)$ only depends on the symbol $\sigma(P)$. By definition, an elliptic operator gives an isomorphism on the $\pi^*E\to\pi^*F$, where $\pi:S(M)\to M$ is the projection from the sphere bundle $S(M)$ to M, we may regard the index as a function

$$\mathrm{ind}\,P:\mathrm{Iso}\,(\pi^*E,\pi^*F)\to\mathbb{Z}$$

mapping the symbol $\sigma(P)$ to $\mathrm{ind}\,(P)$.

By the direct computation, it is easy to show that

1. $\mathrm{ind}\,(\sigma(P)\oplus\sigma(P'))=\mathrm{ind}\,(\sigma(P))+\mathrm{ind}\,(\sigma(P'))$.
2. $\mathrm{ind}\,(\sigma(P)\otimes\sigma(P'))=\mathrm{ind}\,(\sigma(P))\cdot\mathrm{ind}\,(\sigma(P'))$.
3. $\mathrm{ind}\,(\sigma(P))=0$ if $\sigma(P):\pi^*E\to\pi^*F$ extends to an isomorphism $p^*E\to p^*F$ on ball bundle $B(M)$, where $p:B(M)\to M$ is the projection.

Let us define an equivalence relation on the set of all elliptic symbols by $\sigma\sim\sigma'$ if there exist $\alpha_i(i=1,2,3,4)$ which extend to $B(X)$ such that

$$\sigma'\oplus\alpha_1=\alpha_2\,(\sigma\oplus\alpha_3)\,\alpha_4$$

and denote the set of equivalence classes by $\mathrm{Ell}(M)$. It is an abelian semigroup under $\oplus$ and (i)-(iii) above show that ind induces a homomorphism

$$\mathrm{ind}:\mathrm{Ell}(M)\to\mathbb{Z}.$$

The function μ defined by

$$\mu(\sigma)=\int_M \mathrm{ch}(\sigma)\,\mathrm{Td}\,(M)$$

defines another homomorphism

$$\mu : \mathrm{Ell}(M) \to \mathbb{Q}.$$

Then, Atiyah–Singer Index Theorem asserts that $\mu = \mathrm{ind}$.

Now, the first key step in the proof is to determine $\mathrm{Ell}(M)$.

Proposition 23. $\mathrm{Ell}(M)$ is an abelian group and the tensor product makes it into a $K(X)$-module. If $\dim M$ is even, D_0 denotes the operator $d + d^* : \bigwedge^+ T^*M \to \bigwedge^- T^*M$, and $\sigma_0 = \sigma(D_0)$ then $\mathrm{Ell}(M)/K(M)\sigma_0$ is a finite group.

This is proved by showing that[1] $\mathrm{Ell}(M) \cong \widetilde{K}(B(M)/S(M))$, then applying the Chern character and observing that $\mathrm{ch}(\sigma_0)$ is an invertible element of $H^*(X; \mathbb{Q})$.

Proposition 23 reduces us to checking that $\mathrm{ind}(E \otimes \sigma_0) = \mu(E \otimes \sigma_0)$ for all vector bundles E on M. We will write

$$\mu(E \otimes \sigma_0) = \mu(M, E), \quad \mathrm{ind}(E \otimes \sigma_0) = \mathrm{ind}(M, E)$$

to emphasize the base manifold M since we will change the base manifold in the later proof.

Cobordism If $E = 1$ is the trivial vector bundle, then the equality $\mu(M, 1) = \mathrm{ind}(M, 1)$ is just the untwisted signature formula. Now, we introduce the equivalence relation of cobordism on pairs (M, E) where M is a smooth compact oriented manifold of even dimension and E is a complex vector bundle on M:

$$(M_1, E_1) \sim (M_2, E_2)$$

if there exists a smooth compact oriented manifold Y with boundary $\partial Y = M_1 \cup (-M_2)$. ($-M_2$ denotes X_2 with the opposite orientation) and a complex vector bundle U on Y with $U|_{M_i} \cong E_i$. Then, the set of cobordism classes forms an abelian group under the disjoint union operation as the the additive operator. We denote this group by A. Then, the second key step in the proof of Atiyah–Singer Index Theorem is:

Proposition 24. If $(M, E) \sim 0$ in A, then $\mu(M, E) = 0$ and $\mathrm{ind}(M, E) = 0$.

This proof is highly based on the knowledge of singular integral operators on manifolds (cf.[See61]). If T is a singular integral operator from E to F, then we can define the symbol of T, $\sigma(T) : \mathrm{Int}(E, F) \to \mathrm{Hom}(\pi^* E, \pi^* F)$

[1] $\widetilde{K}$ denotes the "reduced" group of K.

given by a Fourier transform. Also, the symbol map on singular integral operators is surjective. The key point in the proof is to find a singular integral operator such that its symbol is the symbol of twisted Dirac operator.

sketch of proof. Let $\partial Y = M$ and $U|_M = E$. On Y, the elliptic differential operator $D = *d + d*$ operates (choose a Riemannian connection ∇ on E) on the vector bundle $\sum_k \bigwedge^{2k} T^*Y \otimes E$. Then, $D^2 = 1$ on M and thus along the boundary M, this vector bundle splits into $E_1 \otimes E$ and $E_2 \otimes E$ as we discussed in the twisted Dirac operator. Then,

$$\begin{aligned} Du = 0, \quad u|_M \in E_1 \otimes E \\ Du = 0, \quad u|_M \in E_2 \otimes E \end{aligned}$$

are well-posed boundary problems in the sense of [ADN59] which give rise to a singular integral operator T on M (see [AD62]) such that $\operatorname{ind}(T) = 0$, and $\sigma(T) = E \otimes \sigma_0$. □

Now, by the following generalized Thom isomorphism theorem, we can determine $A \otimes \mathbb{Q}$.

Proposition 25. $A \otimes \mathbb{Q}$ is the polynomial algebra generated by $\left(\mathbb{CP}^{2i}, 1\right)$, $i = 1, 2, \ldots$, and (S^{2j}, V_j), $j = 1, 2, \ldots$, where $V_j \in K\left(S^{2j}\right)$ has $\operatorname{ch}_j\left(V_j\right)$ a generator of $H^{2j}\left(S^{2j}; \mathbb{Z}\right) \subseteq H^{2j}\left(S^{2j}; \mathbb{Q}\right)$.

To complete the proof of Atiyah–Singer Index Theorem, for $\dim M$ even, we therefore need only check that $\mu = \gamma$ on the generators of $A \otimes Q$. In fact, one can prove that both μ and γ are multiplicative, i.e., that

$$\mu\left(M_1, E_1\right) \cdot \mu\left(M_2, E_2\right) = \mu\left(M_1 \times M_2, E_1 \otimes E_2\right)$$

and similarly

$$\operatorname{ind}\left(M_1, E_1\right) \cdot \operatorname{ind}\left(M_2, E_2\right) = \operatorname{ind}\left(M_1 \times M_2, E_1 \otimes E_2\right).$$

Using this one is finally reduced to checking the following:

1. The Hirzebruch signature of $\mathbb{CP}^{2n}$ is equal to 1.

2. The Euler number of S^{2n} is 2.

By using the multiplicative property of the index the case of an odd-dimensional M can be reduced to that of the even-dimensional manifold $M \times S^1$.

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CONSTRUCTION OF 28 DIFFERENTIAL STRUCTURES ON S^7

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Introduction

We have seen how Milnor construced exotic 7−spheres in class. In this report, our main goal is to give a explicit construction of all 28 differential structures on S^7. The first part introduces the results on group of homotopy spheres done by Kervaire and Milnor [6], which gives us a different understanding toward diffeomorphism class of topological sphere. In the second part, we will follow Brieskorn's work [1]. He considers a series of hypersurfaces and constructs all differential structures on S^7 with some of them.

1. Group of homotopy spheres

In this section, we will go through several results about groups of homotopy sphere (from [6]). From now on, all manifolds are compact, oriented and C^∞−differentiable.

Definition 1.1. Two closed n−manifolds M_1 and M_2 are *h-cobordant* if the disjoint sum $M_1 + (-M_2) = \partial W$ is the boundary of some manifold W, where both M_1 and $(-M_2)$ are deformation retracts of W. Clearly, it is an equivalence relation.

Definition 1.2. For two connected manifolds M_1 and M_2, choose imbeddings $i_1 : D^n \to M_1$ and $i_2 : D^n \to M_2$. The *connected sum* of M_1 and M_2 is obtained from the disjoint sum $(M_1 - i_1(0)) + (M_2 - i_2(0))$ by identifying $i_1(tu) = i_2((1-t)u)$ for each $u \in S^{n-1}$. Denote it by $M_1 \# M_2$. It can be shown that the connected sum is independent of the choice of i_1, i_2.

Remark 1.3. Let W_1 and W_2 are $(n+1)-$ manifolds with connected boundaries. We can construct a manifold W such that $\partial W = \partial W_1 \# \partial W_2$ with the following technique. Let $H^{n+1} = \{x \in \mathbf{R}^{n+1} | |x| \leq 1, x_0 \geq 0\}$ be the half-disk and $D^n = \{x \in H^{n+1} | x_0 = 0\}$.

Choose imbeddings

$$i_q : (H^{n+1}, D^n) \to (W_q, \partial W_q),$$

so that $i_2 \circ i_1^{-1}$ reverse the orientation. Define W from

$$(W_1 - i_1(0)) + (W_2 - i_2(0)), \quad q = 1, 2$$

by identifying $i_1(tu) = i_2((1-t)u)$ for $t \in (0,1), u \in S^n \cap H^{n+1}$. Then W ia a differential manifold with $\partial W = \partial W_1 \# \partial W_2$. We denote $(W, \partial W) = (W_1, \partial W_1) \# (W_2, \partial W_2)$.
Both this operator and connected sum are additive under taking signature.

Theorem 1.4. $(\Theta_n, \#)$ is a group.
We prove the following lemmas first.

Lemma 1.5. Let M_1, M_1' and M_2 be closed and simply connected. If $M_1 \sim M_1'$ then $M_1 \# M_2 \sim M_1' \# M_2$.

Proof. We may assume $n \geq 3$. Let $i_2 : D^n \to M_2$ and $M_1 + (-M_1') = \partial W_1$, where M_1 and $(-M_1')$ are deformation retracts of W_1. Choose $p \in M_1$, $p' \in M_1'$, and an arc $A \subset W_1$ connecting p and p' so that a tubular neighborhood of A is diffeomorphic to $\mathbf{R}^n \times [0,1]$. Thus, we have the imbedding

$$i : \mathbf{R}^n \times [0,1] \to W_1,$$

where $i(\mathbf{R}^n \times \{0\}) \subset M_1$, $i(\mathbf{R}^n \times \{1\}) \subset M_1'$ and $i(\{0\} \times [0,1]) = A$. Define W from

$$(W_1 - A) + (M_2 - i_2(0))$$

by identifying $i(v, s) = i_2(\frac{v}{|v|} - v) \times \{s\}$ for each $v \in D^n - \{0\}$ and $s \in [0,1]$. Then

$$M_1 \# M_2 + (-(M_1' \# M_2)) = \partial W.$$

It remains to show $M_1 \# M_2$ and $-(M_1' \# M_2)$ are deformation retracts of W. By the Mayer-Vietoris sequence, it suffices to show that the inclusion $M_1 - \{p\} \to W - A$ is a homotopy equivalence. Since $n \geq 3$, $M_1' - \{p'\}$ and $W_1 - A$ are simply connected. Consider the homology exact sequence from the pair $(M_1, M_1 - p)$ to $(W_1, W_1 - A)$. Since M_1 is a deformation retract of W_1,

$$H_k(M_1) \xrightarrow{\sim} H_k(W_1).$$

Combined with

$$H_k(M_1, M_1 - p) \xrightarrow{\sim} H_k(W_1, W_1 - A),$$

we have

$$H_k(M_1 - p) \xrightarrow{\sim} H_k(W_1 - A).$$

Thus, $M_1 - \{p\} \to W - A$ is a homotopy equivalence. $\square$

Lemma 1.6. Let M be a simply connected manifold. Then $M \sim S^n$ if and only if M bounds a contractible manifold.

Proof. Suppose $M \sim S^n$ with $M + (-S^n) = \partial W$. Obtain W' by filling in D^{n+1}. Then $\partial W' = M$. Since S^n is a deformation retract of W and D^{n+1} is contractible, W' is contractible.

Conversely, suppose $M = \partial W'$ with W' contractible. Obtain W by removing the interior of an imbedded disk. W is simply connected and $\partial W = M + (-S^n)$. Map the homology exact sequence of the pair (D^{n+1}, S^n) into that of the pair (W', W). Since D^{n+1} and W' are contractible, we can apply the same argument Lemma 1.5 and obtain that S^n is a deformation retract of W. By Poincaré duality,

$$H_k(W, M) \simeq H^{n+1-k}(W, S^n) = 0.$$

Then $H_k(M) \xrightarrow{\sim} H_k(W)$, hence M is a deformation retract of W. $\square$

Proof of Theorem 1.4. From Seifert-Van Kampen theorem, the connected sum of two $n-$homotopy spheres is still an $n-$homotopy sphere. Note that $|\Theta_1| = |\Theta_2| = 1$, so we may assume $n \geq 3$.

- By Lemma 1.5, $\#$ is well-defined on Θ_n.
- S^n is the identity:
  For any $n-$manifold M, take $i_2 : D^n \xrightarrow{\sim} S^n - \{\mathrm{N}\}$, it is easy to see that $M \# S^n$ is diffeomorphic to M. Thus S^n is an identity of Θ_n.

- $-M$ is the inverse of M in Θ_n:
  By Lemma 1.6, it suffices to show that $M\#(-M)$ bounded a contractible manifold. Define W from
  $$\left(M - i\left(\frac{1}{2}D^n\right)\right) \times [0,\pi] + S^{n-1} \times H^2$$
  by identifying
  $$(i(tu),\theta) = (u,((2t-1)\sin\theta,(2t-1)\cos\theta))$$
  for each $\frac{1}{2} < t \leq 1, 0 \leq \theta \leq \pi$. Then W is a contractible differentiable manifold with $\partial W = M\#(-M)$.
- Associativity and commutativity of Θ_n are clear. □

From Theorem 1.7 and Theorem 1.8, we will see that Θ_n is the set of all diffeomorphism classes on topological $n-$sphere, when $n \geq 5$.

Theorem 1.7. (Generalized Poincaré conjecture) Every homotopy $n-$sphere, $n \neq 3,4$, is homeomorphic to S^n. (See [11])

Theorem 1.8. Two homotopy $n-$sphere, $n \neq 3,4$, are $h-$cobordant if and only if they are diffeomorphic. (See [12])

Definition 1.9. A smooth manifold M is parallelizable if TM is trivial and is $s-$parallelizable if $TM \oplus \epsilon$ is trivial, where ϵ is the trivial line bundle over M.

Theorem 1.10. Homotopy spheres are $s-$parallelizable.

Proof. Since the case we concern about is $n = 7$, we will only prove the simpler case $n \equiv 3,5,6,7 \pmod 8$. For the cases remained, see *Theorem 3.1* [6].
Let Σ be homotopy $n-$sphere. By Theorem 1.7, Σ is a topological $n-$sphere. We can trivialize $T\Sigma$ on both hemisphere. This overlap part induces a map $f : S^{n-1} \to SO(n)$. It suffices to show that

$$S^{n-1} \xrightarrow{f} SO(n) \hookrightarrow SO(n+1)$$

is null-homotopic. By Bott's computation [2], the stable group $\pi_{n-1}(SO) = 0$ when $n \equiv 3, 5, 6, 7 \pmod 8$. Moreover, from the homotopy exact sequence of fibration

$$\begin{array}{ccc} SO(n+k) & \longrightarrow & SO(n+k+1) \\ & & \downarrow \\ & & S^{n+k} \end{array}$$

, we have $\pi_{n-1}SO(n+1) = \pi_{n-1}SO(n+k)$ for all $k \geq 1$.
Therefore, Σ is $s-$parallelizable. □

We have proved the following theorem in class.

Theorem 1.11. Let M be a $n-$dimensional submanifold of $S^{n+k}, k > n$, then M is $s-$parallizable if and only if the normal bundle is trivial. Moreover, a connected manifold with nonempty boundary is $s-$parallelizable if and only if it is parallelizable.

We define a subgroup $bP_{n+1} \subset \Theta_n$ as follows. A homotopy $n-$sphere M represents an element of bP_{n+1} if and only if M is the boundary of a parallelizable manifold. We have shown that parallizable is invariant under $h-$cobordism in class.

Definition 1.12. Let X, Y be manifolds and $i : X \hookrightarrow Y$ be an imbedding. Consider the imbedding $\iota : X \hookrightarrow \mathbf{R}^{n+k}$. By tubular neighborhood theorem, we can factor through ι as the zero section imbedding to $N_X Y$ followed by an imbedding into $\mathbf{R}^{n+k}$.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{R}^{n+k} \\ \downarrow & \overset{j}{\nearrow} & \\ N_X Y & & \end{array}$$

We consider the following sequence of maps

$$\begin{array}{l} N_X Y \overset{j}{\hookrightarrow} \mathbf{R}^{n+k} \hookrightarrow (\mathbf{R}^{n+k})^+ \simeq S^{n+k} \xrightarrow{t} T(N_X Y) \simeq D(N_X Y)/S(N_X Y) \\ \uparrow \\ X \end{array}$$

, where t sends points in the image of the disk bundle of $N_X Y$ under j to their preimages in the disk bundle and other points in S^{n+k} to the point made by collapsing the sphere bundle. i.e. the base point of $T(N_X Y)$. t is called *Pontryagin-Thom construction.*

For homotopy $n-$sphere M, by Theorem 1,10, M is $s-$parallelizable and by Theorem 1.11, the normal bundle of M in S^{n+k} is trivial. For a field of normal $k-$frame φ, the Pontryagin-Thom construction yields a map

$$p(M, \varphi) : S^{n+k} \to S^k.$$

The homotopy class of $p(M, \varphi)$ is an element in $\pi_{n+k}(S^k)$. Denote $p(M) = \{p(M, \varphi)\} \subset \pi_{n+k}(S^k)$. From now on, we choose $k \geq n + 2$, then $\pi_{n+k}(S^k)$ is stable and denote $\Pi_n = \pi_{n+k}(S^k)$.

Lemma 1.13. The subset $p(M) \subset \Pi_n$ contains the zero element of Π_n if and only if M bounds a parallelizable manifold.

Proof. Suppose $M = \partial W$ with W is parallizable, then the imbedding $i : M \to S^{n+k}$ can be extended to $W \to D^{n+k+1}$. Let ψ be the normal $k-$frame of W and $\phi = \psi|_M$. Then $p(M, \phi) : S^{n+k} \to S^k$ extends over D^{n+k+1}. Thus, $p(M, \phi) = 0 \in \Pi_n$
Conversely, suppose $p(M, \phi) = 0$. Then there exists W such that $\partial W = M$ and ϕ extends to a normal frames over W. By Theorem 1.11, W is parallelizable. □

Lemma 1.14. If M and M' are $s-$parallelizable then

$$p(M) + p(M') \subset p(M\#M') \subset \Pi_n.$$

Proof. Consider $M \times [0, 1]$ and $M' \times [0, 1]$. Apply the construction in Remark 1.3 to glue only $M \times \{1\}$ and $M' \times \{1\}$ instead. Then we can construct a manifold W with boundary $(M\#M') + (-M) + (-M')$.
Choose an imbedding $W \hookrightarrow S^{n+k}$ such that $M\#M'$ goes to $S^{n+k} \times \{1\}$ and $(-M) + (-M')$ goes to $S^{n+k} \times \{0\}$. For φ, φ', normal frame on $(-M)$ and $(-M')$, it can be extended to a normal frame on W. So $p(M, \varphi) + p(M', \varphi') \in p(M\#M')$. □

Lemma 1.15. $p(S^n) \subset \Pi_n$ is a subgroup. For any $\Sigma \in \Theta_n$, $p(\Sigma)$ is a coset of $p(S^n)$. Thus, $\Sigma \mapsto p(\Sigma)$ defines the homomorphism $p' : \Theta_n \to \Pi_n/p(S^n)$.

Proof. For the first statement, apply Lemma 1.13 to $S^n \# S^n = S^n$, then we get

$$p(S^n) + p(S^n) \subset p(S^n \# S^n) = p(S^n),$$

Also, apply Lemma 1.13 to $S^n \# \Sigma = \Sigma$ and $\Sigma \# (-\Sigma) = S^n$, then we find that $p(\Sigma)$ is a coset of $p(S^n)$. □

By Lemma 1.13, the kernel of p' consists exactly all $M \in \Theta_n$ which bounded a parallelizable manifold. Then bP_{n+1} is a subgroup of Θ_n and Θ_n/bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$.

Remark 1.16. From [4] $p(S^n)$ can be discribed as the image of Hopf-Whitehead homomorphism $J_n : \pi_n(SO(k)) \to \pi_{n+k}(S^k)$. We will not give the details here. $\Pi_n/p(S^n)$ can be computed explicitly. In particular $\Pi_7/p(S^7) = 0$.

Thus, we have $\Theta_7 = bP_8$. This result will help us to construct the differential structures on S^7 later.

In the following, we will consider $4m-$manifolds M bounded by a homotopy $(4m-1)-$sphere Σ. We denote σ_m as the positive generator of signatures of all $s-$parallelizable M_0 bounded by S^{4m-1}. We hope to use $\sigma(M)$ to characterize the $h-$cobordism class of Σ in bP_{4m}. Eventually, we have

Theorem 1.17. Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$, $m > 1$, which bound $s-$parallizable manifolds M_1 and M_2 respectively. Then Σ_1 is $h-$cobordant to Σ_2 if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \pmod{\sigma_m}.$$

The key point of this theorem is a result in another Milnor's work [8]. We will state it below and sketch the proof of Theorem 1.17.

Definition 1.18. Let M be a $n-$dimensional manifold, $n = p + q + 1$ and $\varphi : S^p \times D^{q+1} \to M$ be a differential imbedding. Define M' from

$$(M - \varphi(S^p \times \{0\})) + D^{p+1} \times S^q$$

be identifying $\varphi(u, tv) = (tu, v)$ for $u \in S^p, v \in S^q, t \in (0, 1]$. We say M' is obtained from M by *spherical modification*.

Lemma 1.19. (from [8]) Let M be a parallelizable $4m$-manifold with $4m > 4$ bounded by a homology sphere, then the homotopy group of M can be killed by a sequence of spherical modification is and only if $\sigma(M) = 0$.

Proof of Theorem 1.17.

Let $\sigma(M_0) = \sigma_m$. Suppose $\sigma(M_1) = \sigma(M_2) + \sigma(M_0)$. By Remark 1.3, we construct

$$(M, \partial M) = (-M_1, -\partial M_1)\#(M_2, \partial M_2)\#(M_0, \partial M_0),$$

where $\partial M = -\Sigma_1\#\Sigma_2\#S^{4m-1} = -\Sigma_1\#\Sigma_2$. Since

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) + \sigma(M_0) = 0,$$

from Lemma 1.19, $\partial M = -\Sigma_1\#\Sigma_2 = 0$.

Conversely, let W be an $h-$cobordism between $-\Sigma_1\#\Sigma_2$ and S^{4m-1}. Obtain M from Gluing W to $(-M_1, -\partial M_2)\#(M_2, \partial M_2)$ along the common boundary $-\Sigma_1\#\Sigma_2$. Then M is $s-$parallelizable, bounded by S^{4m-1}. Then

$$0 \equiv \sigma(M) = -\sigma(M_1) + \sigma(M_2) \pmod{\sigma_m}.$$

□

By Theorem 1.17, we obtain that for $n = 4m - 1$, bP_{4m} is a subgroup of a cyclic group of order σ_m. In fact, we have $bP_{4m} = \sigma_m/8$. (See [7])

In [5], Kervaire and Milnor gave a formula to compute σ_m.

$$\sigma_m = 2^{2m-1}(2^{2m-1} - 1)B_m j_m a_m/m,$$

where B_m is the $m-$th Bernoulli number, j_m is the order of the cyclic group $J_k(\pi_{4m-1}(SO))$ and a_m equals 1 or 2 according as m is even or odd. In particular, for $m = 2$ i.e. $M \in \Theta_7$,

$$\sigma_2 = 224 \text{ and } |bP_8| = 28.$$

2. The construction of element in Θ_7

In this section, we follow Brieskorn's paper [1] to see the explicit construction of 28 differential structure on S^7.

Notation. Let $a = (a_1, \dots, a_n)$ be a $n-$tuple of integers with $a_i > 1$. The following are notations we will use in this section.

(1) $X_a = X(a_1, \dots, a_n) := \{z \in \mathbf{C}^n | z_1^{a_1} + \cdots + z_n^{a_n} = 0\}$
(2) $\Sigma_a = \Sigma(a_1, \dots, a_n) := X(a_1, \dots, a_n) \cap S^{2n-1}$.
(3) $\Xi_a(t) = \{z \in \mathbf{C}^n | z_1^{a_1} + \cdots + z_n^{a_n} = t\}$. In particular, $\Xi_a := \Xi_a(1)$.
(4) $M_a(t) := \Xi_a(t) \cap D^{2n}$ and $\Sigma_a(t) = \Xi_a(t) \cap S^{2n-1}$.
(5) $G_a = G(a_1, \dots, a_n)$ is a graph with vertices $a_1, \dots, a_n$ and there is an edge connecting a_i and a_j if and only if $\gcd(a_i, a_j) > 1$.

We will use Σ_a to construct the differential structures on S^7. For every k, one has automorphism ω_k, the multiplication of $k-$th coordinate by $\xi_k = e^{2\pi i/a_k}$, on Ξ_a. Denote Ω_a as the group generated by those ω_k.

$$\Omega_a = \prod_k \mathbf{Z}_{a_k}$$

Let $J_a = \mathbf{Z}[\Omega_a]$ and I_a be the ideal of J_a generated by elements of the form $1 + \omega_k + \cdots + \omega_k^{a_k - 1}$.

Lemma 2.1. The singular homology $H_i(\Xi_a, \mathbf{Z})$ vanishes when $i \neq 0, n-1$, and $H_{n-1}(\Xi_a, \mathbf{Z}) \simeq J_a/I_a$.

Proof. We will construct a simplicail complex $\mathscr{E}$ such that $\mathscr{E}$ is a deformation retract of Ξ_a, then we can compute $H_i(\mathscr{E}, \mathbf{Z})$ instead. Let

$$\mathbf{e} = \{(z_1, \dots, z_n) \in \Xi_a | z_k \in \mathbf{R}_{\geq 0}\},$$

which is homeomorphic to the standard simplex Δ_{n-1}. Let

$$\mathscr{E} = \{(z_1, \dots, z_n) \in \Xi_a | z_k^{a_k} \in \mathbf{R}_{\geq 0}\},$$

which is a simplicial complex consists of images of $\mathbf{e}$ under the action of Ω_a:

$$0 \to J_a \mathbf{e} \to \bigoplus_i J_{a_1, \dots, \hat{a_i}, \dots, a_n}\{\partial_i \mathbf{e}\} \to \bigoplus_{i<j} J_{a_1, \dots, \hat{a_i}, \dots, \hat{a_j}, \dots, a_n}\{\partial_i \partial_j \mathbf{e}\} \to \cdots,$$

where $J_{a_1,\ldots,\hat{a_j},\ldots,a_n} = \mathbf{Z}[\prod_{k\neq j}\mathbf{Z}_{a_k}]$. By computation, we can see that $H_i(\mathscr{E},\mathbf{Z}) = 0$ if $i \neq 0, n-1$. Let

$$e = \prod_{k=1}^{n}(1-\omega_k)\mathbf{e}.$$

Then

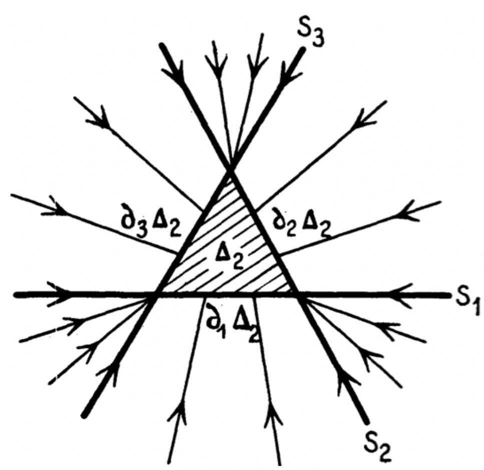
$$H_{n-1}(\mathscr{E},\mathbf{Z}) = J_a e = J_a/I_a.$$

It remains to show that $\mathscr{E}$ is a deformation retract of Ξ_a. First, consider a complex analytic hyperplane

$$X := \{(\eta_1,\ldots,\eta_n) \in \mathbf{C}^n | \sum_i \eta_i = 1\},$$

$$S_i = \{\eta \in X | \eta_i = 0\}.$$

Construct a deformation retraction from the hyperplane system $(X, S_1, \ldots, S_n)$ to the simplicial system $(\Delta_{n-1}, \partial_1\Delta_{n-1}, \ldots, \partial_n\Delta_{n-1})$: this can be done by combining the deformation retraction from complex to its real part and the deformation retraction on Δ_{n-1} symbolized by the below figure (captured from [10]).



By the change of variables, $z_k = \eta_k^{1/a_k}$, we get a deformation retraction from $\mathscr{E}$ to Ξ_a. □

Remark 2.2. $H_{n-1}(\Xi_a,\mathbf{Z}) \simeq J_a/I_a$ is a free $\mathbf{Z}-$module of rank $\prod_{k=1}^{n}(a_k-1)$.

Lemma 2.3. For $n \geq 3$, Ξ_a is simply connected. (And therefore $(n-2)-$connected).

Proof. From Lemma 2.1, we have $\mathscr{E}$ is a deformation retract of Ξ_a, so it suffices to show that $\pi_1(\mathscr{E}_2)$ is trivial. The vertices of $\mathscr{E}_2$ are $p_k^s = (0, \ldots, \xi_k^s, 0, \ldots, 0)$, where ξ_k is the primitive a_k-th root of unity and $0 \leq s \leq a_k$. There is exactly one edge connecting p_i^r, p_k^s for $i \neq k$ and exactly one $2-$simplex passing through triple p_i^r, p_j^s, p_k^t for distinct i, j, k. Then, the edge path connecting p_i^r, p_j^s, p_k^t is homotopic to the edge connecting p_i^r, p_k^t, and the edge path connecting $p_i^{r_1}, p_k^{t_1}, p_i^{r_2}, p_k^{t_2}$ is homotopic to the edge path connecting $p_i^{r_1}, p_j^s, p_k^{t_2}$ for some $j \neq i, k$. Both operations reduce the number of edges of the path by 1. Therefore, one can convert every closed edge path in $\mathscr{E}_2$ into an obviously zero homotopic path by repeatedly using them. So $\mathscr{E}_2$ and thus also Ξ_a is simply connected. $\square$

Consider $\Sigma_a \subset S^{2n-1}$. By Poincaré duality and Alexander duality, we have

$$H_{i-1}(\Sigma_a, \mathbf{Z}) = H^{2n-2-i}(\Sigma_a, \mathbf{Z}) \simeq H_i(S^{2n-1} - \Sigma_a, \mathbf{Z}).$$

Moreover, we have the homeomorphism $\varphi : (S^{2n-1} - \Sigma_a) \times (0, \infty) \to \mathbf{C}^n - X_a$, where

$$\varphi((z_1, \ldots, z_n), \tau) = (\tau^{1/a_1} z_1, \ldots, \tau^{1/a_n} z_n).$$

Therefore, $S^{2n-1} - \Sigma_a$ is a deformation retract of $\mathbf{C}^n - X_a$. To compute the homology of Σ_a, it suffices to look at $Y_a := \mathbf{C}^n - X_a$. Define $p : Y_a \to \mathbf{C}^* \simeq S^1$ by $p(z_1, \ldots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n}$, which is a locally trivial bundle with fiber $\Xi_a(t)$.
Now, consider the action of the generator of $\pi_1(\mathbf{C}^*) = \pi_1(S^1)$ on $H_{n-1}(\Xi_a, \mathbf{Z})$. The action induces a family of diffeomorphism on fibers $h_t : \Xi_a \to \Xi_a(e^{it})$, where

$$h_t(z_1, \ldots, z_n) = (\omega_1^t z_1, \ldots, \omega_n^t z_n).$$

In particular, $h_{2\pi} : \Xi_a \to \Xi_a$ by $h_{2\pi}(z_1, \ldots, z_n) = (\omega_1 z_1, \ldots, \omega_n z_n)$. This induces the linear map $\omega = \prod_{k=1}^n \omega_k$ on $H_{n-1}(\Xi_a, \mathbf{Z})$. Denote the characteristic polynomial of the linear map ω by $\Delta_a(t)$.

Lemma 2.4.

$$\Delta_a(t) = \prod_{0<i_k<a_k} (t - \xi_1^{i_1} \cdots \xi_n^{i_n}), \text{ where } \xi_k := e^{2\pi i/a_k}.$$

Proof. We can regard J_a/I_a as a tensor product $\otimes_{k=1}^n V_k$, where V_k is a $\mathbf{Z}-$module spanned by ω_k^i. Then ω can be regarded as $\omega_1 \otimes \cdots \omega_n$. For every a_k-th root of unity $x_k := \xi_k^{i_k}, 0 < i_k < a_k$, the vector

$$\sum_{r=0}^{a_k-1} x_k^r \omega_k^r \in V_k \otimes \mathbf{C}$$

is an eigenvector of ω_k with eigenvalue x_k^{-1}. Therefore,

$$\prod_{k=1}^{n} \sum_{r=0}^{a_k-1} x_k^r \omega_k^r \in J_a/I_a \otimes \mathbf{C}$$

is an eigenvector of ω with eigenvalue $\xi_1^{-i_1} \cdots \xi_n^{-i_n}$. By calculating the dimension, we can see that all eigenvectors are of this form, so

$$\Delta_a(t) = \prod_{0<i_k<a_k} (t - \xi_1^{i_1} \cdots \xi_n^{i_n}).$$

□

We have the exact sequence (see [9], p. 67)

$$\cdots \to H_{k-1}(\Xi_a, \mathbf{Z}) \stackrel{\omega-1}{\to} H_{k-1}(\Xi_a, \mathbf{Z}) \to H_{k-1}(Y_a, \mathbf{Z}) \to \cdots .$$

Therefore, $H_i(Y_a, \mathbf{Z})$ vanishes when $i \neq 0, 1, n-1, n$ and it vanishes for $i = n-1, n$ if and only if $1 - \omega$ is an isomorphism, that is

$$\Delta_a(1) = \det(1 - \omega) = \pm 1.$$

Lemma 2.5. For $n \geq 4$, Σ_a is at least $(n-3)-$connected.

Proof. Σ_a is a deformation retract of $X_a - \{0\}$. Denote $X_{\hat{a}} = \{z \in X_a | z_n = 0\}$. The inclusion $X_a - X_{\hat{a}} \hookrightarrow X_a - \{0\}$ induces the surjection $\pi_1(X - X_{\hat{a}}) \to \pi_1(X_a - \{0\})$. Define $q : X_a - X_{\hat{a}} \to \mathbf{C}^*$ by $q(z) = z_n$, which is a fibration with fiber $\Xi_a(-z_n^{a_n})$. By Lemma 2.3 and the long exact sequence of homotopy group

$$0 = \pi_1(\Xi_a) \to \pi_1(X_a - X_{\hat{a}}) \to \pi_1(C^*) = \mathbf{Z} \to \pi_0(\Xi_a) = 0,$$

we have $\pi_1(X_a - X_{\hat{a}}) = \mathbf{Z}$. Therefore, $\pi_1(\Sigma_a) = \pi_1(X_a - \{0\})$ is abelian. By the previous argument on $H_i(Y_a, \mathbf{Z})$ and Hurewicz's theorem,

$$\pi_1(\Sigma_a) \simeq H_1(\Sigma_a, \mathbf{Z}) \simeq H_2(Y_a, \mathbf{Z}) = 0$$

and for $i \leq n-3$,

$$\pi_i(\Sigma_a) \simeq H_i(\Sigma_a, \mathbf{Z}) \simeq H_{i+1}(Y_a, \mathbf{Z}) = 0$$

□

Next, we formulate a condition of a component $K \subset G_a$:

K consists of an odd number of points and $\gcd(a_i, a_j) = 2$ for every $a_i, a_j \in K$. $(*)$

Theorem 2.6. If $n > 3$ and $a_k > 1$ for every k, then the following are equivalent

(1) Σ_a is a topological sphere

(2) $\Delta_a(1) = 1$

(3) G_a fulfills one of the following conditions

  (a) G_a has at least two isolated points

  (b) G_a has one isolated point and there exists at least one K satisfying $(*)$.

Proof. (1)⇔(2): By Lemma 2.5, Σ_a is simply connected. By Theorem 1.7, Σ_a is a topological sphere if and only if Σ_a has the homology group of S^{2n-3}. Since $H_{i-1}(\Sigma_a, \mathbf{Z}) = H_i(Y_a, \mathbf{Z}) = 0$ for $i \neq 0, 1, n-1, n$, it suffices to show that $H_{n-1}(Y_a, \mathbf{Z}) = H_n(Y_a, \mathbf{Z}) = 0$. From the previous argument, $H_{n-1}(Y_a, \mathbf{Z}) = H_n(Y_a, \mathbf{Z}) = 0$ if and only if $\Delta_a(1) = \pm 1$. From Lemma 2.4, $\Delta_a(1)$ must be 1.
(2)⇔(3): Also from Lemma 2.4,

$$\Delta_a(t) = \prod_d \Phi_d(t),$$

where $\Phi_d(t)$ is the cyclotomic polynomial and d runs through orders of $\xi_1^{i_1} \cdots \xi_n^{i_n}$. Note that $\Phi_{q^m}(1) = q$ for every prime q and $\Phi_d(1) = 1$ if d is not a prime power. This implies $\Delta_a(1) = 1$ if and only if for every $i = (i_1, \ldots, i_n)$ with $0 < i_k < a_k$, the order of $\xi_1^{i_1} \ldots \xi_n^{i_n}$ is not a prime power.
Let K is a component of G_a. For convenience, write $K = \{a_1, \ldots, a_r\}$. Let

$$\kappa(K) := |\{(i_1, \ldots, i_r) | 0 < i_k < a_k, \xi_1^{i_1} \cdots \xi_r^{i_r} = 1\}|.$$

We see that $\kappa(K) = 0$ if and only if K is an isolated point or K satisfies the condition $(*)$. Moreover, there exists at least two components K_1 and K_2, with $\kappa(K_1) = \kappa(K_2) = 0$ if and only if the order of $\xi_1^{i_1} \ldots \xi_n^{i_n}$ is not a prime power for every $i = (i_1, \ldots, i_n)$ with

$0 < i_k < a_k$. Thus, (2) and (3) are equivalent. $\square$

Next, define

$$\Sigma_a(t) = \Xi_a(t) \cap S^{2n-1} \text{ and } M_a(t) = \Xi_a(t) \cap D^{2n},$$

where $D^{2n} = \{z \in \mathbf{C}^n | |z| \leq 1\}$.
When $|t|$ is small enough, $M_a(t)$ is a differentiable manifold with boundary $\Sigma_a(t)$. Furthermore, when $|t|$ is small enough, $M_a(t)$ has a trivial normal bundle in $\mathbf{C}^n$. Then by Theorem 1.11, $M_a(t)$ is parallelizable. Lastly, from [3] (Also, see the lecture notes in class, Problem 5.16), we have $\Sigma(a_1, \ldots, a_n)$ is diffeomorphic to $\Sigma_a(t)$ when $|t|$ is small enough.
From now on, we choose a small enough t_0 such that all above statements are satisfied, and denote $M_a = M_a(t_0)$. Then we can summarize that

Lemma 2.7. *M_a is a bounded parallelizable manifold, where ∂M_a is diffeomorphic to Σ_a. $M_a - \partial M_a$ is diffeomorphic to Ξ_a.*

Theorem 2.8. *For odd $n \geq 5$, let $\Sigma(a_1, \ldots, a_n)$ be a topological sphere. Then the diffeomorphism type of Σ_a is determined by $\sigma(M_a)$. We have*

$$\sigma(M_a) = \sigma_a^+ - \sigma_a^-,$$

where σ_a^+ = the number of n-tuples of integers

$$j = (j_1, \ldots, j_n), \quad 0 < j_k < a_k$$

with

$$0 < \sum_{k=1}^{n} \frac{j_k}{a_k} < 1 \pmod 2$$

and σ_a^- = the number of n-tuples of integers $(j_1, \ldots, j_n)$ with

$$-1 < \sum_{k=1}^{n} \frac{j_k}{a_k} < 0 \pmod 2.$$

Proof. Use the same notation as in Lemma 2.4. Let

$$v_i = \prod_{k=1}^{n} \sum_{r=0}^{a_k-1} x_k^r \omega_k^r \text{ and } v_j = \prod_{k=1}^{n} \sum_{r=0}^{a_k-1} y_k^r \omega_k^r,$$

where $x_k = \xi_k^{i_k}$ and $y_k = \xi_k^{j_k}$, be eigenvectors on $H_{n-1}(\Xi_a, \mathbf{C}) = J_a/I_a \otimes \mathbf{C}$. Then the intersection number

$$\langle v_i, v_j \rangle = (-1)^{(n-1)(n-2)/2}(1 - x_1 \cdots x_n) \prod_k (1 - x_k^{-1}) \prod_k (1 + x_k y_k + \cdots + (x_k y_k)^{a_k - 1}).$$

This implies $\langle v_i, v_j \rangle \neq 0$ if and only if $i_k + j_k = a_k$ for every k. Therefore, $v_j + v_{a-j}$ and $i(v_j - v_{a-j})$ form an orthogonal basis for $J_a/I_a \otimes \mathbf{R}$, where

$$\langle v_j + v_{a-j}, v_j + v_{a-j} \rangle = \langle i\,(v_j - v_{a-j})\,, i\,(v_j - v_{a-j}) \rangle = 2\,\langle v_j, v_{a-j} \rangle$$

It remains to prove that $\langle v_j, v_{a-j} \rangle > 0$ if and only if $0 < \sum_{k=1}^n \frac{j_k}{a_k} < 1 \pmod 2$, and $\langle v_j, v_{a-j} \rangle < 0$ if and only if $-1 < \sum_{k=1}^n \frac{j_k}{a_k} < 0 \pmod 2$.

$$\begin{aligned} \left(\prod a_k^{-1}\right) \langle v_j, v_{a-j} \rangle &= (-1)^{\frac{n-1}{2}} \left(\prod \left(1 - x_k^{-1}\right) + \prod \left(1 - x_k\right)\right) \\ &= 2\,\mathrm{Re}(-1)^{\frac{n-1}{2}} \prod \left(1 - x_k\right) \\ &= 2\,\mathrm{Re}(-1)^{\frac{n-1}{2}} \prod \left(-2ie^{\pi i \frac{j_k}{a_k}} \sin \pi \frac{j_k}{a_k}\right) \\ &= 2\,\mathrm{Re}\left(-e^{\pi i \left(\frac{1}{2} + \Sigma \frac{j_k}{a_k}\right)} \prod 2 \sin \pi \frac{j_k}{a_k}\right) \end{aligned}$$

Since $\sin \pi \frac{j_k}{a_k}$ is always positive, by discussing the exponential term, the result follows. $\square$

Corollary 2.9. Take $n = 5$ and $a = (3, 6k-1, 2, 2, 2)$, $\Sigma_a(t_0)$ is a topological 7 sphere and

$$\sigma(M_a) = 8k.$$

Proof. The first statement follows from Theorem 2.6. We compute the $\sigma(M_a)$ by Theorem 2.8.

Note that $j_1 = 1$ or 2 and $j_3 = j_4 = j_5 = 1$. Then

$$\sum_{k=1}^{5} \frac{j_k}{a_k} = \frac{j_1}{3} + \frac{j_2}{6k-1} + \frac{3}{2}.$$

If $j_2 = 1$,

$$\sum_{k=1}^{5} \frac{j_k}{a_k} = \frac{1}{3} + \frac{j_2}{6k-1} + \frac{3}{2} \equiv \frac{j_2}{6k-1} - \frac{1}{6} \pmod 2.$$

Then $0 < \sum_{k=1}^{5} \frac{j_k}{a_k} < 1 \pmod 2$ if and only if $k \leq j_2 \leq 6k-2$ and $-1 < \sum_{k=1}^{5} \frac{j_k}{a_k} < 0 \pmod 2$ if and only if $1 \leq j_2 \leq k-1$.

If $j_2 = 2$,

$$\sum_{k=1}^{5} \frac{j_k}{a_k} = \frac{2}{3} + \frac{j_2}{6k-1} + \frac{3}{2} \equiv \frac{j_2}{6k-1} + \frac{1}{6} \pmod 2.$$

Then $0 < \sum_{k=1}^{5} \frac{j_k}{a_k} < 1 \pmod 2$ if and only if $1 \leq j_2 \leq 5k-1$ and $-1 < \sum_{k=1}^{5} \frac{j_k}{a_k} < 0 \pmod 2$ if and only if $5k-1 \leq j_2 \leq 6k-2$.

We conclude that $\sigma_a^+ = 10k-2$, $\sigma_a^- = 2k-2$ and $\sigma(M_a) = 8k$. □

We conclude that $\Sigma_a(t_0)$ with $a = (3, 6k-1, 2, 2, 2)$ are topological $7-$sphere which bounds parallizable manifolds M_a. Apply Theorem 1.17, where $\sigma_2 = 224$, $\Sigma_a(t_0)$ with $a = (3, 6k-1, 2, 2, 2), k = 1, \ldots, 28$ are all differential structures on S^7.

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Kazdan-Warner Problem for Surfaces

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1 Introduction

Main Problem. (Kazdan-Warner) Given (M, g) a compact Riemmanian manifold without boundaries, and a function $\widetilde{R} \in C^\infty(M)$, is there any comformal metric $\widetilde{g} = e^{2u}g$, such that the scalar curvature with respect to $\widetilde{g}$ is exactly $\widetilde{R}$?

In this note, we deal with the case of surfaces ($n = \dim M = 2$)

Observation. In the case $\dim M = 2$, we have $R = 2K$ and $\widetilde{R} = 2\widetilde{K}$, where K is the sectional curvature (Gaussian curvature). By Gauss-Bonnet theorem,

$$\int_M K d\mu = \int_M e^{2u}\widetilde{K} d\mu = 2\pi\chi(M),$$

thus we have the necessary condition:

1. If $\chi(M) < 0$, then $\widetilde{K}$ is negative at some point.
2. If $\chi(M) = 0$, then either $\widetilde{K} \equiv 0$ or $\widetilde{K}$ changes sign.
3. If $\chi(M) > 0$, then $\widetilde{K}$ is positive at some point.

However these conditions are in general not sufficient.

Conformal changes of curvatures. Under local coordinate, we have

$$
\begin{aligned}
\widetilde{\Gamma}_{ij}^{k} &= \frac{1}{2}\widetilde{g}^{kl}\left(\frac{\partial \widetilde{g}_{il}}{\partial x^{j}} + \frac{\partial \widetilde{g}_{jl}}{\partial x^{i}} - \frac{\partial \widetilde{g}_{ij}}{\partial x^{l}}\right) \\
&= \Gamma_{ij}^{k} + \frac{1}{2}\left(\delta_{ik}\frac{\partial u}{\partial x^{j}} + \delta_{jk}\frac{\partial u}{\partial x^{i}} - g_{ij}g^{kl}\frac{\partial u}{\partial x^{l}}\right), \\
\widetilde{R}_{ij} &= \frac{\partial \widetilde{\Gamma}_{ij}^{t}}{\partial x^{t}} - \frac{\partial \widetilde{\Gamma}_{it}^{t}}{\partial x^{j}} + \widetilde{\Gamma}_{ij}^{s}\widetilde{\Gamma}_{st}^{t} - \widetilde{\Gamma}_{it}^{s}\widetilde{\Gamma}_{sj}^{t} \\
&= R_{ij} - (\triangle u)g_{ij}, \qquad (n=2) \\
\widetilde{R} &= \widetilde{g}^{ij}\widetilde{R}_{ij} = e^{-2u}(R - 2\triangle u)
\end{aligned}
$$

Thus Kazdan-Warner problem for surfaces is equivalent to solving the following PDE on M.

$$\triangle u - K + e^{2u}\widetilde{K} = 0. \tag{1}$$

2 Case I: $\chi(M) < 0$

We may try to study this case by sub- and sup- solution method.

Proposition 2.1. On a compact Riemannian manifold (M, g), consider the semilinear PDE $\triangle u + f(x, u) = 0$, where $f \in C^{\infty}(M \times \mathbb{R})$. If there exists $\phi, \psi \in C^{2}(M)$, such that $\phi \leq \psi$ and

$$\triangle \phi + f(x, \phi) \geq 0, \qquad \triangle \psi + f(x, \psi) \leq 0,$$

(We call ϕ and ψ a sub-solution and a sup-solution for the PDE respectively.) Then there exists $u \in C^{\infty}$ s.t. $\phi \leq \psi$ and (1) holds.

Proof. Since M is compact, we can choose $A, c > 0$ such that $-A < \phi \leq \psi < A$ and $ct + f(x, t)$ increasing in $t \in [-A, A]$. We rewrite (1) into

$$Lu = F(x, u),$$

where L is an elliptic operator defined by $Lu \triangleq -\triangle u + cu$, and $F(x, u) \triangleq ct + f(x, t)$. By maximum principle (looking at the minimum), we can see that L is a "positive" operator, in the sense $Lu \geq 0 \Rightarrow u \geq 0$, or equivalently, $Lu_1 \geq Lu_2 \Rightarrow u_1 \geq u_2$.

On the other hand, we have the following result of Schauder estimate:

Proposition 2.2. As an operator $L : C^{2,\alpha} \to C^{0,\alpha}$ between Holder spaces ($\alpha \in (0,1)$), L has a compact inverse L^{-1}.

Consider two sequences of sub- and sup- solutions $\{\phi_k\}, \{\psi_k\}$ given by

$$\phi_0 = \phi, \qquad \phi_{k+1} = L^{-1}(F(x, \phi_k))$$
$$\psi_0 = \psi, \qquad \psi_{k+1} = L^{-1}(F(x, \psi_k))$$

We inductively show that $\phi < \phi_1 < \phi_2 < \cdots < \psi_2 < \psi_1 < \psi$. First, notice that $L\phi_{k+1} = F(x, \phi_k) \geq L\phi_k$, so $\phi_k \leq \phi_{k+1}$ by positiveness of L. Now if $\phi_k \leq \psi_k$, then

$$L\phi_{k+1} = F(x, \phi_k) \leq F(x, \psi_k) = L\psi_{k+1},$$

thus $\phi_{k+1} \leq \psi_{k+1}$ again by positiveness of L.

Thus we have pointwise convergence

$$\{\phi_k\} \to \underline{u}, \{\psi_k\} \to \overline{u}, \phi \leq \underline{u} \leq \overline{u} \leq \psi.$$

Since $\phi_k, \psi_k, L\phi_k, L\psi_k$ are all bounded, using the following L^p-estimate, we can show that $\{\phi_k\}, \{\psi_k\}$ are bounded in the Sobolev space $W^{2,p}$ for $p > n$:

Proposition 2.3. For $u \in W^{2,p}(M)$ with $p > 1$, we have the following inequality[1]:

$$\|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|\triangle u\|_{L^p})$$

for some constant $C > 0$.

Hence by Sobolev embedding theorem[2], $\{\phi_k\}, \{\psi_k\}$ are also bounded in $C^{1,\alpha}$ with $\alpha = 1 - \frac{n}{p}$. As a result, $\{\phi_k\}, \{\psi_k\}$ converges in $C^{1,\alpha}$ to $\underline{u}, \overline{u}$ respectively, and

$$L\underline{u} = F(x, \underline{u}), L\overline{u} = F(x, \overline{u}),$$

and then by elliptic regularity theorem, $\underline{u}, \overline{u}$ are smooth solutions to (1).

[1]Theorem 9.13 of [2]
[2]Theorem 7.26 of [2]

Now we may apply the method to the problem ($n = \dim M = 2$)

Corollary 2.4. In the case of $\chi(M) < 0$, if (1) has a sup-solution ψ in C^2, then it has a smooth solution.

Proof. We only have to show that there is also a sub-solution ϕ such that $\phi \le \psi$. Choose an $f \in C^\infty(M)$ such that

$$\triangle f = K - K_0,$$

where $K_0 = \int K d\mu / \int d\mu$ is the mean value of K, which is negative by Gauss-Bonnet theorem. Note that $\int (K - K_0) d\mu = 0$, so such f exists by Hodge decomposition. Now we consider $\phi = f - c$ for some constant $c > 0$. We have

$$\triangle \phi - K + \tilde{K} e^{2\phi} = -K_0 + \tilde{K} e^{2f-2c} > 0$$

for c sufficient large, and we also take c large enough so that $f - c \le \psi$, then hence we find the ϕ we want.

Theorem 2.5. In the case $\chi(M) < 0$, if $\widetilde{K} \le 0$, then (1) has a smooth solution.

Proof. As previous, we only have to show that there is also a sup-solution ψ for (1). Choose an $f \in C^\infty(M)$ such that

$$\triangle f = \widetilde{K} - \widetilde{K}_0,$$

where $\widetilde{K}_0 = \int \widetilde{K} d\mu / \int d\mu < 0$. Consider $\psi = -af + b$ for some constant $a, b > 0$. We have

$$\triangle \psi - K + \tilde{K} e^{2\psi} = (a\widetilde{K}_0 - K) + (e^{-2af+2b} - a)\widetilde{K} < 0$$

for a, b sufficient large, so we find the ψ we want.

For general $\widetilde{K}$, the problem remains unsolved.

3 Case II: $\chi(M) = 0$

For the case $\chi(M) = 0$, we have a complete criterion for $\widetilde{K}$ that (1) has a smooth solution:

Theorem 3.1. For the case $\chi(M) < 0$, (1) has smooth solution if and only if one of the following condition holds:

(1) $\widetilde{K} \equiv 0$, or

(2) $\widetilde{K}$ changes sign and $\int \widetilde{K}e^{2f}d\mu < 0$,

where $f \in C^\infty(M)$ is a solution of $\triangle f = K$. (Note that such f exists since $\int Kd\mu = 2\pi\chi(M) = 0$.)

Proof. Necessity. Suppose u is a solution of (1), then consider $v = u - f$, then

$$\triangle v = \triangle u - K = -e^{2v+2f}\widetilde{K} \tag{2}$$

Times e^{2v} on both side and integrate, we obtain

$$\int \widetilde{K}e^{2f}d\mu = -\int e^{-2v}\triangle v = -\int 2e^{-2v}|\nabla v|^2 d\mu \leq 0$$

by Green's identity. If the equality holds, then $\nabla v \equiv 0$, so v must be a constant, which implies $\widetilde{K} \equiv 0$ by (2). Thus if $\widetilde{K} \not\equiv 0$, then we must have $\int \widetilde{K}e^{2f}d\mu < 0$, and also $\widetilde{K}$ have to change sign by Gauss-Bonnet theorem.

Sufficiency. If $\widetilde{K} \equiv 0$, then obviously we can take $u = f$, so we only have to deal with the second case. Consider the subset of $W^{2,p}(M)$,

$$\mathcal{S} \triangleq \{u \in W^{2,p}(M), \int ud\mu = \int \widetilde{K}e^{2u+2f}d\mu = 0\}$$

It is nonempty since $\widetilde{K}$ changes sign. We want to minimize in $\mathcal{S}$ the Dirichlet energy

$$J(u) \triangleq \frac{1}{2}\int |\nabla u|^2 d\mu$$

Suppose there is a minimizer $u_0 \in \mathcal{S}$, then by the theory of Lagrange multiplier, u_0 is an emtreme of

$$\int \frac{1}{2}|\nabla u|^2 + \alpha u + \beta\widetilde{K}e^{2u+2f}.$$

Compute the variation in u gives

$$\triangle u_0 + \alpha + 2\beta \widetilde{K} e^{2u_0+2f} = 0.$$

Integrating over M gives

$$\alpha A = -2\beta \int \widetilde{K} e^{2u_0+2f} d\mu = 0,$$

where A is the area of M. Thus $e^{-2u_0}\triangle u_0 + 2\beta\widetilde{K}e^{2f} = 0$. Again integrating over M gives

$$\begin{aligned} 2\beta \int \widetilde{K} e^{2f} d\mu &= \int e^{-2u_0} \triangle u_0 d\mu. \\ &= -2 \int e^{-2u_0} |\nabla u_0|^2 \leq 0 \end{aligned}$$

which implies $\beta > 0$ by the condition.
Thus we have $v_0 = u_0 + \frac{1}{2}\log 2\beta$ is a weak solution of (2), i.e., $u = u_0 + \frac{1}{2}\log 2\beta + f$ is a weak solution of (1). By elliptic regularity, if we can show that $e^u \in L^p(M)$ for all $p > 1$, then u is a smooth solution. To do this we need the following Lemmas.

Lemma 3.2 (Trudinger)**.** For any compact Riemannian surface (M, g), there exists $\beta, C > 0$ such that any $u \in W^{1,2}$ satisfying

$$\int u d\mu = 0, \int |\nabla u|^2 d\mu \leq 1$$

has $\int e^{\beta u^2} d\mu < C$.

Proof. We fix a partition of unity $\{(U_i, \phi_i)\}_{i=1}^k$ with each U_i diffeomorphic to a 2 dimensional Euclidean disc D, and let $u_i = \phi_i u$, thus $u = \sum_i u_i$.

We first prove that on each $U_i \cong D$ with standard Euclidean metric, there exists $c_0 > 0$ such that

$$\|u_i\|_p \leq c_0 \sqrt{p} \|\nabla u_i\|_2, \qquad \forall p \geq 2.$$

For $v \in C_c^1(D)$, we have the following identity,

$$
\begin{aligned}
v(x) &= \frac{1}{2\pi}\int_D \triangle v(y) \cdot \log(|x-y|)dy \\
&= \frac{1}{2\pi}\int_D \nabla v(y) \cdot \frac{x-y}{|x-y|^2}dy
\end{aligned}
$$

By Holder inequality ($\frac{1}{p} + \frac{1}{q} = 1$),

$$
\begin{aligned}
|v(x)| &\leq \frac{1}{2\pi}\int_D \left(|\nabla v(y)|^2 |x-y|^{-q}\right)^{\frac{1}{p}} |x-y|^{-\frac{q}{2}} |\nabla v(y)|^{1-\frac{2}{p}} dy \\
&\leq \frac{1}{2\pi}\left(\int_D |\nabla v(y)|^2 |x-y|^{-q} dy\right)^{\frac{1}{p}} \left(\int_D |x-y|^{-q} dy\right)^{\frac{1}{2}} \left(\int_D |\nabla v(y)|^2 dy\right)^{\frac{1}{2}-\frac{1}{p}}
\end{aligned}
$$

the middle term is controlled by

$$
\int_D |x-y|^{-q} dy \leq \int_{2D} |y|^{-q} dy = 2^{1-q}\pi(p+2).
$$

Taking pth power and integrate over x and we have

$$
\begin{aligned}
&\int_D |v(x)|^p dx \leq c_1 (p+2)^{\frac{p}{2}+1} \left(\int_D |\nabla v(y)|^2 dy\right)^{\frac{p}{2}} \\
\Rightarrow &\left(\int_D |v(x)|^p dx\right)^{\frac{1}{p}} \leq c_2 \sqrt{p} \left(\int_D |\nabla v(y)|^2 dy\right)^{\frac{1}{2}}
\end{aligned}
$$

since C_c^1 is dense in $W^{1,2}$, we have on each (U_i, g),

$$
\|u_i\|_p \leq c_0 \sqrt{p} \|\nabla u_i\|_2,
$$

for some c_0 depend on g.

Now we have

$$
\begin{aligned}
\|u\|_p &\leq \sum \|u_i\|_p \leq c_0\sqrt{p} \sum \|\nabla u_i\|_2 \\
&\leq c_3 \sqrt{p} (\|\nabla u\|_2 + \|u\|_2)
\end{aligned}
$$

by equivalence of sobolev norms. We further use the following estimates:

Proposition 3.3 (Poincare-Wirtinger)**.** For any $u \in W^{1,2}(M)$, there exists $C > 0$ such that

$$\|u - u_0\|_2 \le C\|\nabla u\|_2,$$

where $u_0 = \int u d\mu / A$ is the average of u.

Since we have $u_0 = \int u d\mu = 0$, we obtain

$$\|u\|_p \le c_4 \sqrt{p} \|\nabla u\|_2$$

Now consider the Taylor expansion of $e^{\beta u^2}$, with $\|\nabla u\| \le 1$, we have

$$\begin{aligned} \int e^{\beta u^2} d\mu &= \int \sum_{k=1} \frac{1}{k!} (\beta |u|^2)^k + A \\ &\le \sum_{k=1} \frac{1}{k!} (2k\beta c_4^2)^k + A \le C \end{aligned}$$

for β sufficiently small (Take $p = 2k$.)

Lemma 3.4. There exists $C, \eta > 0$ such that for any $u \in W^{1,2}$,

$$\int e^u d\mu \le C \exp(\eta \|\nabla u\|_2^2 + \frac{1}{A} \int u d\mu)$$

Proof. We may assume $\nabla u \not\equiv 0$, and let $u_0 = u - \frac{1}{A} \int u d\mu$. We write

$$u_0 \le \beta \left(\frac{u_0}{\|\nabla u\|_2} \right)^2 + \frac{1}{4\beta} \|\nabla u\|_2^2,$$

where β is the constant appeared in the last lemma. Note that $\dfrac{u_0}{\|\nabla u\|_2}$ satisfies the condition of the last lemma, thus taking exponent and integrating over M gives

$$\int e^{u_0} d\mu \le C \exp(\frac{1}{4\beta} \|\nabla u\|_2^2),$$

and hence the original statement. (Take $\eta = \dfrac{1}{4\beta}$.)

Lemma 3.5. For $u \in W^{1,2}(M)$, we have $e^u \in L^p$ for all $p > 1$.

Proof. Replace u by pu in the above lemma.

We still need to show the existence of the minimizer of J. Suppose $\{u_k\}$ is a minimizing sequence of J in $\mathcal{S}$, i.e.,

$$J(u_i) \to c_0 = \inf_{u \in S} J(u)$$

Then by Poincare-Wirtinger inequality we have

$$\|u_i\|_2 \leq C\|\nabla u_i\|_2 = 2CJ(u_i)^{\frac{1}{2}}$$

So $\{u_i\}$ is bounded in $W^{1,2}$, and there is a subsequence $\{u_i\}$ weakly converging to some $u_0 \in W^{1,2}$. We have $J(u_0) \leq c_0$ by the weak semi-lower-continuity of J. On the other hand, we can show that $\int \widetilde{K} e^{2u} d\mu$ is a continuous functional of u in $W^{1,2}$-weak topology, and hence

$$\int \widetilde{K} e^{2u_0+2f} d\mu = \lim_{i \to \infty} \int \widetilde{K} e^{2u_i+2f} d\mu = 0$$

and also $\int u_0 d\mu = 0$, hence $u_0 \in \mathcal{S}$ and $J(u_0) \geq c_0$ by the definition of c_0. As a consequence, u_0 is a minimizer of J in $\mathcal{S}$, and we complete the proof of the theorem.

Proof of the continuity of $\int \widetilde{K} e^{2u} d\mu$. Suppose $\{u_i\}$ weakly converges to u, then by Rellich–Kondrachov theorem, they are strongly converge to u in L^p for $p > 1$, thus

$$\int \widetilde{K}(e^{u_i} - e^u) d\mu = \int_0^1 \int (u_i - u)\widetilde{K} e^{u+t(u_i-u)} d\mu dt \to 0$$

as $i \to 0$, by Lemma 3.4. and Holder's inequality.

4 Case III: $\chi(M) > 0$

Only partial results are known for $\chi(M) > 0$. In this case, M must be diffeomorphic to S^2 or $\mathbb{RP}^2$. We first consider the case $M = S^2$ with the standard 2-sphere metric g ($K \equiv 1$).

Proposition 4.1. On (S^2, g), If ϕ is a first eigenfunction of $\triangle$, i.e.,

$$\triangle\phi + 2\phi = 0,$$

then any solution u of (1) must satisfies

$$\int (\nabla \widetilde{K} \cdot \nabla\phi) e^{2u} d\mu = 0$$

Note that ϕ can be seen as a linear function on $\mathbb{R}^3$ restricted to S^2.

Proof. Multiply (1) by $\nabla u \cdot \nabla \phi$ and integrate over S^2, we have

$$\int (\nabla u \cdot \nabla\phi) \triangle u d\mu - \int (\nabla u \cdot \nabla\phi) d\mu + \int (\nabla u \cdot \nabla\phi) \widetilde{K} e^{2u} d\mu = 0$$

we deal with these three terms respectively. Note that $\phi_{,ij} = -\phi g_{ij}$, so we have

$$\begin{aligned}
\int (\nabla u \cdot \nabla\phi) \triangle u d\mu &= -\int \nabla(\nabla u \cdot \nabla\phi) \cdot \nabla u d\mu \\
&= -\frac{1}{2} \int \nabla(|\nabla u|^2) \cdot \nabla\phi d\mu + \int |\nabla u|^2 \phi d\mu \\
&= \frac{1}{2} \int |\nabla u|^2 (\triangle\phi + 2\phi) d\mu = 0, \\
\int (\nabla u \cdot \nabla\phi) d\mu &= -\int \phi \triangle u d\mu = \int \phi(\widetilde{K} e^{2u} - 1) d\mu = \int \phi \widetilde{K} e^{2u} d\mu, \\
\int (\nabla u \cdot \nabla\phi) \widetilde{K} e^{2u} d\mu &= \frac{1}{2} \int (\nabla e^{2u} \cdot \nabla\phi) \widetilde{K} d\mu \\
&= -\frac{1}{2} \int (\nabla(\widetilde{K} e^{2u}) \cdot \nabla\phi) d\mu - \frac{1}{2} \int (\nabla \widetilde{K} \cdot \nabla\phi) e^{2u} d\mu \\
&= -\frac{1}{2} \int \widetilde{K} e^{2u} \triangle\phi d\mu - \frac{1}{2} \int (\nabla \widetilde{K} \cdot \nabla\phi) e^{2u} d\mu
\end{aligned}$$

summing up the three terms then the proposition follows.

With this, consider $\widetilde{K} = 1 + \epsilon\phi$ with $\epsilon > 0$ sufficinet small so that $\widetilde{K} > 0$, then the above proposition says

$$\epsilon \int |\nabla\phi|^2 e^{2u} d\mu = 0,$$

which implies $\nabla\phi = 0$, or ϕ is a constant, contradiction. This shows that even $\widetilde{K} > 0$ is not sufficient for (1) to have a solution.

Nevertheless, we have the following result:

Theorem 4.2. On (S^2, g), if $\widetilde{K}(x) = \widetilde{K}(-x)$ for all $x \in S^2$, and $\widetilde{K}$ is positive somewhere, then there is a solution $u \in C^\infty(S^2)$ such that $u(x) = u(-x)$ for all $x \in S^2$.

Thus for standard $\mathbb{RP}^2$, the condition that $\widetilde{K}$ is positive somewhere is necessary and sufficient.

We need the fact that in this case (S^2 and symmetry) we can actually take the constant η in Lemma 3.4 to be a number close to $1/32\pi$, that is, we can take $\beta = 8\pi - \varepsilon$ with small ε in Lemma 3.2. We first prove for $\beta = 4\pi - \varepsilon/2$ if we drop the condition $u(-x) = u(x)$.

The first step is to symmetrize u into a radially symmetric function $u^\#$ using the following Polya-Szego inequality [4].

Lemma 4.3 (Weil-Polya-Szego)**.** Let (M, g) be a Riemannian surface such that the sectional curvature of M is bounded from above by k, with $B \subset M$ an open subset diffeomorphic to $\mathbb{R}^2$ and with smooth boundary. For $u \in W^{1,2}(B)$ and $u^\#$ is a radially symmetric function on a geodesic ball $B^\#$ on $S = k^{-1/2}S^2$ (2-sphere of radius $k^{-1/2}$) monotone in latitude, such that $|B^\#| = |B|$ and $|(u^\#)^{-1}((t,\infty))| = |u^{-1}(t,\infty)|$ for any $t \in \mathbb{R}$, then we have $u^\# \in W^{1,2}(B^\#)$ and

$$\int_{B^\#} |\nabla u^\#(x)|^2 d\mu_S \leq \int_B |\nabla u(x)|^2 d\mu_M.$$

(and hence we can replace u with $u^\#$ if $M = S^2$ and $k = 1$.)

Now we rewrite things into spherical coordinate. Let θ, ϕ be the longitude and altitude of the 2-sphere, then the metric can be written as

$$ds^2 = d\phi^2 + \cos\theta \, d\theta^2,$$

and the area element is

$$d\mu = \cos\theta \, d\theta \, d\phi.$$

The conditions on u is then

$$\begin{aligned}\int |\nabla u|^2 d\mu &= \int (u_\theta^2 + \cos^{-2}\theta \, u_\phi^2) \cos\theta \, d\theta \, d\phi \\ &= 2\pi \int_{-\pi}^{\pi} u_\theta^2 \, \cos\theta \, d\theta \le 1, \\ \int u \, d\mu &= 2\pi \int u \cos\theta \, d\theta = 0.\end{aligned}$$

Parametrize θ by t such that $e^{-t/2} = \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$, and let

$$w(t) = (4\pi)^{1/2} u(\theta), \qquad \rho(t) = \frac{1}{4}\text{sech}^2 \frac{t}{2}$$

then the conditions become

$$\begin{aligned}\int |\nabla u|^2 d\mu &= \int_{-\infty}^{\infty} w'(t)^2 dt \le 1, \\ \int u \, d\mu &= \int_{-\infty}^{\infty} w(t)\rho(t) dt = 0,\end{aligned}$$

and we want a bound for

$$\int e^{(4\pi - \varepsilon/2)u^2} d\mu = 4\pi \int_{-\infty}^{\infty} e^{(1-\varepsilon/8\pi))^2 w(t)^2} \rho(t) dt.$$

Note that $\rho(t)$ has the properties

$$\rho(t) < C_0 e^{-|t|}, \qquad \int_{-\infty}^{\infty} \rho(t) dt = 1$$

for some constant C_0. By Cauchy's inequality we have

$$(w(r) - w(s))^2 = \left(\int_s^r w'(t) dt\right)^2 \le \left(\int_s^r w'(t)^2 dt\right) \left(\int_s^r 1 dt\right) \le |r - s|.$$

Write $(1 - \varepsilon/8\pi)^2 = 1 - \tau$, then

$$\int_{-\infty}^{\infty} e^{(1-\varepsilon/8\pi)^2 w(t)^2} \rho(t) dt \le C_0 e^{(1-\tau)C_1} \int_{-\infty}^{\infty} e^{-\tau|t|} dt = 2C_0 e^{(1-\tau)C_1} \tau^{-1}$$

if we have the following estimate

$$w(t)^2 < |t| + C_1$$

for some constant C_1. To show this inequality, we write

$$\rho(s)(w(t) - w(s)) \leq \rho(s)|(t-s)|^{1/2}$$

integrate over s and we have

$$w(t) \leq \int_{-\infty}^{\infty} \rho(s)|(t-s)|^{1/2}\, ds \leq (|t| + C_1)^{1/2}$$

for some $C_1 \geq (\int_{-\infty}^{\infty} \rho(s)|(t-s)|^{1/2}\, ds)^2 - |t|$ for all t.

Remark. In fact the boundedness is also true for $\beta = 4\pi$, but that requires a longer argument controlling the case when

$$\delta \triangleq 1 - \max_{r>s} (w(r) - w(s))^2 / |r - s|$$

is small.

We return to the case $\beta = 8\pi - \epsilon$ with $u(-x) = u(x)$. In Lemma 4.3, we take B to be a hemisphere on S^2, and $k = \sqrt{2}$, then the lemma implies

$$\int_S |\nabla u^{\#}(x)|^2 d\mu_S \leq \frac{1}{2} \int_{S^2} |\nabla u(x)|^2 d\mu_{S^2},$$

where $S = \dfrac{1}{\sqrt{2}} S^2$, and thus

$$\int_{S^2} |\nabla u^{\#}(\sqrt{2}x)|^2 d\mu_{S^2} \leq \frac{1}{2} \int_{S^2} |\nabla u(x)|^2 d\mu_{S^2}$$

if we scale $u^{\#}$ to the standard 2-sphere. Now we can run the same estimate for the function $\sqrt{2}u^{\#}(\sqrt{2}x)$, and we have $\int e^{(8\pi - \varepsilon)u^2} d\mu$ bounded.

Now consider the set $\mathcal{S} \triangleq \{u \in W^{1,2}(S^2), \int u d\mu = 0, \int \widetilde{K} e^{2u} d\mu > 0\}$. This is nonempty since $\widetilde{K}$ is positive somewhere. We want to minimize

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - 2\pi \log \int \widetilde{K} e^{2u} d\mu$$

Note that $c_0 \triangleq \inf_{u \in S} J(u) \geq \left(\frac{1}{2} - \frac{2\pi}{8\pi - \varepsilon}\right) \|\nabla u\|_2^2 + c_1 \geq -\infty$ by Lemma 3.4 and $\eta = \dfrac{1}{32\pi - 4\varepsilon}$. A similar argument of case II shows that J has a minimizer $u_0 \in S$. The theory of Lagrange multiplier and calculus of variation then gives

$$\triangle u_0 + \frac{4\pi \widetilde{K} e^{2u_0}}{\int \widetilde{K} e^{2u_0} d\mu} - \lambda = 0$$

for some multiplier λ. Integrate over S^2 gives $4\pi - 4\pi\lambda$, hence $\lambda = 1$ and we have $u = u_0 + \frac{1}{2}\log(\frac{1}{4\pi}\int \widetilde{K} e^{2u_0} d\mu)$ is a solution of (1).

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WITTEN'S PROOF OF THE POSITIVE ENERGY THEOREM

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Abstract. We mainly follow the presentation by Parker[4] and put an emphasis on computations left out (we omit those clearly presented in his paper) with a digression on the asymptotic behaviour of solutions of certain elliptic operators in $\mathbb{R}^n$[3], which is later used to estimate the boundary terms in the Bochner formula.

1. Introduction

The positive energy theorem, in mathematical terms, is a theorem concerning an inequality of certain integrals (known as energy E_l and momentum p_{lk} to physicists), and the flatness of an oriented 3-dim spacelike complete hypersurface M in 4-dim Lorentzian manifold N with signature $(-+++)$ when some asymptotic data and positivity of certain curvature are assumed. More precisely, given Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

we assume for an orthonormal basis $\{e_\alpha\}$ on M (where e_0 is a timelike vector normal to M and $\{e_1, e_2, e_3\}$ tangent to M), $T_{00} \geq 0$ and T_{0j} is a non-spacelike covector. In particular $T_{00} \geq (-T_{0j}T^{0j})^{1/2}$ where j ranges from 1 to 3 (known as dominant energy condition to physicists. Later we will see this corresponds to the positivity of certain curvature). The asymptotic data is to assume M the shape of a compact set K joining by a finite number of pieces $M_l \cong \mathbb{R}^3 - B$ where B is a contractible compact set, and the diffeomorphism being given has the specific properties that the metric on each piece only differ from standard metric on $\mathbb{R}^3$ by $g_{ij} = \delta_{ij} + a_{ij}$ with the asymptotic

$$a_{ij} = O(r^{-1}), \partial_k a_{ij} = O(r^{-2}), \text{and } \partial_l \partial_k a_{ij} = O(r^{-3})$$

also the second fundamental form of M shall be $h_{ij} = O(r^{-2})$ and $\partial_k h_{ij} = O(r^{-3})$.

Positive Energy Theorem. *Under these assumption, $E_l \geq |P_l|$ on each M_l, and M consists of only one M_l with N flat along M if $E_l = 0$ is further assumed.*

The integrals are defined as

$$E_l = \lim_{R\to\infty} \frac{1}{16\pi G} \int_{S_{R,l}} (\partial_j g_{ij} - \partial_i g_{jj}) d\Omega^i$$

$$p_{lk} = \lim_{R\to\infty} \frac{1}{16\pi G} \int_{S_{R,l}} 2(h_{ik} - \delta_{ik} h_{jj}) d\Omega^i$$

2. The Spinor of $Cl_{3,1}(\mathbb{R})$

The proof relies on what is called hypersurface Dirac operator in Witten's words, but before that we shall specify which spinor bundle on M we are working on. At the linear algebra level, recall that in class we use a complex structure J to obtain our spinor for $\mathbb{R}^n$ case through tensoring $\mathbb{C}$. There is not much difference here except we adapt the idea of isotropic space, where $V \oplus \overline{V}$ is changed to $W \oplus W'$ with the property that $g(w, w) = 0$ for w in W or W', and the Clifford action is given by restricting $\mathbb{C}$ back to $\mathbb{R}$. Explicitly we exhibit a basis

$$\begin{array}{lll} W & e_1 = \frac{1}{\sqrt{2}}(e_x + e_t) & e_2 = \frac{1}{\sqrt{2}}(e_y + ie_z) \\ W' & e_3 = \frac{1}{\sqrt{2}}(e_x - e_t) & e_4 = \frac{1}{\sqrt{2}}(e_y - ie_z) \end{array}$$

where $\{e_t, e_x, e_y, e_z\}$ is the standard basis for the Lorentzian metric. Now we set $\{1, e_1 \wedge e_2, e_1, e_2\}$ to be an ordered basis for the spinor, the real vector $te_t + xe_x + ye_y + ze_z$ in matrix form is

$$U = \begin{bmatrix} 0 & \tilde{A} \\ A & 0 \end{bmatrix} \text{ with } A = \begin{bmatrix} t - x & y + iz \\ y - iz & t + x \end{bmatrix}$$

where $\tilde{A} = A - \operatorname{tr}(A)I$ is called the time reversal matrix and $-\det(A)$ corresponds to the length of this vector under Lorentzian metric, and the universal covering $\text{Spin}(3,1) \to SO(3,1)$ is still the twisted adjoint[1] given in the textbook. For our purpose we shall now devise a hermitian inner product on the spinor, and it is better to be invariant under $\text{Spin}(1,3)$ (or equivalently U is hermitian under this inner product) so that when lifting the connection ∇, the inner product will be compatible with ∇. In matrix form above we let our inner product to be $(\phi, \psi) = \bar{\phi}^t B\psi$, then we require

$$BU = \bar{U}^t B \text{ so that } B = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Unfortunately this inner product is not positive definite, but recall that we are working on a spacelike hypersurface M so e_t is special among others. Specifically we let $\langle \phi, \psi \rangle = (e_t\phi, \psi)$ and associated with this choice of e_t is an inclusion of $\text{Spin}(3) \subseteq \text{Spin}(3,1)$. This inner product is positive definite and is invariant under matrix U above. Now the point is that instead of lifting the full $SO(3,1)$ frame bundle $F(N)$ we only lift $SO(3)$ to its $\text{Spin}(3)$, since we are essentially working on a spacelike hypersurface where only $SO(3)$ is required when patching up different local charts. From the theory of Stiefel-Whitney class, it is known that this patching is possible for oriented 3-dim manifold[2]. The connection ∇ in the ambient space N then also lifted to a connection on spinor bundle $\widetilde{i^*F(N)} \times_\rho S$ where $i^*F(N)$ is the patching of $F(N)$ only by $SO(3)$ and ρ is the spin representation of $\text{Spin}(3,1)$. As remark above, ∇ is compatible with the inner product $(,)$. As usual, there is an induced connection $\widehat{\nabla}$ on hypersurface so we can also lift this to be a connection on the spinor bundle. Fixing an orthonormal frame with e_0 normal to M

$$e^\alpha \omega_{\alpha j}(\partial_i) = e^k \widehat{\omega}_{kj}(\partial_i) + e^0 h_{ij}$$

[1]In fact $\text{Spin}^+(3,1) \cong SL(2,\mathbb{C})$ though we do not use this

[2]This seems to be an exercise in Milnor's book[2] although I do not have time to study through.

where ω are the connection forms, then under the lifting $e_i \wedge e_j \to -\frac{1}{2}e^i e^j$ we have

$$\nabla_i = \widehat{\nabla}_i - \frac{1}{2}h_{ij}e^0 e^j$$

and for similar reason $\langle , \rangle$ is compatible with $\widehat{\nabla}$.

3. The Hypersurface Dirac Operator

The Dirac operator we consider here is

$$\mathscr{D}\psi = \sum_{i=1}^{3} e^i \nabla_i \psi$$

the interesting idea of using ∇ is that it entails the second fundamental form, which shall tell us the flatness of the hypersurface. In the following, we will do our local calculation with $\widehat{\nabla}_i e_j|_p = 0$ and $\nabla_0 e_i|_p = 0$ where i, j range from 1 to 3. First notice that $\mathscr{D}$ is formal self-dual

$$\begin{aligned} d[\langle \phi, e^i\psi\rangle e_i \lrcorner \mu] &= [\langle \widehat{\nabla}_i \phi, e^i\psi\rangle + \langle \phi, e^i \widehat{\nabla}_i \psi\rangle] \\ &= [\langle \phi, e^i(\widehat{\nabla}_i - \frac{1}{2}h_{ij}e^0e^j)\psi\rangle - \langle e^i(\widehat{\nabla}_i - \frac{1}{2}h_{ij}e^0e^j)\phi, \psi\rangle]\mu \\ &= [\langle \phi, \mathscr{D}\psi\rangle - \langle \mathscr{D}\phi, \psi\rangle]\mu \end{aligned}$$

with $\mu = e^1 \wedge e^2 \wedge e^3$ the volume form. Now as computed in class

$$\mathscr{D}^2 = -\nabla_i\nabla_i - \frac{1}{8}R_{\alpha\beta ij}e^ie^je^\alpha e^\beta - h_{ij}e^ie^0\nabla_j$$

except there is an additional last term which arise naturally since $\nabla_i e^j|_p = -h_{ij}e^0$ as i, j only range from 1 to 3 (while α, β range from 0 to 3). Now notice the formal dual $\nabla_i^* = -\nabla_i - h_{ij}e^je^0$ (see Parker for details), so the last term combines with $-\nabla_i\nabla_i$ into

$$\mathscr{D}^2 = -\nabla^*\nabla + \mathscr{R}$$

a Bochner formula where $\mathscr{R}$ denotes the curvature terms. Later we shall inspect the integral form of this formula to obtain the postive energy theorem in the spirits of vanishing theorem. We have once computed $\mathscr{R}$ in an exercise for the full $R_{\alpha\beta\gamma\lambda}e^\alpha e^\beta e^\gamma e^\lambda$ case which gives $2R$, now i, j only range from 1 to 3 so in this case $\mathscr{R} = \frac{1}{4}(R - R_{\alpha\beta 0j}e^\alpha e^\beta e^0 e^j)$. First

$$\begin{aligned} R_{\alpha\beta 0j}e^\alpha e^\beta e^0 e^j = R_{0j\alpha\beta}e^\alpha e^\beta e^0 e^j &= 2R_{0jj\beta}e^je^\beta e^0 e^j \\ &= 2R_{0j\beta j}e^\beta e^0 = -2R_{00} - 2R_{0j}e^0e^j \end{aligned}$$

where the second equality comes from the 1st Bianchi identity as for $\{j, \alpha, \beta\}$ distinct those terms cancel out, while the third equality comes from $j \neq \{\beta, 0\}$. Miraculously, from Einstein equation $R_{0j} = 8\pi G T_{0j}$ and $R_{00} + \frac{1}{2}R = 8\pi G T_{00}$ we see the curvature term is exactly $\mathscr{R} = 4\pi G(T_{00} + T_{0j}e^0e^j)$ and by the dominant energy condition we have the positivity of curvature $\mathscr{R} \geq 0$.

4. Proof of the Positive Energy theorem

The proof relies on what is called a constant spinor on the asymptotic ends M_l. Remember our M_l is defined by a special choice of coordinates, we define the constant spinor ψ_{0l} to mean $\partial_i\psi_{0l} = 0$ under this special choice[3] of coordinates on the asymptotic end M_l. Now we first state an existence theorem where the positive energy theorem follows directly from this

Theorem 4.1. *Let ψ_{0l} be constant spinors on each M_l, then there exist a harmonic smooth spinor ψ (that is $\mathscr{D}\psi = 0$) on M with the asymptotic*

$$\lim_{r\to\infty} r^{1-\epsilon}|\psi - \psi_{0l}| = 0$$

for any $\epsilon > 0$ in each M_l and the formula

$$\int_M (\langle\nabla\psi, \nabla\psi\rangle + \langle\psi, \mathscr{R}\psi\rangle)\mu = 4\pi G \sum_{i=1}^{k} (E_l\langle\psi_{0l}, \psi_{0l}\rangle + \langle\psi_{0l}, p_{lj}dx^0dx^j\psi_{0l}\rangle)$$

where dx^j is the standard basis[4] *on each $M_l \cong \mathbb{R}^3 - B$ and dx^0 is from $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$.*

Proof of the Positive Energy Theorem. We choose our constant spinors $\psi_{0l} = 0$ on each end except M_m where we let ψ_{0m} to be the eigenvector of the matrix $P_{mj}dx^0dx^j$ with eigenvalue[5] $-|P|$. The formula then directly shows $E_m - |P_m| \geq 0$.

For the second claim, let $E_m = 0$ and notice $|P_m|$ as well. We choose our constant spinor $\{\psi_m^\alpha\}$ to form a basis in M_m while $\psi_l^\alpha = 0$ on all the other M_l, then we obtain smooth $\{\psi^\alpha\}$ with $\mathscr{D}\psi^\alpha = 0$ as stated in the above theorem. Now the RHS is identically zero so it follows $\nabla\psi^\alpha = 0$ as $\mathscr{R} \geq 0$. The asymptotic of ψ^α then shows $\psi^\alpha \to 0$ on each M_l except M_m, but this contradicts to the following elementary lemma unless there is only one end M_m.

Lemma 4.2. [6] *If $\nabla\psi = 0$ and $\lim_{\gamma\to\infty}\psi(\gamma) = 0$ along a path γ in M_l then $\psi = 0$ identically on all M.*

Proof. This follows from $|d|\psi|^2| \leq |h|\cdot|\psi|^2$ and $h = O(r^{-2})$ so that $|d\ln|\psi|| \leq Cr^{-2}$ and hence $|\psi(x)| \geq |\psi(x_0)|$ for $|x| \geq |x_0|$(see Parker for details). □

It remains to prove N is flat along M. From the lemma above we see ψ^α is linear independent everywhere (Let $\psi = \sum c_\alpha\psi^\alpha$ and apply the lemma) on M. Now $\nabla\psi^\alpha = 0$ shows $R_{\alpha\beta ij}e^\alpha e^\beta\psi^\alpha = 0$ and notice spinor representation is faithful (as $\mathrm{Spin}(3,1)$ is a covering map so their Lie algebra differential map must has zero kernel) and ψ^α is a basis so $R_{\alpha\beta ij} = 0$ (beware i, j only range from 1 to 3). Now at least $R = -2R_{0j0j}$ and $R_{00} = R_{0j0j}$ (be careful with g signature), then the Einstein equation shows $0 = R_{00} + \frac{1}{2}R = 8G\pi T_{00}$ so from $R_{\alpha\beta} = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)$ and the dominant energy condition $T_{00} \geq |T_{\alpha\beta}|$, $R_{\alpha\beta} = 0$ and hence all $R_{\alpha\beta\gamma\lambda} = 0$. □

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[3] In fact the spinor bundle on $\mathbb{R}^3 - B$ is trivial up to isomorphism.

[4] Notice dx^j as a Clifford action is constructed from the metric δ_{ij} while e^j is from $g_{ij} = \delta_{ij} + a_{ij}$, but as noted before we choose that from δ_{ij} though they are all isomorphic to the trivial one.

[5] As $(P_{mj}dx^0dx^j)^2 = |P|^2$ but the eigenvalue is not all $|P|$ as it is not a multiple of identity matrix as dx^0 is hermitian while dx^j is skew-hermitian.

[6] This is possibly the only place where complete M assumption is used

5. Proof of Theorem 4.1

From the nature of M we shall expect to divide our discussion into compact set K and M_l. As it is known that the Rellich lemma no longer hold on non-compact domain in general, we consider a modified Sobolev norm defined as the following

Definition 5.1. *For $p \geq 2$ and $\frac{1}{2} - \frac{3}{p} \leq \delta \leq 2 - \frac{3}{p}$, $s = 0$ or 1 we consider the space $\mathscr{H}_{s,\delta,p}$ the completion of C_0^∞ compact support smooth sections under the norm*[7]

$$\|\psi\|_{s,\delta,p} = s\|\sigma^{1+\delta}\widehat{\nabla}\psi\|_p + \|\sigma^\delta \psi\|_p$$

where $\|\cdot\|_p$ is the standard L^p norm

$$\|\psi\|_p = \left(\int_M \langle\psi,\psi\rangle^{p/2}\mu\right)^{1/p}$$

and $\sigma \geq 1$ is a smooth function with the properties

$$\sigma = r \text{ on the asymptotic end } M_{l,2R}$$
$$\sigma = 1 \text{ in } M - \cup_l M_{l,R}$$

where we identify M_l as $\mathbb{R}^3 - B$ so $M_{l,r} = M_l - B_r$ a ball $B \subset B_r$ of radius r.

Proof of Theorem 4.1. We consider a cutoff function $\beta_{l,R}$ for each M_l where $\beta_{l,R} = 1$ on $M_{l,3R}$ and 0 outside $M_{l,2R}$ and construct a spinor ψ_0 as

$$\psi_0 = \sum_{l=1}^{k} \beta_{l,R}\psi_{0l}$$

Now from the formula

$$\nabla_j\psi = \partial_j\psi - \frac{1}{4}\Gamma_{kjl}e^k e^l\psi - \frac{1}{2}h_{jk}e^0 e^k\psi$$

and $h = O(r^2)$, $\Gamma = O(r^{-2})$ we see $\mathscr{D}\psi_0 = O(r^{-2})$ (remember the constant spinor ψ_{0l} is defined to satisfy $\partial_j\psi_{0l} = 0$). We now need a technical result

Theorem 5.2. *For $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$ or $p = 2$ and $\delta = -1$ the operator $\mathscr{D}$ is an isomorphism between $\mathscr{H}_{1,\delta,p}$ and $\mathscr{H}_{0,\delta+1,p}$ with bounded inverse.*[8]

Observe that $\mathscr{D}\psi_0 \in \mathscr{H}_{0,1+\delta,p}$ for those $p \geq 2$ and $0 < \delta < 1 - \frac{3}{p}$ or $p = 2$ and $\delta = -1$, we see the following equation

$$\mathscr{D}\psi_1 = -\mathscr{D}\psi_0$$

has a unique solution $\psi_1 \in \mathscr{H}_{1,\delta,p}$ and hence $\psi = \psi_0 + \psi_1$ is a harmonic spinor. This spinor satisfies the desired asymptotic behaviour. To see this, notice $\psi_1 \in \mathscr{H}_{1,\delta,p}$ so in particular $\sigma^\delta\langle\psi_1,\psi\rangle^{1/2} \in W^{1,p}$ for all $0 < \delta < 1 - \frac{3}{p}$. From Morrey's inequality (see Evans 5.6.2) we see for $p > 3$ (larger than dimension 3) that $W^{1,p}$ is continuous embedded into C^0, which shows the desired asymptotic decay.
Now we shall derive the formula from the integral Bochner

$$\int_M (\langle\nabla\psi,\nabla\psi\rangle + \langle\psi,\mathscr{R}\psi\rangle)\mu = -\int_M [d\eta + \frac{1}{2}d(\langle\psi_0,[e^i,e^j]\nabla_j\psi_0\rangle e_i \lrcorner \mu]$$

[7]From $h = O(r^{-2})$ and $\Gamma = O(r^{-2})$ we can replace $\widehat{\nabla}$ by the usual coordinate derivative ∂ on $\mathbb{R}^3$ and the norm is still equivalent when restricted ψ to a M_l. Here we want to define it globally so $\widehat{\nabla}$ is used

[8]The smoothness of solution ψ is a local property so the elliptic theory on compact set suffices.

As noted by Parker η can be replaced by $\hat{\eta}$, but there is a typographical error which shall correct to

$$d[\langle \psi_0, [e^i, e^j]\psi_1\rangle e_k \lrcorner \mu] = [\langle \nabla_j \psi_0, [e^i, e^j]\psi_1\rangle + \langle \psi_0, [e^i, e^j]\nabla_j \psi_1\rangle] e_i \lrcorner \mu$$

where $\{i, j, k\}$ on the LHS is the sum of all cyclic permutation. The computation is essentially the same done in Parker (3.5). For example, the additional term in $e_1 \wedge e_3$ are

$$\langle \psi_0, h_{1j} e^j e_0 [e^1, e^2]\psi_1\rangle + \langle \psi_0, \nabla_1([e^1, e^2])\psi_1\rangle - \langle \psi_0, h_{3j} e^j e_0 [e^1, e^2]\psi_1\rangle - \langle \psi_0, \nabla_3([e^1, e^2])\psi_1\rangle$$

where $\nabla_1([e^1, e^2]) = -2h_{11}e^0e^2 - 2h_{12}e^1e^0$ cancel out with $(h_{11}e^1e^0 + h_{12}e^2e^0)[e^1, e^2]$ from the first term and the rest term are also similarly cancel out. By doing so it is only required to study $\{\psi_1, \nabla\psi, \nabla\psi_0\}$. Now it is tempting to use the Stoke's theorem to study this integral, but this may not work out since our knowledge on $\{\psi_1, \nabla\psi, \nabla\psi_0\}$ is all about integral, which seems unlikely to tell us anything about them on $\partial M_{l,r}$ (it evidently has measure zero in whole M_l). An alternative way is to smear out $\partial M_{l,r}$ over r to $2r$ (so it no longer has measure zero in M_l). This idea is facilitated by a cutoff function w_r, say $w_r = 0$ on $\cup_l M_{l,2r}$ and 1 outside $\cup_l M_{l,r}$ with the addition property $|dw_r| < 2r^{-1}$ (easy to achieve by rescaling), then by dominated convergence theorem (we shall use $d\hat{\eta} \in L_1$, see Parker for details) and integration by parts

$$\int_M d\hat{\eta} = \lim_{r\to\infty} - \int_M dw_r \wedge \hat{\eta}$$

By an elementary inequality

$$\int_M |dw_r \wedge \hat{\eta}|\, \mu \le 2\|\psi_1 dw_r\|_2 (\|\nabla\psi\|_2 + \|\nabla\psi_0\|_2)$$

and from $\|\psi_1 dw_r\|_2 \le 2\|\sigma^{-1}\psi(1 - w_{r/2})\|_2$ (the reason why we require $|dw_r| < 2r^{-1}$ is that σ^{-1} will come out) by dominated convergence theorem again the RHS tends to 0 (as we know that $\psi_1 \in \mathscr{H}_{1,-1,2}$). Now we observe that $|e^i - dx^i| = O(r^{-1})$ (this norm is understood as some matrix norm for example) then it is safe to replace e^i by dx^i and the integral only differs by $O(r^{-1})$, which under $r \to \infty$ is the same.

$$-\frac{1}{2}\int_{\partial M_{l,r}} \langle \psi_0, [dx^i, dx^j]\nabla_j \psi_0\rangle \partial_i \lrcorner \mu$$

We write $dx^k dx^l = \frac{1}{2}[dx^k, dx^l]$ and plug in all the terms in the expression of $\nabla_j \psi_0$ in coordinates, and we use the following two formula (i, j is from 1 to 3)

$$[dx^i, dx^j][dx^k, dx^l] = -4(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})$$
$$[dx^i, dx^j]dx^0 dx^k = 2(\delta^{ik} dx^0 dx^j - \delta^{jk} dx^0 dx^i)$$

The first formula follows from noticing that $i \ne j$ and $k \ne l$ so this leaves us with either $i = k$ and $j = l$ or $i = l$ and $j = k$. As first case $[dx^i, dx^j][dx^i, dx^j] = 4(dx^i dx^j)^2 = -4$ we see the formula holds. Similarly $i \ne j$ for the second formula so either $i = k$ or $j = k$, and in the first case $2(dx^i dx^j)dx^0 dx^i = 2dx^0 dx^j$. Finally using the formula for Γ_{kjl} and its symmetry $\Gamma_{kjl} = \Gamma_{klj}$ we have proved the formula shown in Theorem 4.1. □

6. Proof of Theorem 5.2

As usual we shall first establish a weak solution before any further discussion. Here we consider $\mathscr{H}_{1,-1,2}$ (by construction a Hilbert space) and apply a well known theorem for weak form on Hilbert space.

Theorem 6.1 (Lax-Milgram)**.** *Given a bounded bilinear form $B(u,v)$ which is also coercive on a Hilbert space V, then $B(u,v) = f(v)$ has a unique solution $u \in V$ for any bounded linear functional f.*

We now let[9] $B(u,v) = \langle \mathscr{D}u, \mathscr{D}v\rangle_2$ and $f(v) = \langle \eta, v\rangle_2$ (weak form of $\mathscr{D}^2 u = \eta$).

Lemma 6.2. *For (p,δ) stated in Theorem 5.2, $\mathscr{D} : \mathscr{H}_{1,\delta,p} \to \mathscr{H}_{0,\delta+1,p}$ is bounded.*

Proof. From the inequality

$$\|\sigma^{1+p}\mathscr{D}\psi\|_p \le 2\|\sigma^{1+p}\nabla\psi\|_p \le 2\|\sigma^{1+p}\widehat{\nabla}\psi\|_p + 2\|\sigma^{1+p}|h|\psi\|_p \tag{6.1}$$

and $|h| \le C\sigma^{-2}$ so the last integral converge for $\psi \in \mathscr{H}_{1,\delta,p}$. □

To prove the coercive part, we shall derive some estimate for asymptotic part.

Lemma 6.3. *For sufficiently large R we have the following estimate of $\psi \in \mathscr{H}_{1,-1,2}$*

$$\|\sigma^{-1}\psi\|^2_{2;M_{l,2R}} \le 16\|\widehat{\nabla}\psi\|^2_{2;M_{l,2R}}$$

$$\|\widehat{\nabla}\psi\|^2_{2;M_{l,2R}} \le \frac{6}{5}\|\nabla\psi\|^2_{2;M_{l,2R}}$$

Here the subscript $M_{l,2R}$ denotes integration over $M_{l,2R}$.

Proof. It suffices to verify this for C_0^∞, but we first work on usual $\mathbb{R}^n$. In this case $d\mu_3 = r^2 dr d\Omega$ in spherical coordinates so from integration by parts[10]

$$\int_{2R}^\infty r^{-2}|\psi|^2(r^2dr) \le 2\int_{2R}^\infty r(2|\psi|d|\psi|)dr \le \left(\int_{2R}^\infty 4|\psi|^2dr\right)^{1/2}\left(\int_{2R}^\infty |d|\psi||^2(r^2dr)\right)^{1/2}$$

and the last inequality is from Cauchy-Schwarz and gives the formula

$$\int_{M_{l,2R}} r^{-2}|\psi|^2 d\mu_3 \le 16\int_{M_{l,2R}} |d|\psi||^2 d\mu_3$$

Here $d|\psi|$ can be further estimated

$$|d|\psi|| < \frac{d\langle\psi,\psi\rangle}{2|\psi|} = \frac{\langle\widehat{\nabla}\psi,\psi\rangle}{|\psi|} \le |\widehat{\nabla}\psi|$$

At minimum $|\psi| = 0$ if it is differentiable (almost everywhere, but we are going to integrate anyway so this suffices) then LHS is zero so inequality still holds. Now g_{ij} differs from δ_{ij} by $O(r^{-1})$ so for R large enough this inequality still holds if $d\mu_3$ is replaced by the volume form $\sqrt{g}d\mu_3$. The second inequality is using $|h| \le C_1 r^{-2}$

$$\|\nabla\psi\|_{2;M_{l,2R}} \ge \|\widehat{\nabla}\psi\|_{2;M_{l,2R}} - \||h|\psi\|_{2;M_{l,2R}} \ge \|\widehat{\nabla}\psi\|_{2;M_{l,2R}} - \frac{C_1}{R}\|\sigma^{-1}\psi\|_{2;M_{l,2R}}$$

and use first inequality. □

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[9] Notice there is no boundary term encoded in our weak formulation, this is because we are working on $\mathscr{H}_{1,-1,2}$ and in fact the boundary term will vanish (similar calculation as $d\hat{\eta}$).

[10] The boundary term is collected in the integral $|r|\psi|^2|_{2R}^\infty| \le 2R\int_{2R}^\infty d|\psi|^2dr \le \int_{2R}^\infty r(2|\psi|d|\psi|)dr$

Proposition 6.4. *For sufficiently large R, we have a constant c_R so $\psi \in \mathscr{H}_{1,-1,2}$*

$$\|\psi\|_{1,-1,2}^2 \le c_R \|\mathscr{D}\psi\|_2^2 \tag{6.2}$$

Proof. As before we use the cutoff function β_R define in proof of theorem 4.1 to write $\psi_{\text{in}} = (1-\beta_R)\psi$ and $\psi_{\text{ex}} = \beta_R\psi$, we also pick R large enough so that the estimate in the above lemma holds. First we notice for the compact portion ψ_{in} Rellich lemma still works, we shall show the inequality

$$\|\psi_{\text{in}}\|_2^2 \le c_1 \|\nabla\psi_{\text{in}}\|_2^2 \tag{6.3}$$

Suppose not, let ψ_k be a sequence such that $k\|\nabla\psi_k\|_2^2 \le \|\psi_k\|_2^2$ and by rescaling we may assume $k\|\nabla\psi_k\|_2^2 \le 1$ so $\psi_k \in W^{1,2}$ and Rellich lemma then provide a subsequence $\psi_{k_j} \to \psi$ in L^2 which in terms show $\psi \in W^{1,2}$ and $\nabla\psi = 0$ as well (see Evans 5.8.1) From Lemma 4.2 for C_0^∞ that $\nabla\psi = 0 \implies \psi = 0$ and by a density argument we see this inequality holds. From this we have

$$\|\widehat{\nabla}\psi_{\text{in}}\|_2 \le \|\nabla\psi_{\text{in}}\|_2 + \|h\|_\infty \cdot \|\psi_{\text{in}}\|_2 \le c_2\|\nabla\psi_{\text{in}}\|_2$$

For the asymptotic portion, $\nabla\psi_{\text{ex}} = d\beta_R \cdot \psi + \beta_R \cdot \nabla\psi$ and we also set $|d\beta| \le c_3\sigma^{-1}$ so from the estimate in the above lemma $\|\nabla\psi_{\text{ex}}\| \le c_4\|\nabla\psi\|_2$, now[11]

$$\|\nabla\psi\|_2^2 \ge \frac{1}{2}\|\nabla\psi_{\text{in}}\|^2 + \|\nabla\psi_{\text{ex}}\|_2^2 - c_5\|\nabla\psi\|_2^2$$

so it follows $c_6\|\widehat{\nabla}\psi\|_2^2 \le \|\nabla\psi\|_2^2$. Finally we know from integral Bochner that

$$\|\mathscr{D}\psi\|_2^2 = \|\nabla\psi\|_2^2 + \langle\psi, \mathscr{R}\psi\rangle$$

for C_0^∞ so by a density argument it is also true for $\mathscr{H}_{1,-1,2}$. In particular since $\mathscr{R} \ge 0$ so we have

$$\|\mathscr{D}\psi\|_2^2 \ge \|\nabla\psi\|_2^2 \ge \frac{c_6}{2}\|\widehat{\nabla}\psi\|_2^2 + \frac{c_6}{10}\|\sigma^{-1}\psi\|_2^2 \ge c_7\|\psi\|_{1,-1,2}^2$$

the last inequality is also from Lemma 6.3. □

At first sight one may wonder whether we can solve for $\eta \in L^2$ as it is not immediate $\langle\eta, \cdot\rangle$ defines a bounded functional over $\mathscr{H}_{1,-1,2}$, but remember C_0^∞ is dense in L^2 and we surely can solve for C_c^0, the theorem then follows from a short argument.

Proof for $\eta \in L^2$. Given $\eta \in L^2$ we find a sequence η_i in C_0^∞ to approximate and let $\mathscr{D}\psi_i = \eta_i$ be their solutions. Since we have the estimate in Proposition 6.4, ψ_i is also a Cauchy sequence so has a limit ψ, and by triangle inequality we see

$$\|\mathscr{D}\psi - \eta\|_2 \le c_1\|\psi - \psi_i\|_{1,-1,2} + c_2\|\eta - \eta_i\|_2$$

the two terms on RHS follows from inequality (6.1) and (6.2) respectively. □

It is not clear from this argument that our solution is unique or not, to this end

Proposition 6.5. *$\mathscr{D} : \mathscr{H}_{1,-1,2} \to L^2$ is injective.*

Proof. The solution $\mathscr{D}\psi = 0$ being smooth and $\nabla\psi = 0$ is clear (C^∞ regularity is local and by Bochner). In view of Lemma 4.2, it remains to show ψ decay to zero[12] by observing that

$$\int_{2R}^{\infty} |r^{-1}\psi|^2 (r^2 dr) < \infty$$

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[11]From this inequality $c^{-1}|a|^2 + c|b|^2 \ge 2\langle a, b\rangle$ and that for $\nabla\psi_{\text{ex}}$.

[12]Though not necessary, for $\delta > 0$ we can obtain a better exponent decay as demonstrate above by Morrey inequality

so ψ must tends to zero (this does not show uniform but along some path). $\square$

This completes $\mathscr{H}_{1,-1,2}$ case. For other $\mathscr{H}_{1,\delta,p}$, we first note that $\mathscr{H}_{1,\delta,p} \subset \mathscr{H}_{1,-1,2}$ is a continuous embedding, which can be shown by Hölder inequality (see Parker). Now if we have the following estimate

$$\|\psi\|_{1,\delta,p} \leq c_{\delta,p}\|\eta\|_{0,\delta+1,p} \tag{6.4}$$

for the weak solution $\psi \in \mathscr{H}_{1,-1,2}$ obtained above, then by an almost identical argument to $\eta \in L^2$ we complete the proof, so let us finish this.

7. Proof of $\|\cdot\|_{1,\delta,p}$ Estimate

As before, we divide our discussion into ψ_{in} and ψ_{ex}. For the compact portion we can use L^p-estimate and notice that we know $\mathscr{D}$ has unique solution if it exists, so by similar argument (but this time we use L^p-regularity here instead of Rellich inequality, see Gilbarg-Trudinger Lemma 9.17) in inequality (6.3) we can in fact show (remember $\psi = \mathscr{D}u$)

$$\|\widehat{\nabla}\psi_{\text{in}}\|_{p;K} \leq c_1\|\eta\|_{p;K} \text{ and } \|\psi_{\text{in}}\|_{p;K} \leq c_2\|\eta\|_{p;K}$$

Sketch of L^p-estimate on K. (from Gilbarg-Trudinger) As in class, we decompose an elliptic operator into $L = L_0 + L_1 + L_2$ on a small ball where L_0 is the constant coefficients 2nd-order terms, L_1 its 2nd-order variation, and L_2 the lower order terms. We then estimate on each small ball and patch up to the compact set K. As in class, we use an L^p estimate for L_0 and an interpolation between $W^{0,p}$ and $W^{2,p}$, so it all boils down to L^p-estimate on L_0, which is the standard laplacian Δ on $\mathbb{R}^n$ after some linear transformation.

7.1. L^p-Estimate for Standard Laplacian on $\mathbb{R}^n$.

For $\Delta u = f$ classically we have Green representation formula

$$u(x) = \int_B G(x-y)f(y)dy$$

where G is the fundamental solution and $f \in C_0^\infty(B)$, then we extend this to $f \in L^2(B)$ by a density argument. Now we can view $\partial_i\partial_j u$ as an operator T on L^2, with ∂_i understood as weak derivative that

$$Tf = \partial_i\partial_j(\int_B G(x-y)f(y)dy)$$

so the L^p-estimate is equivalent to $\|Tf\|_p \leq c\|f\|_p$, T is called strong type p. For $p = 2$, using Green 1st identity one can show

$$\int_{\mathbb{R}^n}\sum_{i,j}|\partial_i\partial_j u|^2 = \int_{\mathbb{R}^n} f^2 \tag{7.1}$$

where the boundary terms vanish from an estimate on G. On the other hand for $p = 1$ we remark that the strong type no longer holds. An example is provided by

$$u(r) = \int_r^1 \frac{1}{x^{n-1}(1-\ln x)}dx$$

in unit ball B. From $\Delta = r^{1-n}\partial_r(r^{n-1}\partial_r u) + \cdots$ and $|\partial^2 u| \geq \partial_r^2 u$ we require

$$r^{n-1}\partial_r u \to 0 \text{ but } r^{n-2}\partial_r u \notin L^1(B)$$

near origin, the first corresponds to $\Delta u \in L^1(B)$ and the 2nd $|\partial^2 u| \notin L^1(B)$. Thus, we can at most anticipate a weak type estimate for $p = 1$, namely

$$\mu_f(t) := |\{x \in B | f(x) > t\}| \leq \frac{c_1}{t} \|f\|_1$$

To prove this, a subdivision of space is used. For a fixed $t > 0$,

1. (starter) Pick P_0 such that $K \subset P_0$ and $\displaystyle\int_{P_0} f \leq t|P_0|$
2. Apply bisection on P_0
3. Pick those sub-cubes P that $\displaystyle\int_P f \leq t|P|$ and repeat the process.

There is some quick observation on this subdivision, for those P cannot be further subdivided by this procedure

$$t \leq \frac{1}{|P|} \int_P f \leq 2^n t \tag{7.2}$$

$$|f| \leq t \text{ a.e. on } P_0 - \cup_{j=1}^{\infty} P_j \tag{7.3}$$

the LHS is from P cannot be further subdivided and RHS from P predecessor, and $|f| < t$ can be seen from Lebesgue differentiation theorem. We write function $f = g + h$ where g is defined by

$$g(x) = \begin{cases} \dfrac{1}{|P_j|} \displaystyle\int_{P_j} f & \text{for } x \in P_j \\ f(x) & \text{otherwise} \end{cases}$$

with P_j the sequence of cubes described by the procedure above. First

$$\mu_{Tf}(t) \leq \mu_{Tg}(t/2) + \mu_{Tb}(t/2)$$

and the estimate for Tg is

$$\mu_{Tg}(t/2) \leq \frac{4}{t^2} \int |g|^2 \leq \frac{2^{n+2}}{t} \int |g| \leq \frac{2^{n+1}}{t} \|f\|_1$$

where the first is Chebyshev inequality and (7.1), the second is from $|g| \leq 2^n t$ almost everywhere by (7.2) and (7.3) while the third by construction of g. For the part h, we shall study it on each P_l. Let $h_{kl} \in C_0^{\infty}(P_l)$ be a sequence converges to $h\chi_{P_l}$ in L^2-norm with the additional property that its integral over P_l is identically zero (easy to achieve by adding some small constant), then we have

$$Th_{kl} = \int_{P_l} (\partial_{ij}^2 G(x, y) - \partial_{ij}^2 G(x, y_l)) h_{kl}(y) dy$$

since the last term average out by the construction on h_{kl}, where y_l is the center of P_l, then we use mean value theorem and the fundamental solution to obtain

$$|Th_{kl}| \leq \frac{Cr_l}{(\text{dist}(x, P_l))^{n+1}} \int_{P_l} |h_{kl}(y)| dy$$

where r_l is the size of P_l. Now if we integrate over the region with P_l scooped out,

$$\int_{P_0 - P_l} |Th_{kl}| \leq C_1 \int_{P_l} |h_{kl}|$$

Consequently, we have by letting $k \to \infty$ and summing over l

$$\int_{P_0 - \cup_{i=1}^{\infty} P_i} |Th| \leq C \int |h|$$

This is a strong type on $P_0 - \cup_{j=1}^{\infty} P_j$, but we do not obtain any strong type on $\cup_{j=1}^{\infty} P_j$, so for that part we could only say its measure is bounded by $C_2\|f\|_1/t$ from (7.2) to obtain a weak type. Finally, we introduce an analytic tool

Theorem 7.1 (Marcinkiewicz interpolation)**.** *If T is of weak type p and weak type q, then T is of strong type r for $p < r < q$.*

This establishes L^p-estimate for $1 < p \le 2$. For $p > 2$ we use the idea of L^p dual space

$$\|Tf\|_p = \sup_{\|g\|_{p^*}=1} \int_B (Tf)g$$

and the inequality follows from integration by parts

$$\int_B (Tf)g = \int_B f(Tg) \le \|f\|_p \|Tg\|_{p^*}$$

so apply the estimate $1 < p^* \le 2$ for g we also obtain $p > 2$ case. □

7.2. A SMALL DEVIATION FROM CONSTANT COEFFICIENTS. Now we shall exhibit a $\mathscr{H}_{1,\delta,p}$-estimate for ψ_{ex}. In the same spirit of compact case, we expect to relate our operator to the constant coefficients operator, and build the estimate on it. To be more explicit, we shall construct a series of $\mathscr{D}_R$ and g^R on $\mathbb{R}^3$ parametrized by R, where $g^R_{ij}(x) = \delta_{ij} + \beta(3x)(g_{ij}(x) - \delta_{ij})$ and the Dirac operator $\mathscr{D}_R$ is defined by

$$\mathscr{D}_R = \sum_{i=1}^{3} e^i_R \widehat{\nabla}^R_i - \frac{1}{2}\beta(3x)h_{ij}(x)e^0 e^j_R$$

where[13] $\widehat{\nabla}^R_i = \partial_i + \beta(3x)\Gamma_i(x)$. From this construction we have $\mathscr{D}_0 = \sum_{i=1}^3 dx^i \partial_i$ (the most beloved $\mathscr{D}_0^2 = -\Delta_{\mathbb{R}^3}$), while $\mathscr{D}_R = \mathscr{D}$ on $M_{l,R}$, and the perturbation of coefficients has that

$$\sup_{\mathbb{R}^3} |e^i_R - dx^i| < \varepsilon \text{ and } \sup_{\mathbb{R}^3} \sigma \cdot \beta(3x)(|\Gamma_i| + |h_{ij}|) < \varepsilon$$

This is exactly $\|\mathscr{D}_R - \mathscr{D}_0\| < \varepsilon$ when R is sufficiently large and if we happen to show $\mathscr{D}_0$ is a Banach isomorphism, then as having inverse is an open condition we are done. Now we state the estimate on $-\Delta$, which is a special case of [3] Thm 2.1.

Theorem 7.2. *For $p \ge 2$ and $0 < \delta < 1 - \frac{3}{p}$ we have the estimate*

$$\sum_{|\alpha| \le m} \||x|^{|\alpha|+\delta}\partial_\alpha u\|_p \le c\||x|^{2+p}\Delta_{\mathbb{R}^3} u\|_p$$

for all $u \in \mathscr{H}_{1,-1,2}$ and $|x|^{2+\delta}\Delta_{\mathbb{R}^3} u \in L^p(\mathbb{R}_3)$.

Proof. We begin by noticing the fundamental solution is $G(x,y) = c_1|x-y|^{-1}$, then

$$\left||x|^\delta u\right| \le c_1 \int_{\mathbb{R}^3} \left(\frac{1}{|x|^{-\delta}|x-y||y|^{2+\delta}}\right) \left||y|^{2+\delta} f\right| dy$$

for $f = -\Delta_{\mathbb{R}^3} u$. We shall remark that this integral viewed as an integral operator with the kernel $K(x,y)$ defined as in the parentheses, is a bounded operator over $L^p(\mathbb{R}^3)$. It then follows that

$$\||x|^\delta u\|_p \le c_1 \||x|^{m+\delta}\Delta_{\mathbb{R}^3} u\|_p \tag{7.4}$$

[13]Notice the connection form is not changed, we just slightly perturb the matrix e^i_R

To see this, we first observe by $|x| \le |x-y| + |y|$ that

$$K(x,y) \le \frac{1}{|x-y|^{1-\delta}|y|^{2+\delta}} + \frac{1}{|x-y||y|^2}$$

so we just estimate the two terms. By AM-GM inequality $|x| \ge (\prod_{i=1}^3 |x_i|)^{1/3}$ we see this can be further reduced to 1-dim case, which is exactly the following lemma

Lemma 7.3 (Schur). *Let $K(x,y) \ge 0$ and $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$, then*

$$\int_0^\infty y^{-1/p} K(1,y)dy = C < \infty \implies \Big\| \int_0^\infty K(x,y)f(y)dy \Big\|_p \le C\|f\|_p$$

Proof. Write $K(x,y) = x^{-1}K(1, y/x)$ and apply Minkowski integral inequality. □

Now from the standard L^p-estimate

$$R^{|\alpha|}\|\partial_\alpha u\|_{p;A_R} \le c_2(R^2\|\Delta u\|_{p;A_R} + \|u\|_{p;A_R})$$

(those R arise naturally by rescaling) for an annulus $A_R = \{R \le |x| \le 2R\}$. It then follows by multiply both sides R^δ and with some manipulations that

$$\||x|^{\delta+|\alpha|}\partial_\alpha u\|_{p;A_R} \le c_3(\||x|^{2+\delta}\Delta u\|_{p;A_R} + \||x|^\delta u\|_{p;A_R})$$

finally we sum over $R = 2^j$, then use (7.4) to conclude. □

Finally we remark that $\mathscr{D}u = \psi$, so the estimate for u establishes (6.4) for ψ_{ex}.

8. Higher Dimensional Generalization

The idea above can be extended to higher dimensions[1], but first we shall ask ourselves why one shall do this after all. In fact, if one can exhibit locally conformally flat coordinates in a neighborhood B, then by $r \to r^{-1}$ this will flip the coordinates to

$$g_{ij} = \left(1 + \frac{m}{r^{n-2}}\right)\delta_{ij} + a_{ij}$$

where a_{ij} is the higher order terms, and notice m is related to the integral

$$m(n-1)\text{vol}(S^n) = \lim_{R\to\infty} \int_{S_R} (\partial_j g_{ij} - \partial_i g_{jj})d\Omega^i$$

so the positive mass theorem will tell us $m \ge 0$, and $m = 0$ if and only if $M - B$ is isometric to $\mathbb{R}^n$. This is interesting in itself, or we shall remark that this is a step in proving compact Yamabe problem. Now we state relevant assumptions.

Assumption (Bartnik). *Let M be a complete Riemannian manifold which admits a spin structure with scalar curvature $R \ge 0$, and require some decay*[14] *condition say $g_{ij} - \delta_{ij} \in W_{2,p,-\tau}(M_R)$ where $\tau \ge \frac{1}{2}(n-2)$.*

First notice if we take up this route of proving positive mass theorem, we need a spin structure though it is not always true. Nevertheless, we briefly describe how to extend the above calculation to this specific version in higher dimension, and the following will be sketchy.

1. The constant spinor ψ_0 is still subjected to a particular choice of coordinates, but we will show this choice is superficial in some sense later.

[14]the exponent τ here of Bartnik is slightly different from that δ of Parker, $-\tau - n/p = \delta$

2. The boundary term in Bochner formula is still the mass, and here we do not bother ourselves with hypersurface so $\nabla\psi = 0$ will directly imply $\psi = 0$ so the argument is much more direct.
3. The injectivity of $\mathscr{D}$ follows from $\|\mathscr{D}\psi\|_2^2 = \|\nabla\psi\|_2^2 + \langle \psi, \frac{1}{4}R\psi\rangle$ and $R \geq 0$, so as before $\nabla\psi = 0$ but now it shows $\psi = 0$ directly.
4. The relevant estimates over weighted Sobolev space are essentially the same where they are done similarly by summing over annulus A_R, and an estimate of Green function $G(x,y) = c_n|x-y|^{2-n}$ of the standard Laplacian $\Delta_{\mathbb{R}^n}$ followed by using Schur would give weighted Rellich–Kondrachov.

We close this section with two theorems showing the mass and the choice of coordinates are more or less intrinsic to M itself.

Theorem 8.1 (Bartnik [1] 3.2)**.** *Let x^i and z^i be two coordinates with asymptotic decay of τ or higher, then*

$$|x^i - (A^i_j z^j + b^i)| = o(r^{1-\tau})$$

where A^i_j is a constant orthogonal matrix and b^i some constants.

Sketch of proof. First we know $\Delta x^i = g^{jk}\Gamma^i_{jk} \in W_{0,p,1-\tau}$ so we can solve $\Delta v^i = \Delta x^i$ and obtain harmonic coordinates $x^i - v^i$, and similarly for $z^i - w^i$. Notice $\dim\ker\Delta = n+1$ (Roughly speaking, using an expansion at infinity for elements in $\ker\Delta$, then $\dim\ker\Delta$ can be computed from the dimension of harmonic polynomial of degree k) this implies both $\{1, x^i - v^i\}$ and $\{1, z^i - w^i\}$ consists a basis for $\ker\Delta$ and from this we can relate them by a constant orthogonal matrix. □

Theorem 8.2 (Bartnik [1] 4.2)**.** *Given two coordinates x^i and z^i with decay $\tau \geq \frac{1}{2}(n-2)$ then their mass are the same.*

Sketch of proof. Notice from the asymptotic assumption

$$\sqrt{|g|}\, g^{ij}(\Gamma_j - \frac{1}{2}\partial_j \log|g|) = \partial_j g_{ij} - \partial_i g_{jj} + o(r^{-1-2\tau})$$

and the LHS appear in the divergence term in scalar curvature R, so this gives a geometric interpretation of mass in terms of a limit of an integral of scalar curvature. Now notice the change of frame is $A^j_i + o(r^{-\tau})$ from above, so one may check all additional terms are in correct order so that one can discard them in the limit and obtain the same integral. □

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Note for the Solutions to the Yamabe Problem

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1 Introduction to Yamabe Problem

To introduce what does the Yamabe problem mean, we first recall some knowledges about conformal deformation of Riemannian metric.

Definition 1.1. Let (M, g) be an $n(\geq 2)-$dimenstional smooth Riemannian manifold. Given $\tilde{g}$ another Riemannian metric on M, we say $\tilde{g}$ is conformal to g iff there exist a diffeomorphism f of M onto M and a porsitive function $\rho \in C^{\infty}(M)$ such that $\tilde{g} = \rho f^{*}g$. In the case f is the identity map, we say $\tilde{g}$ is point wise conformal to g. On the other hand, if $\tilde{g} = g$, i.e. $g = \rho f^{*}g$ for some positive function ρ, then f is called a conformal transformation of (M, g).

Let $\mathcal{C}_g = \{\rho g \mid \rho \in C^{\infty}(M), \rho > 0\}$ be the set of Riemannian metrics on M pointwise conformal to g. A nature problem arising is Given (M, g) and a function $K \in C^{\infty}(M)$, does there exist $\tilde{g} \in \mathcal{C}_g$ such that the scalar curvature $\tilde{R}$ of $\tilde{g}$ is equal to the given function K?

For arbitrary function K, this problem is easily disproved since we all know there are some topological constraint on the scalar curvature (for example, the Gauss-Bonnet theorem for surface). Hence we restrict the problem to the most interesting case which $\tilde{K}$ is a constant. For dimension 2, this problem is just the unifromization theorem of Riemann surface. For dimension ≥ 3, this problem is the so-called **Yamabe conjecture**, which was studied by H. Yamabe(1960), N. Trudinger(1968), T. Aubin(1976), and has been completely solved by R. Schoen(1984). The answer is also affirmative when the manifold is compact (there are counterexapmles for noncompact case, see [5]).

The goal of this note is to show the proof of Yamabe problem. In this note, we consider that case that (M, g) is a compact, smooth, Riemannian manifold without boundary (in general, we can also discuss this problem in the category of complete, non-compact Riemannian manifold, but the study in this direction is in complete).

To start the disscussion of the proof of Yamabe problem, we have to first investigate how the scalar curvature transform when the metric transform conformally:

Proposition 1.2.(the conformal deformation of scalar curvature) Given a Riemannian manifold $(M.g)$ and $\rho \in C^\infty(M)$, $\rho > 0$. If $\tilde{g} = \rho g$, then we have $\tilde{R} = \rho^{-1}R - (n-1)\rho^{-2}\Delta\rho - \frac{1}{4}(n-1)(n-6)\rho^{-3}|\nabla\rho|^2$.

Proof: We have:

$$\begin{aligned}\tilde{\Gamma}_{ij}^k &= \frac{1}{2}\tilde{g}^{kl}(\frac{\partial\tilde{g}_{il}}{\partial x^j} + \frac{\partial\tilde{g}_{jl}}{\partial x^i} - \frac{\partial\tilde{g}_{ij}}{\partial x^l})\\ &= \frac{1}{2}\rho^{-1}g^{kl}(\frac{\partial\rho g_{il}}{\partial x^j} + \frac{\partial\rho g_{jl}}{\partial x^i} - \frac{\partial\rho g_{ij}}{\partial x^l})\\ &= \Gamma_{ij}^k + \frac{1}{2}(\delta_{ik}\frac{\partial\log\rho}{\partial x^j} + \delta_{jk}\frac{\partial\log\rho}{\partial x^i} - g_{ij}g^{kl}\frac{\partial\log\rho}{\partial x^l})\end{aligned}$$

Take the result into the formula of Ricci curvature. After some tedius calculation , we get:

$$\tilde{R}_{ij} = R_{ij} - \frac{n-2}{2}(\log\rho)_{,ij} + \frac{n-2}{4}(\log\rho)_{,i}(\log\rho))_{,j} - \frac{1}{2}(\Delta(\log\rho) + \frac{n-2}{2}|\nabla\rho|^2)g_{ij}$$

Taking trace, we get the desired result.□

For the case $n > 3$, we define $\rho = u^{\frac{4}{n-2}}$, then we have:

$$\tilde{R} = u^{-\frac{n+2}{n-2}}(Ru - \frac{4(n-1)}{n-2}\Delta u)$$

So our problem reduces to solving the equation:

$$\frac{4(n-1)}{(n+2)}\Delta u - Ru + \lambda u^{\frac{n+2}{n-2}} = 0$$

for some $u > 0$ and constant λ.

2 Conformal Invariant $\lambda(M)$

From now on, we using the notation:

$$p = \frac{2n}{n-2}, a = \frac{n-2}{4(n-1)}, L = -\Delta + aR$$

The operator L is called conformal Laplacian of (M, g). then the equation becomes:

$$Lu = \lambda u^{p-1}$$

Yamabe observed that this is the Euler-Lagrange equation of the functional

$$Q_0(\tilde{g}) = \frac{\int_M R_{\tilde{g}}d\mu_{\tilde{g}}}{(\int_M d\mu_{\tilde{g}})^{2/p}}$$

restricted to the conformal class $\mathcal{C}_g$. Indeed, if we define $Q(u) = aQ_0(u^{p-2}g) = aQ_0(\tilde{g})$. Since we know the scalar curvaure of $\tilde{g}$ is $R_{\tilde{g}} = a^{-1}u^{1-p}Lu$, and $d\mu_{\tilde{g}} = u^p d\mu$, so we have:

$$Q(u) = aQ_0(u^{p-2}g) = \frac{E(u)}{||u||_p^2}$$

for functional:

$$E(u) = \int_M uLud\mu = \int_M (|\nabla u|^2 + aRu^2)d\mu$$
$$||u||_p^2 = (\int_M |u|^p d\mu)^{2/p}$$

called the Yamabe queotient. Then we can compute the variation for positive u:

$$\delta E(u) = \frac{(\int_M (2\nabla u \cdot \nabla \delta u) + 2aRu\delta ud\mu)}{||u||_p^2} - \frac{E(u)(2/p)(\int_M |u|^p d\mu)^{2/p-1}(\int_M pu^{p-1}\delta ud\mu)}{||u||_p^4}$$
$$= 2\frac{\int_M (-\Delta u + aRu - \lambda u^{p-1})\delta ud\mu}{||u||_p^2}$$

By integration by part and $\lambda = E(u)/||u||_p^p$.

Since $2/p = \frac{2n-4}{2n} < 1$, we can find $q > 0$ such that $1/q + 2/p = 1$ and by Hölder's inequality, we get:

$$|\int_M Ru^2 d\mu| \leq (\int_M R^q d\mu)^{1/q}(\int_M (u^2)^{p/2}d\mu)^{2/p} = c||u||_p^2$$

Hence the functional $Q(y = u)$ is bounded from below, Define

$$\lambda(M) = \inf\{aQ_0(\tilde{g}) \mid \tilde{g} \in \mathcal{C}_g\} = \inf\{Q(u) \mid u \in C^\infty(M), u > 0\}$$

By definition, $\lambda(M)$ only depends on the conformal class of g not g itself; hence $\lambda(M)$ is a conformal invariant.More over since we have the inequality:

$$||\nabla|u|||_2 \leq ||\nabla u||_2$$

the restriction of u to be positive is not necessary, and since $C^\infty(M)$ is densed in $L_1^2(M)$ we can also defined:

$$\lambda(M) = \inf\{Q(u) \mid u \in L_1^2(M), u \neq 0\}$$

The reason why the conformal invariant play a key role in the Yamabe problem is the theorem:

Theorem 2.1. Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. If $\lambda(M) < \lambda(S^n)$, then the Yamabe problem is solvable for

(M, g). Here S^n is the n-spherewith the standard metric.

To prove the theorem, we needs some lemmas. First, we consider the Sobolev inequality:

$$\Lambda(\int_{\mathbb{R}^n} |u|^p)^{2/p} \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx, \forall u \in C_0^\infty(\mathbb{R}^n)$$

It can be shown (ref.[1]) that the best constant Λ can be defined as:

$$\Lambda = \inf\{Q_{\mathbb{R}^n}(u) \mid u \in C_0^\infty(\mathbb{R}^n)\backslash\{0\}\} = n(n-1)\omega_n^{2/n}$$

for ω_n being the volume of S^n.

Secoond, we consider the stereographic projection $\pi : S^N\backslash\{P\} \to \mathbb{R}^n$, then we have:

$$(\pi^{-1})^* g_0 = \frac{4}{(1+|x|^2)^2} ds^2 = \rho(x)^{p-2} ds^2$$

for g_0, ds^2 the standard metric on n sphere and Euclidean space respectively; hence, π is an conformal transformation. So we have $Q_{S^n}(u) = Q_{\mathbb{R}^n}(\bar{u})$ for $\bar{u} = \rho u \circ \pi^{-1}$. Since for all $u \in C^\infty(S^n)$, we can approximate it by $u_i \in C_0^\infty(S^n\backslash\{P\})$, and corresponding $\bar{u}_i$ is compact support; therefore $\lambda(S^n) \geq \Lambda$.

In the section 4, we will prove the result:

Lemma 2.2. For any compact Riemannian manifold (M, g) without boundary, we always have $\lambda(M) \leq \Lambda$. As a corollary, we get $\lambda(S^n) = \Lambda$.

Note that the number $p = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $L_1^2(M) \hookrightarrow L^q(M)$ to be compact. In other word, we have the embedding is compact for $1 \leq q < p$.(ref. Rellich–Kondrachov theore, [2] and other L^p estimate) This makes it difficult to using the minimization method to obtain minimal critical points of Q. Yamabe's method is to decrease the power p to s and using the limiting procedure (corrected by Trudinger).

For all $s \in [1, p)$ we deinfe:

$$Q_s(u) = \frac{E(u)}{||u||_s^2}$$

By the same analysis as Q, we find that Q_s is bounded from below and hence we can define:

$$\lambda_s = \inf\{Q_s(u) \mid u \in L_1^2(M)\backslash\{0\}\}$$

Note that the signature of λ_s is the signature of $E(u)$ which is same as the first eigenvalue of L.

For the next, we considering the two lemma:

Lemma 2.3. $\limsup_{s\to p}\lambda_s \le \lambda(M)$. If $\lambda_s \ge 0$, then $\lambda_s \to \lambda(M)$ as $s \to p$.

Proof. It is easy to see that for a fixed sequence $0 \ne u_i \in L_1^2(M)$ such that $Q_p(u_i) = Q(u) \to \lambda(M)$, we have $\lambda_s \le Q_s(u_i) \to Q_p(u_i)$ as $s \to p$. This shows the first assertion. When $\lambda_s \ge 0$, $Q_s(u) \ge 0$ for all u. The we get:

$$Q_p(u) = Q_s(u)\cdot\frac{||u||_s^2}{||u||_p^2}$$

The latter term can be estimated by Hölder ineqeuality:

$$||u||_s^2 = (\int_M |u|^s d\mu)^{2/s} \le ((\int_M (|u|^s)^{p/s} d\mu)^{s/p}(\int_M 1 d\mu)^{1-s/p})^{2/s} = ||u||_p^2 \cdot \mathrm{vol}(M)^{2(1/s-1/p)}$$

Hence, we get $\lambda_p = \lambda(M) \le \lambda_s V^{2(1/s-1/p)}$; therefore $\liminf_{s\to p}\lambda_s \ge \lambda(M)$, proving the lemma.□

Lemma 2.4. Let $2 < s < p$. Then there exists $u_s \in C^\infty(M)$, with $u_s > 0$ amd $||u_s||_s = 1$, such that $Q_s(u_s) = \lambda_s$ and u_s satisfies the equation

$$Lu_s = \lambda_s u_s^{s-1}$$

Proof: Taking a minimizing sequence $\{u_i\} \subset L_1^2(M)$, such that $Q_s(u_i) \to \lambda_s$. We may assume $u_s \ge 0$ as before and since $Q_s(tu) = Q_s(u)$ for all positive constant t, we may also assume $||u_i||_s = 1$. Thus,

$$Q_s(u_i) = E(u_i) = ||\nabla u_i||_2^2 + a\int_M Ru_i^2 d\mu \to \lambda_s$$

Hence $||\nabla u_i||_2^2 \le c_1 + c_2||u_i||_2^2$. But we also have $||u_i||_2^2 \le c||u_i||_2^s$ by Hölder ineqeuality. So $\{u_i\}$ is a bounded sequence in $L_1^2(M)$; Therefore, we can assume $\{u_i\}$ converges weakly in $L_1^2(M)$ to some u_s. As weak limit, u_s satisfyes $||\nabla u_s||_2 \le \liminf_{i\to\infty}||\nabla u_i||_2$. Since the embedding $L_1^2(M) \hookrightarrow L^s(M)$ is compact , we have:

$$\int_M Ru_i^2 d\mu \to \int_M Ru_s^2 d\mu,\ \text{and}\ ||u_i||_s \to ||u_s||_s$$

it follows that $Q_s(u_s) \le \lim Q_s(u_i) = \lambda_s$; hence $Q_s(u_s) = \lambda_s$. There fore u_s is a L_1^2- weak solution of the Euler-Lagrange equation. To apply the regularity for the elliptic operator to the equation, we must have to show the smoothness of the function $(\cdot)^{s-1}$; i.e. we need to show that u_s is strickly positive.

Since $u_i \ge 0$, we may assume $u_s \ge 0$. From the Euler-Lagrange equation, we can find $c \ge 0$ such that $\Delta u_s - cu_s \le 0$. By the maximum principle of Laplacian if $u_s = 0$ at some point , then $u_s \equiv 0$, which is impossible since $||u_s||_s = 1$. Hence $u_s > 0$ and prove the theorem.□

Now we can proof the key theorem in this section:

Proof of theorem 2.1.: It is easy to see that if u_s jas an uniform upper bound: $u_s \leq c$, then by the qrgument in the proof of Lemma 2.4. we can obtain a subsequnce of $u_s \in C^{k,\alpha}$, where k is any positive integer and $0 < \alpha < 1$, then there exist a subsequence $s_i \to p$ such that u_{s_i} converges in $C^k(M)$ to some $u \in C^\infty(M)$ positive which satisfies:

$$Lu = \lambda u^{p-1}, \text{ and } Q(u) = \lambda$$

(The existence part is by Ascoli-Arzela theorem and smoothness is from regularity theorem) where $\lambda = \lim \lambda_{s_i}$. By lemma 2.3., we have $\lambda \leq \lambda(M)$ but $Q(u) = \lambda$. Hence $\lambda = \lambda(M)$ and u is an absolute minimizer of Q. Therefore, it suffices to show that u_s has a uniformly bounded.

Suppose not, there exist $s_k \to p$, $u_k = u_{s_k}$ and $z_k \in M$ such that $u_k(z_k) = \max u_k = m_k \to \infty$. By compactnesss, we may assume $z_k \to z_0 \in M$. Take R.N.C. centered at z_0 and the corresponding coordinate of z_k is x_k. Then $x_k \to 0$. In coordinate, the equation of u_k is:

$$\frac{1}{\sqrt{g(x)}}\partial_j(\sqrt{g(x)}g^{ij}(x)\partial_i u_k) - aR(x)u_k + \lambda_k u_k^{s_k-1} = 0$$

for $\lambda_k = \lambda_{s_k}$, we may assume the equation is defined on $|x| < 1$. Now set

$$v_k(x) = m_k^{-1}u_k(\delta_k x + x_k)$$

where $\delta_k = m_k^{(2-s_k)/2} \to 0$ (the scaling trick). Then v_k is defined on a ball of radius $\rho_k = (1 - |x_k|)/\delta_k \to \infty$ and satisfies the equation:

$$\frac{1}{b_k}\partial_j(b_k a_k^{ij}\partial_i v_k) - c_k v_k + \lambda_k v_k^{s_k-1} = 0$$

where we use the notation:

$$\begin{aligned} a_k^{ij}(x) &= g^{ij}(\delta_k x + x_k) \to \delta_{ij} \\ b_k(x) &= \sqrt{g(\delta_k x + x_k)} \to 1 \\ c_k(x) &= am_k^{1-s_k}R(\delta_k x + x_k) \to 0 \end{aligned}$$

The convergence is C^1- convergence on any compact subset of $\mathbb{R}^n$. Notice that by definition $v_k \leq v_k(0) = 1$. Hence we may know apply the L^p Schauder interio estimate (ref. [3]) for elliptic operator to get:

$$||v_k||_{C^{2,\alpha}(\bar{B}_R)} \leq C(R), \, \forall k \geq k(R)$$

Take a sequence $R_m \to \infty$. By the argument of diagonal subsequence, we obtain a subsequence $\{v_m\}$ such that $v_m \to v \in C^2(\mathbb{R}^k)$ and converges on $\bar{B}_{R_m}$ in C^2 convergence. then we get v is a non-negative solution of

$$\Delta_0 v + \lambda v^{p-1} = 0$$

on $\mathbb{R}^n$ with $v(0)=1$. By maximum principle again we have $v>0$. By Lemma 2.3, if $\lambda(M)\geq 0$ we have $\lambda=\lambda(M)$, otherwise $\lambda\leq 0$.

Now by definition we have:

$$\int_{|x|\leq\frac{1}{2}\delta_k^{-1}} v_k^{s_k} b_k dx = \int_{B_{1/2}(x_k)} u_k^{s_k}\sqrt{g}dx\delta_k^{\alpha_k} \leq ||u_k||_{s_k}^{s_k}\delta_k^{\alpha_k} = \delta_k^{\alpha_k}$$

Hence by Fatou's lemma,

$$\int_{\mathbb{R}^n} v^p dx \leq 1$$

Similariy, we have:

$$\int_{\mathbb{R}^n} |\nabla v|^2 dx \leq \infty$$

(by the L_1^2 estimating we doing in Lemma2.4.). Now let $\eta\in c_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $0\leq\eta\leq 1$, and $\eta=1$ on B_1 and $\eta=0$ outside B_2 for B_1, B_2 two open ball in $\mathbb{R}^n$. Define $v_R(x)=\eta(x/R)v(x)$. then we have:

$$\int_{\mathbb{R}^n}(|\nabla(v-v_R)|^2+|v-v_R|^p)dx\to 0$$

as $R\to\infty$ by the standard analysis of cut-off function. Integrating the differential equation of v, we get:

$$\int_{\mathbb{R}^n}\lambda v^{p-1}v_R dx = -\int_{\mathbb{R}^n} v_R\Delta_0 v dx = \int_{\mathbb{R}^n}\nabla v\cdot\nabla v_R dx$$

Taking $R\to\infty$, we get:

$$\int_{\mathbb{R}^n}|\nabla v|^2 dx = \lambda\int_{\mathbb{R}^n} v^p dx$$

For the case $\lambda\leq 0$, we have v is a constant, but $v\in L^p$ hence $v=0$ contradict to $v>0$. For $\lambda>0$, we get $\lambda=\lambda(M)$. Then by Sobolev inequality, we get:

$$\Lambda(\int_{\mathbb{R}^n} v^p dx)^{2/p} \leq \int_{\mathbb{R}^n}|\nabla v|^2 dx = \lambda(M)\int_{\mathbb{R}^n} v^p dx$$

Since $1>\int_{\mathbb{R}^n} v^p dx>0$ and $2/p<1$, we get:

$$\lambda(S^n)=\Lambda\leq\lambda(M)(\int_{\mathbb{R}^n} v^p dx)^{1-2/p}\leq\lambda(M)$$

contradict the assumption, hence prove the theorem.□

The theorem reduce the Yamabe problem to the estimate of $\lambda(M)$. And to do so, we take the method formulated by J. Lee and T. Paker, which will be introduced in the next section:

3 Conformal Normal Coordinates and Asymptotic Expansion of Green's Function

In this section, we fixed (M, g_1) an compact Riemannian manifold without boundary. And for all other metric g on M, we say g is conformal iff $g \in \mathcal{C}_{g_1}$.

In this section, we need to defined the Weyl tensor which was not mentioned in the class:

$$W_{iklm} = R_{iklm} + \frac{1}{2}(R_{im}g_{kl} - R_{il}g_{km} + R_{kl}g_{im} - R_{km}g_{il}) + \frac{1}{(n-1)(n-2)}R(g_{il}g_{km} - g_{im}g_{kl})$$

The special property of Weyl tensor is that it is an conformal invariant. The purpose of this secion is developing a appropriate coordinate system and conformal metric such that we can expand the Green's function of the conformal Laplacian in such coordinate system to do further estimate. The coordinat system we want is:

Theorem 3.1.(existence of conformal normal coordinate) Let M be a Riemannian manifold of dimension $n \geq 3$, and let $P \in M$. For any integer $N \geq 2$, there exists a conformal metric g on M such that in a normal coordinate system for g at P

$$\det(g_{ij}) = 1 + O(r^N)$$

where $r = |x|$. Furthermore, if $N \geq 5$ then we may require:

$$R = O(r^2) \text{ and } \Delta R(P) = -\frac{1}{6}|W(P)|^2,$$

where R and W are the scalar curvature and Weyl tensor respectively. Such metric together with its normal coordinate is called a conformal normal coordinate system at P

To prove the theorem, we need three lemma:

Lemma 3.2. Let $P \in M$, and let T be a symmetric $(k+2)$-tensor on T_pM, $k \geq 0$. There exists a unique homogeneous polynomial f of order $(k+2)$ such that in the normal coordinate system of g the metric $\tilde{g} = e^{2f}g$ satisfies

$$\text{Sym}(\tilde{\nabla}^k \tilde{R}_{ij})(P) = T$$

where Sym$(\cdot)$ denote the symmetrization operator for tensors.

Proof: Let x_j be the normal coorinates of g at P, and $r = |x|$. We denote the space of homogeneous polynomials of order m by $\mathcal{P}_m$. Let $F(x) = R_{ij}(x)x^ix^j$ be the homogeneous polynomial correspond to the symmetric tensor R_{ij}. By Taylor expansion, we have:

$$F(x) = \sum_{m=2}^{k+2} F^{(m)}(x) + O(r^{k+3})$$

where

$$F^{(m)} = \frac{1}{(m-2)!}\sum_{|K|=m-2}\sum_{i,j}\partial_K(R_ij(P)x^ix^j)x^K$$

Notice that we have:

$$R_{ij,K} = \partial_K R_{ij}(P) + S_{ijK}$$

Where S_{ij_K} is a polynomial whose coefficients consist of derivatives of order less than $|K|$ of R_{ij} at P. Hence if $f \in \mathcal{P}_{k+2}$, then we have $\tilde{S}_{ijK} = S_{ijK}$ for $|K| = k$. Notice that the lemma is equivialent to:

$$0 = \sum_{|K|=k}(\tilde{R}_{ij,K}(P) - T_{ijK})x^ix^jx^K = k!\tilde{F}^{(k+2)}(x) + \sum_{|K|=k}(S_{ijK} - T_{ijK})x^ix^jx^K$$

and the Euler's formula for homogeneous polynomials is:

$$x^ix^j\partial_i\partial_j f = (x^i\partial_i)^2 f - x^i\partial_i f = (k+2)(k+1)f$$

And $\Delta f = \Delta_0 f + O(r^k + 1)$, hence we get by the of R_{ij} and $\tilde{R}_{ij}$ we derived in the begining:

$$\begin{aligned}\tilde{F}^{(k+2)}(x) &= F^{(k+2)}(x) + x^ix^j[(2-n)\partial_i\partial_j f - \Delta_0 f\delta_{ij}] \\ &= F^{(k+2)}(x) - (n-2)(k+2)(k+1)f - r^2\Delta_0 f\delta_{ij}]\end{aligned}$$

To prove the lemma , we just need to show that the operator $r^2\nabla_o + (n-2)(k+2)(k+1)$ is invertible on $\mathcal{P}_{k+2}$, then we get the unique f, which follows from the next lemma:

Lemma 3.3. The nonzero eigenvalues of $r^2\Delta_0$ on $\mathcal{P}_m$ are:

$$\{\lambda_j = 2j(n-2+2m-2j) \mid j = 1, ..., [m/2]\}$$

The eigenfunction corresponding to λ_j has the form $r^{2j}\psi$, where $\psi \in \mathcal{P}_{m-2j}$ is a harmonic polynomial.

Proof:The lemma obviously holds for $m = 0$ or 1. Assumme now $m \geq 2$ and $f \in \mathcal{P}_m$ satisfies $r^2\Delta_0 f = \lambda f$, the we have $\Delta_0 f \in \mathcal{P}_{m-2}$ and:

$$\lambda\Delta_0 f = \Delta_0(r^2\Delta_0 f) = \Delta_0(r^2)\Delta_0 f + 4x^i\partial_i\Delta_0 f + r^2\Delta_0^2 f = (2n+4(m-2))\Delta_0 f + r^2\Delta_0^2 f$$

So we have:

$$r^2\Delta_0 f(\Delta_0 f) = (\lambda - 2n - 4m + 8)\Delta_0 f$$

This implies that, either $\Delta_0 f = 0$ are $(\lambda - 2n - 4m + 8)$ is an eigenvalue of $r^2\Delta_0$ on $\mathcal{P}_{m-2}$. In the latter case, we can write $f = \lambda^{-1}r^2\Delta_0 f$the proof is now by induction.□

Lemma 3.4. In the normal coordinate system of g, $\det(g_{ij})$ can be written as:

$$\det(g_{ij}) = 1 - \frac{1}{3}R_{ij}x^ix^j - \frac{1}{6}R_{ij,k}x^ix^jx^k - (\frac{1}{20}R_{ij},kl + \frac{1}{90}R_{hijm}R_{hklm} - \frac{1}{18}R_{ij}R_{kl})x^ix^jx^kx^l + O(r^5)$$

Proof: The geneous proof of this equation is by Jacobi field. Note that in the Normal coordiante, the variation of geodesic is just $\gamma_s(t) = t(\tau + s\xi)$, hence the Jacobi field is nothing but $X(\gamma_s(t)) = \frac{\partial}{\partial}\gamma_s(t) = t\xi$. And we have the Jacobi euqation $\nabla_T^2 X = R_T(X)$, where $R_T(X) = R(T,X)T$. Consider $f(t) = |X(\gamma_0(t))|^2$. Using Jacobi equation and initial condition $X(0) = 0$ and $\Delta_T X(0) = \xi$, we have

$$\nabla_T f(0) = 0, \nabla_T^2 f(0) = 2\langle \xi, \xi\rangle(0), \nabla_T^3 f(0) = 0$$
$$\nabla_T^4 f(0) = 8\langle R_\tau \xi, \xi\rangle(0), \nabla_T^5 f(0) = 20\langle(\nabla_\tau R\tau)\xi, \xi\rangle(0)$$
$$\nabla_T^6 f(0) = 36\langle(\nabla_\tau^2 R\tau)\xi, \xi\rangle(0) + 32\langle R_\tau \xi, R_\tau \xi\rangle(0)$$

Hence we get:

$$\begin{aligned}\langle \xi, \xi\rangle(t\tau) &= t^{-2}|X(\gamma_0(t))|^2 \\ &= \langle \xi, \xi\rangle + \frac{t^2}{3}\langle R_\tau \xi, \xi\rangle + \frac{t^3}{6}\langle(\nabla_\tau R_\tau)\xi, \xi\rangle + \frac{t^4}{20}\langle(\nabla_\tau^2 R\tau)\xi, \xi\rangle + \frac{2t^4}{45}\langle R_\tau \xi, R\tau\xi\rangle + O(t^5)\end{aligned}$$

all term are evaluate at 0. By polarization formula, we get:

$$g_{pq}(x) = \delta_{pq} + \frac{1}{3}R_{pijq}x^i x^j + \frac{1}{6}R_{[ijq,k}x^i x^j x^k + (\frac{1}{20}R_{pijq,kl} + \frac{2}{45}R_{pijm}R_{qklm})x^i x^j x^k x^l + O(r^5)$$

and the lemma follows.□

Now we can prove the theorem:

Proof of Theorem 3.1: We prove it by induction. Assume that g satisfies:

$$\det(g_{ij}) = 1 + O(r^N), N \geq 2$$

Consider the proof of Lemma 3.4., each term in the expansion of the determinent of order k takes the form:

$$c_k t^k[\langle(\nabla_\tau^{k-2}R_\tau)\xi, \xi\rangle + B_k(\xi, \xi)]$$

where c_k is a number, and B_k is a bilinear form with coefficients consisting of derivatives of R_τ of order less than $k-2$, hance the expansion can be written as:

$$\det(g_{ij}) = 1 + \sum_{|K|=N-2} c_K(R_{ij,K} - T_{ijK})x^i x^j x^K + O(r^{N+1})$$

where T_ijK is a symmetric tensor depending only on the drivatives of order less than $k-1$ of the curvature. Than we can using the defromation of Lemma 3.2 to kill T (since $f \in \mathcal{P}_N$, we have $T = \tilde{T}$).

Now assume that $N \geq 5$, Then from $det(g_ij) = 1 + O(5)$ we know the coefficients in Lemma 3.4 vanish. This means at P we have:

$$(a) R_{ij} = 0,$$
$$(b) R_{ij,k} + R_{jk,i} + R_{ki,j} = 0$$
$$(c) \mathrm{Sym}(R_{ij,kl} + \frac{2}{9}R_{pijm}R_{pklm}) = 0$$

(a) give us $R_{ijkl} = W_{ijki}$ and

$$R_{ij,kl} - R_{ij,lk} = R^m_{ikl} R_{mj} + R^m_{jkl} R_{im} = 0$$

together with (c) gives us:

$$\begin{aligned}
&(R_{ij,kl} + R_{kl,ij} + 2R_{ik,kl} + 2R_{jl,ik})x^i x^j + \\
&\frac{2}{9}(W_{pijm}W_{pklm} + W_{pikm}W_{pjlm} + W_{pkim}W_{pjlm} + \\
&W_{pjkm}W_{plim} + W_{pkjm}W_{plim} + W_{plkm}W_{pjim})x^i x^j = 0
\end{aligned}$$

Contracting i, j and by the symmetry of the Weyl tensor and Bianchi identity $(R_{,j}(P) = 2R_{ij,i})$ we get:

$$(3R_{,ij} + R_{ij,kk} + \frac{2}{3}W_{ipkm}W_{jpkm})x^i x^j = 0$$

Contract again, we get:

$$\Delta R = R_{,ii} = \frac{-1}{6}|W|^2$$

Finally, by (a) we have $R(P) = R_{ii}(P) = 0$ and by (b) we have $R_{,j}(P) = -2R_{ij,i}$ together with the Bianchi identity we get $R_{,j} = 0$.□

Now we discuss the expansion of Green's function in conformal coordinates when N sufficient large. Due to our goal, we can restrict on the case $\lambda(M) > 0$. In this case, by Lemma 2.4 there exists $u_s > 0$ satisfying $Lu_s = \lambda_s u_s^{s-1}$ for $2 < s < p$. Therefore, the scalar curvature of $g' = u_s^{p-2} g$ is $R' = a^{-1} u_s^{1-p} Lu_s = a^{-1}\lambda_s u_s^{s-p} > 0$. Hence the conformal Laplacian:$L' = -\Delta' + aR'$ has unique Green's function $G'_P \in C^\infty(M\backslash\{P\})$ such that:

$$L'G'_P = \delta_P, \text{ and } G'_P > 0$$

And we have:

$$L(u_s v) = u_s^{p-1} L'v$$

Hence

$$G_P = u_s(P) u_s G'_P$$

will be the Green's function of L. Hence we know that the Green's function of conformal Laplacian exist if $\lambda M > 0$. In particular, we can expand it in the conformal normal coordinate at P, then it is equal to(the asymptotic Euclidean Green's function):

$$G_P(x) = \frac{1}{(n-2)\omega_{n-1}} r^{2-n}(1 + o(1))$$

Defined $G(x) = (n-2)\omega_{n-1} G_p(x)$, thenw we get the following asymptotic expansion:

Theorem 3.5. In a conformal normal coordinate system, G has the following asymptotic expansion:

1. If n=3,4,5 or M is locally conformally flat in a neighborhood of P, then:

$$G = r^{2-n} + A + \alpha(x)$$

   where A is a constant, $\alpha = O(r)$, and $\alpha \in C^{2,\mu}$ unless $n = 4$; for $n = 4$, $\alpha = P_2(x)\log r + \alpha_0$, where $P_2(x)$ is a homogeneous polynomial of degree 2 and $\alpha_0 \in C^{2,\mu}$

2. For $n = 6$

$$G = r^{-4} - \frac{a}{288}|W(P)|^2 \log r + \alpha(x),$$

   where $\alpha(x) = P(x)\log r + \alpha_0$ for some polynomial P with $P(0) = 0$ and $\alpha_0 \in C^{2,\mu}$.

3. For $n \geq 7$,

$$G = r^{2-n}[1 + \frac{a}{12(n-4)}(\frac{r^4}{12(n-6)})|W(P)|^2 - R_{,ij}(P)x^i x^j r^2)] + \alpha(x),$$

   where $\alpha = (P(x)\log r + \alpha_0)r^{2-n}$ for some polynomial $P(x)$ and $\alpha_0 \in C^{2,\mu}$

Proof:We may write $G = r^{2-n}(1+\psi)$ the asymptotic expansion of G. If a function depends on r only, then in a normal coordinate system on has:

$$\Delta f = \frac{1}{r^{n-1}\sqrt{g}}\partial_r(r^{n-1}\sqrt{g}\partial_r f)$$

Indeed, using polar coordinate (r, ξ), where $\xi \in S^{n-1}$, the metric has expression by Gauss lemma:

$$g = dr^2 + h_{ij}(r,\xi)d\xi_i d\xi_j$$

and we have $\sqrt{h} = r^{n+1}\sqrt{g}$; hence the expression above. Using $g = 1 + O(r^N)$, we find that:

$$\Delta r^{2-n} = \Delta_0 r^{2-n} + \theta,$$

Where $\theta \in \mathcal{C}_{N'}$, the set of smooth functions on a neighborhood of the origin whose derivatives of up to N'-th order vanish at the origin. We can make N' arbitrary larger if we let N large enough. Since we have:

$$\Delta_0 r^{2-n} = -(n-2)\omega_{n-1}\delta_P \Rightarrow \Delta r^{2-n} = -(n-2)\omega_{n-1}\delta_P + \theta.$$

Thus the equation of Green's function becomes:

$$L(r^{2-n}\psi) + aRr^{2-n} = \theta$$

Using the notation:

$$L_0 = -r^2\Delta_0 + 2(n-2)r\partial_r$$

and

$$K = r^2(\Delta - \Delta_0) + 2(n-2)(r\partial_r - g^{ij}x^i\partial_j)$$

The the equation is equivalent to:

$$L_0\psi = K\psi + aRr^2(1+\psi) + \theta$$

To fund the asymptotic expansion, we want to fund $\bar{\psi} \in C^\infty(B\backslash\{0\})$ of $o(1)$ such that:

$$L(r^{2-n}\bar{\psi}) + aRr^{2-n} \in \mathcal{C}_1 \oplus \mathcal{C}_1 \log r$$

Which is equivalent to:

$$L_0\bar{\psi} - K\bar{\psi} - aRr^2(1+\bar{\psi}) \in \mathcal{C}_{n-1} \oplus \mathcal{C}_{n+1} \log r,$$

If we define $\phi = \psi - \bar{\psi}$, then:

$$L(r^{2-n}\phi) \in C^\mu(B)$$

we want to conclude $r^{2-n}\phi \in C^{2,\mu}$ by regularity. Let v be the solution of the following Dirichlet problem:

$$Lv = L(r^{2-n}\phi), \text{ and } v\mid_{\partial B} = r^{2-n}\phi$$

By reguarity againg, $v \in C^{2,\mu}$. But $w = r^{2-n}\phi - v$ satisfies $Lw = 0$ and $w|_{\partial B} = 0$. Since $\phi = o(1)$, we have $w = o(r^{2-n})$. Hence for any $\epsilon > 0$, we have $\epsilon G > w$ for $r = 1$(i.e. ∂B) and r sufficient small. Notice also $LG = 0$ on $B\backslash\{0\}$. By maximum principle apply to $\epsilon G - w$, we get $\epsilon G > w$. In particular $w \leq 0$. If we consider $-\epsilon G$ instead, we get $w \geq 0$. Hence $w = 0$ and $r^{2-n}\phi = v \in C^{2,\mu}$. As a result, we get $G = r^{2-n}(1+\bar{\psi}) + v$, where $v \in C^{2,\mu}$.

Now, we are aiming to find an appropriate $\bar{\psi}$ which satisfies the condition. Consider the first case n is odd. Suppose $\bar{\psi} = \psi_1 + ... + \psi_n$, where $\psi_k \in \mathcal{C}_k$. Consider the equation:

$$L_0\bar{\psi} - K\bar{\psi} - aRr^2(1+\bar{\psi}) \in \mathcal{C}_k$$

Since $R = O(r^2)$, we get $aRr^2 \in \mathcal{C}_4$, we can take $\psi_1 = \psi_2 = \psi_3 = 0$ satisfy the equation for $k \leq 4$. By induction, if we have $\bar{\psi} = \psi_1 + ... + \psi_{k-1}$ satisfies above equation, we can first write the right hand side by its Taylor expansion, i.e. $b_k + \mathcal{C}_{k+1}$ for $b_k \in \mathcal{P}_k$.. Since Lemma 3.3 asserts that L_0 is invertible on P_k for odd n (where $2k(n-2)$ is not equal to eigenvalues of $r^2\Delta_0$). Let $\psi_k = L_0^{-1}b_k$, Then $\bar{\psi} = \psi_1 + ... + \psi_k$ satisfies (3.11) with $\mathcal{C}_k$ replaced by C_{k+1}. By induction we are done.

Consider the case n is even. The above construction still holds for $k < n-2$ where L_0 is still invertible on $\mathcal{P}_k$. but not holds for $k \geq n-2$, where the invertibility is violated. The trick is that L_0 is self-adjoint on P_k with respect to the inner product $\langle \sum a_I x^I, \sum b_I x^I \rangle = \sum a_I b_I$. So we have $P_k = \mathrm{Im}L_0 \oplus \mathrm{Ker}L_0$. Now $\mathrm{Ker}L_0 \neq \{0\}$, we can take $\psi_k = p_k + q_k \log r$, where $p_k + q_k \in P_k$. Computations show:

$$L_0(p_k + q_k \log r) = L_0 p_k + (n-2-2k)q_k + (L_0 q_k)\log r$$

Since any $b_k \in P_k$ can be written as $b_k = L_0 + q_k$, where $L_0 q_k = 0$, hence we can take:

$$\psi_k = p_k + (n-2-2k)^{-1} q_k \log r.$$

When $k = n-2$, by Lemma 3.3, we have $\mathrm{Ker} L_0$ is spanned by r^{n-2}, Therefore:

$$\psi_{n-2} = p_{n-2} + c r^{n-2} \log r$$

Now, for $f = f(r)$, we haveL

$$Kf = r^2 \partial_i [(\frac{1}{3} R_{iklj} x^k x^l + \theta_1) r^{-1} x^j f'(r)]$$

where $\theta \in \mathcal{C}_3$. By symmetry of the curvature, $R_{iklj} x^l x^j = 0$. Hence if $f(r) = cr^{n-2} \log r$; Then we have $Kf \in \mathcal{C}_{n+1} \oplus \mathcal{C}_{n+1} \log r$. Hence, $K\psi_{n-2} \in \mathcal{C}_m \oplus \mathcal{C}_{n+1} \log r$.. Finally, we get:$\bar{\psi} = \psi_1 + ... + \psi_n$ satisfies:

$$L_0 \bar{\psi} - K\bar{\psi} - aRr^2(1+\bar{\psi}) \in \mathcal{C}_{n-1} \oplus \mathcal{C}_{n+1} \log r.$$

For $n = 3$, we get $\bar{\psi} = 0$, for $n = 5$, we get $\bar{\psi} = p_4 + q_4$, and for $n = 4$, $\bar{\psi} = \psi_4 = p_2(x) \log r$. Those cases are easy.

In the case where M is conformally flat near P, one may take Euclidean neighborhood of p, then $\Delta_0(r^{2-n}\psi) = 0$ in that neighorhood. By regularity, we have $r^{2-n}\psi \in C^\infty$. Hence (a) holds trivially.

Now for $n \geq 6$, $\bar{\psi} = \psi_4 + ... + \psi_n$, and we need only to find the leading term ψ_4. By previous analysis, we have:

$$L_0 \psi_4 = \frac{-a}{2} r^2 \partial_k \partial_l x^k x^l$$

For $n > 6$, using $\Delta R(P) = \frac{-1}{6}|W(P)|^2$, one substitues $\phi = r^2 b_{kl} x^k x^l$ to get:

$$\psi_4 = \frac{a}{12(n-4)} [\frac{r^4}{12(n-6)} |W(P)|^2 - R_{,kl}(P) x^k x^l r^2]$$

which is the result of (c). For $n = 6$. we using $\psi_4 = r^2(b_k l + c_k l \log r) x^k x^l$ instead, then we get:

$$\psi_4 = \frac{-a}{24} [R_{,kl}(P) x^k x^l r^2 + \frac{r^4}{12} |W(P)|^2 \log r],$$

which is the result of (b). □

For the next step, we need to using the positive mass theorem, to do so, we need some definition:

Definition 3.6. A Riemannian manifold (M, g) is called an asymptonically flat manifold of order r if $M = M_0 \cup M_\infty$, where M_0 is compact, and M_∞ is diffeomorphic to $\mathbb{R}^n \backslash B_R$ for some $R > 0$ and the differomophism provieds a coordinate system y^i on M^∞ such that:

$$g_{ij} = \delta_{ij} + O(|y|^{-r}), \partial_k g_{ij} = O(|y|^{-r-1}), \partial_k \partial_l g_{ij} = O(|y|^{-r-2})$$

This coordinate system is called an symptotic coordinate system.

Then we can assert the generalized positive mass theorem(ref. [4]):

Theorem 3.7. Let (M, g) be an n-dimensional asymptotic flat manifold of order $(n-2)$. Assume that in the asymptotic coordinates we have:

$$g_{ij} = (1 + \bar{A}\rho^{2-n})\delta_{ij} + h_i j$$

where $\bar{A}$ is a constant, $\rho = |y|$, $h_{ij} = O(\rho^{1-n}), \partial_k h_{ij} = O(\rho^{-n})$ and $\partial_k \partial_l h_{ij} = O(\rho^{-n-1})$. Assume further that the scalar curvature $R \geq 0$, $R \in L^1(M, g))$. Then $\bar{A} \geq 0$. $\bar{A} = 0$ iff (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.

Using the positve mass theorem, we can prove the result below:

Theorem 3.8. In $n = 3, 4, 5$ or M is conformally flat in a neighborhood of P, the constant A in the expansion of G is non negative. Moreover, $A = 0$ iff M is conformally equivalent to the standard S^n.

Proof: Consider $\hat{g} = G^{\frac{4}{n-2}} g$. Then $(M \backslash \{P\}, \hat{g})$ has scalar curvature $\hat{R} = 0$ since $LG = 0$. Futher, consider the expansion of G in conformal normal coordiantes and that $g_{ij} = \delta_{ij} + f_{ij}$, where $f_{ij} \in C^\infty$, $\partial_k f_{ij}(P) = 0$, we know that:

$$\hat{g}_{ij}(x) = r^{-4}(1 + \frac{4}{n-2} A r^{n-2})\delta_{ij} + \beta_{ij}(x)$$

where $\beta = O(r^{n-5})$, $\partial\beta = (r^{n-6})$, $\partial\partial\beta = O(r^{n-7})$. now take the asymptotic coordinates $\{y^i\}$ by $y_i = \frac{x^i}{|x|^2}$. Then in the new cooridnate system $\hat{g}_{ij}(y) = r^4 \hat{g}_{ij}(x)$, i.e.

$$\hat{g}_{ij}(y) = (1 + \frac{4}{n-2} A\rho^{2-n})\delta_{ij} + \beta_{ij}(x)$$

where $\bar{\beta}_{ij(y)} = \rho^{-4}\beta_{ij}(\frac{y}{|y|^2})$ satifies the condition of positive mass theorem. Hence $A \geq 0$ and $A = 0$ iff $(M \backslash \{P\}, \hat{g})$ is isometric to $\mathbb{R}$. Since $\mathbb{R}^n$ is conformal to $S^n \backslash \{\text{a point}\}$. not hard to see that (M, g) must be conformal to S^n. $\square$

4 Resolution of Yamabe Problem

Finally, we are equipped enough to prove the Yamabe problem:

Theorem 4.1. Let (M, g) be an $n(\geq 3)$-dimensional compact Riemannian manifold without boundary. There exists a conformal metric $\tilde{g} = \rho g$ such that the scalar curvature of $\tilde{g}$ is a constant.

Thanks to Theorem 2.1, it is suffice to prove the following result:

Theorem 4.2 Let (M, g) be an $n(\geq 3)$-dimensional compact Riemannian manifold without boundary. Suppose that (M, g) is not conformally equivalent to the standard S^n. Then $\lambda(M) < \lambda(S^n)$.

Proof: To prove the theorem, it suffice to construct a text function ϕ on M such that $Q(\phi) < \lambda(S) = \Lambda$. For this purpose, we defined a family of funtion on $\mathbb{R}^n$:

$$u_\epsilon(x) = (\frac{\epsilon}{\epsilon^2 + |x|^2})^{(n-2)/2}$$

By direct computation, we find that those function achieve the best Sobolev constant on $\mathbb{R}^n$ and we have:

$$\Delta u_\epsilon + n(n-2)u_\epsilon^{p-1} = 0$$

Thus we have:

$$\int_{\mathbb{R}^n} -u_\epsilon \Delta u_\epsilon dx = n(n-2)\int_{\mathbb{R}^n} u_\epsilon^p dx = \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 dx$$

So we have

$$\lambda(S^n) = \Lambda = \frac{\int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 dx}{\int_{(\mathbb{R}^n} u_\epsilon^p dx)^{2/p}} = n(n-2)(\int_{\mathbb{R}^n} u_\epsilon^p dx)^{2/n}$$

case (1): $n \geq 6$ and (M, g) is not a locally conformally flat manifold.

In this case, there exists a point $P \in M$ such that the Weyl tensor is not vanishing at P (ref.[1] p.235). Let x be a conformal normal coordinate system at P, and use this system to difined a cut-off function η such that $\eta = \eta(r)$ and $0 \leq \eta \leq 1$ with $\eta = 1$ on B_ρ and $\eta = 0$ ouside $B_{2\rho}$ for sufficiently small ρ. Furthermore, we require that $|\nabla\eta| \leq c\rho^{-1}$. Defined $\phi = \eta u\epsilon$, we claim that it is the desired test function for ϵ sufficiently small. Since ϕ depens on r only.

$$\begin{aligned}
\int_M |\nabla\phi|^2 d\mu &= \int_{B_{2\rho}} |\partial_r\phi|^2 \sqrt{g} dx \\
&\leq \int_{B_{2\rho}} |\partial_r\phi|^2 (1 + cr^N) dx \\
&= \int_{B_\rho} |\nabla u_\epsilon|^2 dx + c\int_{B_\rho} r^N |\nabla u_\epsilon|^2 dx + \int_{B_{2\rho}\setminus B_\rho} |\nabla(\eta u_\epsilon)|^2 (1 + cr^N) dx
\end{aligned}$$

where N is sufficiently big. By direct computation, we find that the second integral is of order $O(\epsilon^N)$ and the third integral is of order $O(\epsilon^{N-2})$ and the for first integral, we get:

$$\int_{B_\rho} |\nabla u_\epsilon|^2 dx = n(n-2)\int_{B_\rho} u_\epsilon^p dx + \int_{\partial B_\rho} u_\epsilon \frac{\partial u_\epsilon}{\partial r} ds$$

by the differential equation of u_ϵ and since $\partial_r u_\epsilon < 0$, we get:

$$\begin{aligned}\int_{B_\rho} |\nabla u_\epsilon|^2 dx &\le n(n-2)\int_{B_\rho} u_\epsilon^p dx \\ &= n(n-2)(\int_{B_\rho} u_\epsilon^p dx)^{2/n}(\int_{B_\rho} u_\epsilon^p dx)^{2/p} \\ &< \lambda(S^n)(\int_{B_\rho} u_\epsilon^p dx)^{2/p}\end{aligned}$$

Hence we get the first estimation:

$$\int_M |\nabla\phi|^2 d\,u < \lambda(S^n)(\int_{B_p} u_\epsilon^p dx)^{2/p} + c\epsilon^{n-2}$$

On the other hand,

$$\begin{aligned}\int_M \phi^p d\mu &= \int_{B_\rho} u_\epsilon^p \sqrt{g} dx + \int_{B_{2\rho}\setminus B_\rho} (\eta u_\epsilon)^p \sqrt{g} dx \\ &\ge \int_{B_\rho} u_\epsilon^p dx - c\int_{B_\rho} r^N u_\epsilon^p dx - \int_{B_{2\rho}\setminus B_\rho} (u_\epsilon)^p (1 + cr^N) dx \\ &\ge \int_{B_\rho} u_\epsilon^p dx - c\epsilon^n\end{aligned}$$

And in conformal normal coordinate, we have $R = O(r^2)$ amd $\nabla R(P) = \frac{-1}{6}|W(P)|^2$:

$$\begin{aligned}\int_M R\phi^2 d\mu &= \int_{B_{2\rho}} [\frac{1}{2}\partial_i\partial_j R(P)x^i + x^j + O(r^3)]\eta^2 u_\epsilon^2 dx \\ &\le \frac{1}{2}\int_{B_{2\rho}} \partial_i\partial_j R(P)x^i + x^j \eta^2 u_\epsilon^2 dx + c\int_{B_{2\rho}} \eta^2 u_\epsilon^2 r^3 dx \\ &= \frac{1}{2}\int_0^{2\rho} \eta^2 u_\epsilon^2 dr \int_{|x|=r} \partial_i\partial_j R(P)x^i x^j ds + c\int_{B_{2\rho}} \eta^2 u_\epsilon^2 r^3 dx \\ &= \frac{\omega_{n-1}}{2n}\Delta R(P)\int_0^{2\rho} \eta^2 u_\epsilon^2 r^{n+1} dr + c\omega_{n-1}\int_0^{2\rho} \eta^2 u_\epsilon^2 r^{n+2} dr \\ &\le -c_1|W(P)|^2 \int_0^{2\rho} \eta^2 u_\epsilon^2 r^{n+1} dr \\ &< -c_1|W(P)|^2 \int_0^{\rho} u_\epsilon^2 r^{n+1} dr\end{aligned}$$

And we have:

$$\int_0^\rho u_\epsilon^2 r^{n+1} dr = \int_0^\rho (\frac{\epsilon}{\epsilon^2 + r^2})^{n-2} r^{n+1} dr = \epsilon^4 \int_0^{\rho\epsilon} \frac{t^{n+1} dt}{(1+t^2)^{n-2}}$$

Directly compute the integral, we get: For $n = 6$

$$\int_M R\phi^2 d\mu \le -c|W(P)|^2 \epsilon^4 |\log\epsilon|$$

and $n \geq 7$

$$\int_M R\phi^2 d\mu \leq -c|W(P)|^2\epsilon^4$$

Combine all three estimate, we get for $n = 6$:

$$E(\phi) = \int_M |\nabla\phi|^2 d\mu + a\int_M R\phi^2 d\mu \leq \lambda(S^n)||\phi||_p^2 - c|W(P)|^2\epsilon^4|\log\epsilon| + O(\epsilon^4)$$

and for $n \geq 7$:

$$E(\phi) \leq \lambda(S^n)||\phi||_p^2 - c|W(P)|^2\epsilon^4 + O(\epsilon^(n-2))$$

Since $|W(P)| > 0$, we have $Q(u) < \lambda(S^n)$ for ϵ small enough.
case(2). $N \geq 6$ and (M, g) locally conformally float.

Let $P \in M$. Since M is locally conformally float, we may find a conformal normal coordinate system at P such that $g_{ij} = \delta_{ij}$. By Theorem 3.5., we have $G = r^{2-n} + A + \alpha(x)$, for $\alpha \in C^\infty$, and $\alpha(x) = O(r)$. Let $\rho > 0$ be sufficiently small, η be the cut-off function, then we define:

$$\phi(x) = \begin{cases} u_\epsilon(x) & \text{if } r \leq \rho, \\ \epsilon_0(G(x) - \eta(x)\alpha(x)) & \text{if } \rho \leq r \leq 2\rho, \\ \epsilon_0 G(x) & \text{otherwise} \end{cases}$$

Here, $\epsilon_0 > 0$, $\epsilon \ll \rho$ and we require:

$$\epsilon_0(\rho^{2-n} + A) = (\frac{\epsilon}{\epsilon^2 + \rho^2})^{\frac{n-2}{2}}$$

for the continuity. Since $R = 0$ in $B_{2\rho}$, we have:

$$\int_{M\setminus B_\rho}(|\nabla\phi|^2 + aR\phi^2)d\mu = \epsilon_0^2\int_{M\setminus B_\rho}(|\nabla G|^2 + aRG^2)d\mu + \int_{B_{2\rho}\setminus B_\rho}\epsilon_0^2(|\nabla(\eta\alpha)|^2 - 2\nabla G\cdot\nabla(\eta\alpha))d\mu$$

Since $\alpha = O(r)$, we have $\nabla\alpha = O(1)$ and $|\nabla(\eta\alpha) < C|$. Therefore, form $|\nabla G| \leq cr^{1-n}$in the ball, we see that the last integral in the above equation $\leq c\rho\epsilon_0^2$. Since $-\Delta G + aRG = 0$ in the region, we can take integration by parts to get:

$$\epsilon_0^2\int_{M\setminus B_\rho}(|\nabla G|^2 + aRG^2)d\mu \leq -\epsilon\int_{\partial B_\rho} G\frac{\partial G}{\partial r}ds$$

Hence:

$$\int_{M\setminus B_\rho}(|\nabla\phi|^2 + aR\phi^2)d\mu \leq -\epsilon\int_{\partial B_\rho} G\frac{\partial G}{\partial r}ds + \leq c\rho\epsilon_0^2$$

An, similarly to the case (1), we have:

$$\begin{aligned} \int_{B_\rho}(|\nabla(\phi)|^2 + aR\phi^2)d\mu &= \int_{B_\rho}|\Delta u_\epsilon|^2 dx \\ &= n(n-2)\int_{B_\rho} u_\epsilon^p dx + \int_{\partial B_\rho} u_\epsilon\frac{\partial u_\epsilon}{\partial r}ds \\ &\leq \lambda(S^n)(\int_{B_\rho} u_\epsilon^\rho dx)^{2/p} + \int_{\partial B_\rho} u_\epsilon\frac{\partial u_\epsilon}{\partial r}ds \end{aligned}$$

And a easy estimate:

$$\int_M \phi^p d\mu \geq \int_{B_\rho} \phi^p d\mu = \int_{B_\rho} u_\epsilon^p dx$$

Conbine them all, we get:

$$E(\phi) \leq \lambda(S^n)||\phi||_p^2 + c\rho\epsilon_0^2 + \int_{\partial B_\rho} (u_\epsilon \frac{\partial u_\epsilon}{\partial r} - \epsilon_0^2 G \frac{\partial G}{\partial r} ds)$$

At $r = \rho$, we get:

$$\epsilon_0^2 G \frac{\partial G}{\partial r} = -(n-2)\epsilon_0^2(\rho^{3-2n} + A\rho^{1-n} + O(\rho^{2-n}))$$

and:

$$u_\epsilon \frac{\partial u_\epsilon}{\partial r} = -(n-2)\epsilon_0^2(\rho^{3-2n} + 2A\rho^{1-n} + O(\rho^{-1}))$$

Hence, the integral $\leq -(n-2)\omega_{n-1}A\epsilon_0^2 + c\rho\epsilon_0^2$; hence we get:

$$E(\phi) \leq \lambda(S^n)||\phi||_p^2 + (-(n-2)\omega_{n-1}A + c\rho)\epsilon_0^2$$

Since c is indepedent of ρ and $A > 0$, we have $Q(\phi) < \lambda(S^n)$ for ρ small.
case(3). n=3,4 or 5

Let $P \in M$ and let x be a conformal normal coordinate system at P. By Theorem 3.5., $G = r^{2-n} + A + \alpha(x)$, with $\alpha(x) = O(r)$ and $\nabla\alpha = O(1)$. Take the same test function as in case 2, but we cannot assume conformally flat in this case, i.e., we only have:

$$g_{ij} = \delta_{ij} + O(r^2),\ g = 1 + O(r^N) \text{ and } R = O(r^2)$$

Then the calculation is modified by:

$$\begin{aligned}
&\int_{M\setminus B_\rho} (|\nabla\phi|^2 + aR\phi^2)d\mu \\
=&\int_{M\setminus B_\rho} \epsilon_0^2(|\nabla G|^2 + aRG^2)d\mu + \epsilon_0^2 \int_{M\setminus B_\rho} (|\nabla(\eta\alpha)|^2 - 2\nabla G \cdot \nabla(\eta\alpha) + aR(\eta^2\alpha^2 - 2\eta\alpha G)d\mu \\
\leq&\epsilon_0^2 \int_{M\setminus B_\rho} (|\nabla G|^2 + aRG^2)d\mu + c\rho\epsilon_0^2
\end{aligned}$$

and the integration by parts and $LG = 0$ give us:

$$\begin{aligned}
\int_{M\setminus B_\rho} (|\nabla G|^2 + aRG^2)d\mu &= -\int_{\partial B_p} G\sqrt{g}g^{ij}\partial_i G n_j ds \\
&= -\int_{\partial B_\rho} G\frac{\partial G_0}{\partial r}\sqrt{g}d\mu - \int_{\partial B_\rho} G\sqrt{g}g^{ij}\partial_i\alpha n_j ds
\end{aligned}$$

where $G_0 = r^{2-n} + A$ and n is the normal vector. Hence we have:

$$\int_{M\setminus B_\rho} (|\nabla\phi|^2 + aR\phi^2)d\mu \leq -\epsilon_0^2 \int_{\partial B_\rho} G\frac{\partial G_0}{\partial r}\sqrt{g}d\mu + c\rho\epsilon_0^2$$

And we have:

$$\int_{B_\rho} (|\nabla\phi|^2 + aR\phi^2)d\mu = \int_{B_\rho} (|\nabla\phi|^2 + aR\phi^2)\sqrt{g}dx \le \int_{B_\rho} |\nabla u_\epsilon|^2 dx + c\rho^{6-n}\epsilon_0^2$$

Therefore by the smilialr argument as in case(2), we get:

$$E(\phi) \le \lambda(S^n)||\phi||_p^2 + c\rho\epsilon_0^2 + \int_{\partial B_\rho} (u_\epsilon \frac{\partial u_\epsilon}{\partial r} - \epsilon_0^2 G \frac{\partial G}{\partial r})ds$$

which is the same as case (2) hence we get $Q(\phi) < \lambda(S^n)$.

In summary, in each cases, we indeed have $\lambda(M) < \lambda(S^n)$ hence prove the theorem.□

5 Reference

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Calabi conjecture

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Contents

Work distribution:

Shuang-Yen Lee: Section 2, 3, 8, 9.

Yi-Heng Tsai: Section 4, 5, 6, 7.

1 Introduction

This survey is mainly based on S-T Yau, On The Ricci Curvature of a Compact Kahler Manifold and the Complex Monge-Ampère Equation.

Our first goal is to solve the Calabi conjecture:

Theorem (Calabi conjecture)**.** Let M be a compact Kähler manifold with Kähler metric g. Let

$$\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$$

be a tensor whose associated $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta$ represents $\mathrm{c}_1(M)$. Then we can find a Kähler metric $\tilde{g}$ whose Ricci tensor is given by $\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$.

Furthermore, we can require that $\tilde{g}$ has the same Kähler class as g. In this case, $\tilde{g}$ is unique.

To solve this conjecture, we will see (in Section 4) that it suffices to prove the following theorem:

Theorem. Let $F \in C^{k\geq 3}(M)$ and $⨍_M e^F = 1$. Then there is $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ such that $\tilde{g} = (g_{i\overline{j}} + \varphi_{i\overline{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric and

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = e^F \det(g_{i\overline{j}}).$$

In the first three sections, we are going to use Schauder theory and continuity method to find a solution of this partial differential equation. Hence, we must have the second and third order estimates, which will be completely computed in Section 2 and 3. Similar to what we establish Hodge theory through Gårding's inequality, we can find a solution.

After proving the theorem and the Calabi conjecture, we consider the complex Monge-Ampère equation. In section 5, we will solve the equation

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = |s|^{2k} e^F \det(g_{i\overline{j}}),$$

where s is a nontrivial holomorphic section of a line bundle L. The main difference between these equations is whether the functions on the right-hand side vanish or not. To solve this problem, we consider the equation

$$\det(g_{i\overline{j}} + \varphi_{i\overline{j}}) = C_\varepsilon(|s|^2 + \varepsilon)^k e^F \det(g_{i\overline{j}}),$$

where C_ε is a suitable constant that will be determined later. Then by estimate the differentiability of φ_ε, we will get a solution when ε tends to zero.

In Section 6 $\sim$ 9, we consider more general right-hand side of the complex Monge-Ampère equation. For instance, we will replace the function $F(x)$ by $F(x, \varphi)$ and apply iteration method to solve it. In the end, we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\varphi)} \det(g_{i\bar{j}}),$$

where $t_i = \sum_{j=1}^{\ell} |s_j|^{2k_j}$ with $k_j \geq 0$ and s_j being a section of some holomorphic line bundle.

2 Estimates up to Second Order

Consider the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}) \tag{2.1}$$

where $F \in C^3(M)$.

We are going to find solutions φ of (2.1) such that $\tilde{g}_{i\bar{j}}\, dz^i \otimes d\bar{z}^j = (g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric on M.

Before proving the existence of φ, we need a priori estimates of φ. Since $F \in C^3(M)$, we assume that $\varphi \in C^5(M)$. We will give second order estimates of φ up to second derivatives under the normalization

$$\int_M \varphi = 0.$$

Differentiating (2.1), we get

$$F_k = \tilde{g}^{i\bar{j}} \left(g_{i\bar{j},k} + \varphi_{i\bar{j}k}\right) - g^{i\bar{j}} g_{i\bar{j},k} = \tilde{g}^{i\bar{j}} \varphi_{;i\bar{j}k}.$$

We differentiate the above equation again and obtain

$$\begin{aligned} F_{k\bar{\ell}} &= -\tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}} \left(g_{t\bar{n},\bar{\ell}} + \varphi_{t\bar{n}\bar{\ell}}\right) \left(g_{i\bar{j},k} + \varphi_{i\bar{j}k}\right) \\ &\quad + \tilde{g}^{i\bar{j}} \left(g_{i\bar{j},k\bar{\ell}} + \varphi_{i\bar{j}k\bar{\ell}}\right) + g^{t\bar{j}} g^{i\bar{n}} g_{t\bar{n},\bar{\ell}} g_{i\bar{j},k} - g^{i\bar{j}} g_{i\bar{j},k\bar{\ell}}. \\ &= \tilde{g}^{i\bar{j}} \varphi_{;i\bar{j}k\bar{\ell}} - \tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{;t\bar{n}\bar{\ell}} \varphi_{;i\bar{j}k}. \end{aligned} \tag{2.2}$$

Let $\widetilde{\triangle}$ be the Laplacian associated with the metric $\tilde{g}$. Then

$$\begin{aligned} \widetilde{\triangle}(\triangle\varphi) &= \tilde{g}^{k\bar{\ell}} \partial_k \overline{\partial}_\ell \left(g^{i\bar{j}} \varphi_{i\bar{j}}\right) \\ &= \tilde{g}^{k\bar{\ell}} g^{i\bar{j}} \varphi_{i\bar{j}k\bar{\ell}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,k\bar{\ell}} \varphi_{i\bar{j}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,k} \varphi_{i\bar{j}\bar{\ell}} + \tilde{g}^{k\bar{\ell}} g^{i\bar{j}}_{,\bar{\ell}} \varphi_{i\bar{j}k}. \end{aligned} \tag{2.3}$$

Since M is Kähler, we may take $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j},k} = g_{i\bar{j},\bar{\ell}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$. Then inserting (2.2) into (2.3), we have

$$\widetilde{\triangle}(\triangle\varphi) = \triangle F + \tilde{g}^{k\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{k\bar{n}\ell} \varphi_{i\bar{j}\bar{\ell}} + \tilde{g}^{i\bar{j}} R_{i\bar{j}\ell\bar{\ell}} - R_{i\bar{i}\ell\bar{\ell}} + \tilde{g}^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}}. \tag{2.4}$$

Since $\tilde{g}^{i\bar{j}} = \delta_{ij}(1+\varphi_{i\bar{i}})^{-1}$,

$$\begin{aligned}
\tilde{g}^{i\bar{j}} R_{i\bar{j}\ell\bar{\ell}} - R_{i\bar{i}\ell\bar{\ell}} + \tilde{g}^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}} &= -R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} + R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} \\
&= -R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \\
&= \frac{1}{2}\left(-R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{i\bar{i}}(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} - R_{i\bar{i}\ell\bar{\ell}} \frac{\varphi_{\ell\bar{\ell}}(\varphi_{i\bar{i}} - \varphi_{\ell\bar{\ell}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \right) \\
&= \frac{1}{2} R_{i\bar{i}\ell\bar{\ell}} \frac{(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \\
&\geq \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1}{2} \cdot \frac{(\varphi_{\ell\bar{\ell}} - \varphi_{i\bar{i}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\ell\bar{\ell}})} \right) \\
&= \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right).
\end{aligned}$$

Combining (2.4) and the above equation, we see that

$$\widetilde{\triangle}(\triangle\varphi) \geq \triangle F + \tilde{g}^{k\bar{j}} \tilde{g}^{i\bar{n}} \varphi_{k\bar{n}\bar{\ell}} \varphi_{i\bar{j}\ell} + \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right). \tag{2.5}$$

Let C be a positive constant. We want to estimate $e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi))$. Using (2.5) and Schwarz inequality, we have

$$\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &= \widetilde{\triangle}(\triangle\varphi) + C^2 |\widetilde{\nabla}\varphi|^2 (m+\triangle\varphi) \\
&\quad - C\left(2\langle \widetilde{\nabla}\varphi, \widetilde{\nabla}(\triangle\varphi) \rangle + (\widetilde{\triangle}\varphi)(m+\triangle\varphi) \right). \\
&\geq \widetilde{\triangle}(\triangle\varphi) - \frac{|\widetilde{\nabla}(\triangle\varphi)|^2}{m+\triangle\varphi} - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi) \\
&\geq \triangle F + \frac{\varphi_{k\bar{i}\bar{j}} \varphi_{i\bar{k}\bar{j}}}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})} - \frac{1}{m+\triangle\varphi} \sum_i \frac{|\sum \varphi_{k\bar{k}i}|^2}{1+\varphi_{i\bar{i}}} \\
&\quad + \left(\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \right) \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right) - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi).
\end{aligned} \tag{2.6}$$

By Schwarz inequality,

$$\begin{aligned}
\frac{1}{m+\triangle\varphi} \sum_i \frac{|\sum \varphi_{k\bar{k}i}|^2}{1+\varphi_{i\bar{i}}} &\leq \frac{1}{m+\triangle\varphi} \left(\sum \frac{\varphi_{k\bar{k}i} \varphi_{\bar{k}k\bar{i}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \right) \sum (1+\varphi_{k\bar{k}}) \\
&\leq \frac{\varphi_{k\bar{i}\bar{j}} \varphi_{i\bar{k}j}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})}.
\end{aligned} \tag{2.7}$$

Inserting the above equation into (2.6), we obtain

$$e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) \geq \triangle F + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \cdot \left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}} - m^2 \right) - C(\widetilde{\triangle}\varphi)(m+\triangle\varphi).$$

Note that

$$\widetilde{\triangle}\varphi = \sum \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} = m - \sum \frac{1}{1+\varphi_{i\bar{i}}}.$$

So, we get

$$
\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \cdot \left(\sum \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{\ell\bar{\ell}}}\right) \\
&\quad - Cm(m+\triangle\varphi) + C(m+\triangle\varphi)\sum \frac{1}{1+\varphi_{i\bar{i}}}. \\
&= \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi)\sum \frac{1}{1+\varphi_{i\bar{i}}}. \qquad (2.8)
\end{aligned}
$$

By AM-GM inequality,

$$
\sum \frac{1}{1+\varphi_{i\bar{i}}} \geq \left(\frac{\sum(1+\varphi_{i\bar{i}})}{\prod(1+\varphi_{i\bar{i}})}\right)^{1/(m-1)} = (m+\triangle\varphi)^{1/(m-1)} e^{-F/(m-1)}. \qquad (2.9)
$$

Choose C so that

$$
C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq 1.
$$

Then

$$
\begin{aligned}
e^{C\varphi}\widetilde{\triangle}(e^{-C\varphi}(m+\triangle\varphi)) &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{-F/(m-1)}(m+\triangle\varphi)^{1+1/(m-1)}. \qquad (2.10)
\end{aligned}
$$

By maximum principle, at some point x that $e^{-C\varphi}(m+\triangle\varphi)$ achieve its maximum, we have

$$
\begin{aligned}
0 &\geq \triangle F - m^2 \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - Cm(m+\triangle\varphi) \\
&\quad + \left(C + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{-F/(m-1)}(m+\triangle\varphi)^{1+1/(m-1)}.
\end{aligned}
$$

Hence $(m+\triangle\varphi)(x)$ has an upper bound C_1 depending only on $\sup(-\triangle F)$, $\sup|\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}|$, Cm and $\sup F$.

Since $e^{-C\varphi}(m+\triangle\varphi)$ achieves its maximum at x, we have the following inequality

$$
0 < m + \triangle\varphi \leq C_1 e^{C(\varphi - \inf\varphi)}. \qquad (2.11)
$$

We want to estimate $\sup|\varphi|$. Since

$$
m + \triangle\varphi = \sum_i (1+\varphi_{i\bar{i}}) = g^{i\bar{j}}\tilde{g}_{i\bar{j}} > 0,
$$

we can estimate $\sup\varphi$ by using the Green's function.

Let $G(p,y)$ be the Green's function of the operator $\triangle$ on M. Let A be a constant (depending only on M) such that $G(p,y)+A\geq 0$. Then

$$\varphi(p) = -\int_M G(p,y)\triangle\varphi(y)\,dy = -\int_M (G(p,y)+A)\triangle\varphi(y)\,dy$$

by the normalization of φ (which gives $\varphi \in \operatorname{Im}\triangle$). Therefore,

$$\sup\varphi \leq m\sup_p \int_M (G(p,y)+A)\,dy.$$

The inequality and the normalization also imply

$$\begin{aligned} \fint_M |\varphi| &\leq \fint_M |\sup\varphi - \varphi| + \fint_M |\sup\varphi| \\ &\leq 2m\sup_p \int_M (G(p,y)+A)\,dy. \end{aligned} \tag{2.12}$$

Let us now give an estimate of $-\inf\varphi$. Choose N large enough so that $N+\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq N/2$. Then, by (2.9),

$$\left(N+\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right) \geq \frac{N}{2}e^{-F/(m-1)}(m+\triangle\varphi)^{m/(m-1)}.$$

There is a constant C_1 depending only on $\sup F$ and m such that

$$\frac{N}{2}e^{-F/(m-1)}(m+\triangle\varphi)^{m/(m-1)} \geq 2Nm(m+\triangle\varphi) - NC_1.$$

Inserting above inequalities into (2.7) with C replaced by N, we get

$$e^{N\varphi}\widetilde{\triangle}(e^{-N\varphi}(m+\triangle\varphi)) \geq \triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_1 + Nm(m+\triangle\varphi).$$

Therefore,

$$\begin{aligned} &e^{N\varphi+F}\widetilde{\triangle}(e^{-N\varphi}(m+\triangle\varphi)) \\ &\geq e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3\right) + Ne^{\inf F}m(m+\triangle\varphi) \\ &= e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3 + m^2Ne^{\inf F - F}\right) + mNe^{\inf F}\triangle\varphi \\ &= e^F\left(\triangle F - m^2\inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} - NC_3 + m^2Ne^{\inf F - F}\right) + me^{\inf F}(-e^{N\varphi}\triangle e^{-N\varphi} + N^2|\nabla\varphi|^2) \\ &\geq me^{\inf F}(-e^{N\varphi}\triangle e^{-N\varphi} + N^2|\nabla\varphi|^2) - C_2, \end{aligned}$$

where C_2 depends only on N, F and M. Multiplying the above inequality by $e^{-N\varphi}$ and integrating, we get the inequality

$$\int_M |\nabla e^{-N\varphi/2}|^2 = \frac{N^2}{4}\int_M e^{-N\varphi}|\nabla\varphi|^2 \leq \frac{C_2}{4m}e^{-\inf F}\int_M e^{-N\varphi}.$$

Claim. We have an estimate of $\int_M e^{-N\varphi}$ (depending on N, F and M).

Proof of Claim. We are going to prove this statement by contradiction. Suppose there exists a sequence $\{\varphi_i\}$ satisfying the above inequality and (2.12) such that

$$\lim \int_M e^{-N\varphi_i} = \infty.$$

Then we define

$$e^{-N\tilde{\varphi}_i} = e^{-N\varphi_i}\left(\int_M e^{-N\varphi_i}\right)^{-1} \tag{2.13}$$

so that $\int_M e^{-N\tilde{\varphi}_i} = 1$.

It follows that $\int_M |\nabla e^{-N\tilde{\varphi}_i/2}|^2$ is uniformly bounded from above by a constant. Since $W^{1,2} \subset\subset L^2(M)$, there exists a subsequence of $e^{-N\tilde{\varphi}_i/2}$, which we may assume is itself, converges to $f \in L^2(M)$.

For any $\lambda > 0$,

$$\mathrm{Vol}\{x \mid \lambda \le e^{-N\tilde{\varphi}_i/2}\} = \mathrm{Vol}\left\{x \;\middle|\; \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i} \le -\varphi_i\right\}.$$

Since $\lim \int_M e^{-N\varphi_i/2} = \infty$, we conclude that, for i large enough,

$$\begin{aligned}\mathrm{Vol}\{x \mid \lambda \le e^{-N\tilde{\varphi}_i/2}\} &\le \mathrm{Vol}\left\{x \;\middle|\; \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i} \le |\varphi_i|\right\}\\ &\le \left(\frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M e^{-N\varphi_i}\right)^{-1}\int_M |\varphi_i|.\end{aligned}$$

By (2.12), $\int_M |\varphi_i|$ is uniformly bounded and thus,

$$\mathrm{Vol}\{x \mid \lambda \le e^{-N\tilde{\varphi}_i/2}\} \to 0$$

for all $\lambda > 0$. For all $\lambda > 0$, we get

$$\begin{aligned}\mathrm{Vol}\{x \mid \lambda \le f\} &\le \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \le |f - e^{-N\tilde{\varphi}_i/2}|\right\} + \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \le e^{-N\tilde{\varphi}_i/2}\right\}\\ &\le \frac{4}{\lambda^2}\int_M |f - e^{-N\tilde{\varphi}_i/2}|^2 + \mathrm{Vol}\left\{x \;\middle|\; \frac{\lambda}{2} \le e^{-N\tilde{\varphi}_i/2}\right\} \to 0.\end{aligned} \tag{2.14}$$

Since f is the L^2-limit of $e^{-N\tilde{\varphi}_i/2}$, f is zero almost everywhere. This is a contradiction because $\int_M f^2 = 1$. ■

Using (2.11) and the Schauder estimate, there are constants C_3 and C_4 depending only on M such that

$$\sup|\nabla\varphi| \le C_3\left(e^{-C\inf\varphi} + \int_M |\varphi|\right) \le C_4(e^{-C\inf\varphi} + 1). \tag{2.15}$$

We introduce the geodesic ball trick. Let q be a point in M where $\varphi(q) = \inf \varphi$. Then in the geodesic ball, with center q and radius

$$\frac{-\frac{1}{2}\inf \varphi}{C_4(e^{-C\inf \varphi}+1)},$$

φ is not greater than $\frac{1}{2}\inf \varphi$. Since we may assume $-\inf \varphi$ to be large (otherwise we get an upper bound), we may assume that the radius is smaller than $\mathrm{inj}(M)$. Then we choose N larger so that $N \geq 4mC$. Since

$$\int_B e^{-N\varphi} \geq e^{-N\inf \varphi/2}\,\mathrm{Vol}(B) \gtrsim e^{-N\inf \varphi/2}\left(\frac{-\frac{1}{2}\inf \varphi}{C_4(e^{-C\inf \varphi}+1)}\right)^{2m},$$

we have an estimate of $-\inf \varphi$.

Together with the estimate of $\sup \varphi$, we get an estimate of $\sup |\varphi|$. The inequalities (2.15) and (2.11) then give estimates of $\sup |\nabla \varphi|$ and $\sup(m + \triangle \varphi)$. Since $(\delta_{ij} + \varphi_{i\bar{j}})$ is positive definite, we can find upper estimates of $(1 + \varphi_{i\bar{i}})$ for each i. The equation $\prod_i (1 + \varphi_{i\bar{i}}) = e^F$ then gives a positive lower estimate of $(1 + \varphi_{i\bar{i}})$ for each i. Hence, the metric $\tilde{g}$ is uniformly equivalent to g.

So we get

Proposition 1. Let M be a compact Kähler manifold with metric g. Let φ be a real-valued function in $C^4(M)$ such that $\int_M \varphi = 0$ and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines another metrix tensor on M. Suppose

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Then there are positive constants $C_1 \sim C_4$, depending on $\inf F$, $\sup F$, $\inf \triangle F$ and M such that $\sup |\varphi| \leq C_1$, $\sup |\nabla \varphi| \leq C_2$ and $C_3 \cdot g \leq \tilde{g} \leq C_4 \cdot g$.

3 Third-Order Estimates

We now estimate the third derivatives $\varphi_{;i\bar{j}k}$ assuming φ solves the equation (2.1) and F is $C^3(M)$. Consider the function

$$S = \sum \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{;i\bar{j}k}\varphi_{;\bar{r}s\bar{t}} \geq 0.$$

We are going to compute $\widetilde{\triangle} S$. We say that

- $A \simeq B$ if $|A - B| \lesssim \sqrt{S} + 1$,
- $A \cong B$ if $|A - B| \lesssim S + \sqrt{S} + 1$.

Since $\tilde{g}$ is uniformly equivalent to g, we see that $\varphi_{;i\bar{j}k} \simeq 0$.

Claim. Take $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j},k} = g_{i\bar{j},\bar{\ell}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$ at a point. We have the following estimate:

$$\widetilde{\triangle} S \cong \frac{\left|\varphi_{;i\bar{j}k\alpha} - \dfrac{\varphi_{;\bar{i}p\bar{k}}\varphi_{;\bar{p}j\alpha}}{1+\varphi_{;p\bar{p}}}\right|^2 + \left|\varphi_{;i\bar{j}k\alpha} - \dfrac{\varphi_{;\bar{p}ia}\varphi_{;p\bar{j}k} + \varphi_{;\bar{p}ik}\varphi_{;p\bar{j}\alpha}}{1+\varphi_{;p\bar{p}}}\right|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{\alpha\bar{\alpha}})} \tag{3.1}$$

Proof of Claim. Since $\tilde{g}$ is uniformly equivalent to g,

$$\begin{aligned}
\widetilde{\triangle} S &= \tilde{g}^{\alpha\bar{\beta}} S_{\bar{\beta}\alpha} \\
&= \tilde{g}^{\alpha\bar{\beta}} \Big(-\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}\bar{\beta}}\tilde{g}^{k\bar{t}}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad -\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{p}}\tilde{g}^{q\bar{t}}\varphi_{\bar{p}q\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}} \Big)_\alpha \\
&\simeq \tilde{g}^{\alpha\bar{\beta}} \Big(-2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{p\bar{q}\beta}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad +\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}}) \Big)_\alpha \\
&\simeq \tilde{g}^{\alpha\bar{\beta}} \Big(2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad - 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{q\bar{p}\bar{\beta}\alpha}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\alpha}) \\
&\qquad + \tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}p}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}p}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\
&\qquad - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}} \left(\varphi_{p\bar{q}\bar{\beta}\alpha}\varphi_{i\bar{j}k}\varphi_{\bar{r}st} + \varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\alpha}\right) \\
&\qquad - (2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha})(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}st} + \varphi_{i\bar{j}k}\varphi_{\bar{r}st\bar{\beta}}) \\
&\qquad + g^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}\alpha}\varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}\alpha} + \varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}\bar{\beta}} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}\alpha}) \Big).
\end{aligned} \tag{3.2}$$

From the commutation formula, we have

$$\begin{aligned}\varphi_{i\bar{j}k\bar{\beta}\alpha} &= \varphi_{i\bar{j}\bar{\beta}k\alpha} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{j}\bar{\beta}k} - \varphi_{p\bar{j}}R^{\bar{p}}_{ik\bar{\beta}}\right)_{\alpha} \\ &= \varphi_{i\bar{\beta}\alpha\bar{j}k} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{\beta}\bar{j}\alpha} - \varphi_{p\bar{\beta}}R^{p}_{i\alpha\bar{j}}\right)_{k} + \left(\varphi_{i\bar{p}}R^{\bar{p}}_{\bar{j}\bar{\beta}k} - \varphi_{p\tilde{j}}R^{p}_{ik\bar{\beta}}\right)_{\alpha} \\ &\simeq \varphi_{i\bar{\beta}\alpha\bar{j}k}. \end{aligned} \tag{3.3}$$

We can see from (2.2) that

$$\tilde{g}^{i\bar{j}}\varphi_{;i\bar{j}k\bar{\ell}} = F_{k\bar{\ell}} + \tilde{g}^{t\bar{j}}\tilde{g}^{i\bar{n}}\varphi_{;t\bar{n}\bar{\ell}}\varphi_{;i\bar{j}k}. \tag{3.4}$$

Differentiating this one more time, we get

$$\tilde{g}^{i\bar{j}}\varphi_{;i\bar{j}k\bar{\ell}s} = \tilde{g}^{i\bar{t}}\tilde{g}^{n\bar{j}}\varphi_{;n\bar{t}s}\varphi_{;i\bar{j}k\bar{\ell}} + F_{k\bar{\ell}s} + \left(\tilde{g}^{t\bar{j}}g^{i\bar{n}}\varphi_{;t\bar{n}\bar{\ell}}\varphi_{;i\bar{j}k}\right)_{s}.$$

By (3.3),

$$\begin{aligned}\tilde{g}^{\alpha\beta}\varphi_{i\bar{j}k\bar{\beta}\alpha} &\simeq \tilde{g}^{\alpha\beta}\varphi_{\alpha\bar{\beta}i\bar{j}k} \\ &= \tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}i\bar{j}} + F_{i\bar{j}k} + \left(\tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i}\right)_{k} \\ &= \tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}i\bar{j}} + F_{i\bar{j}k} - \tilde{g}^{p\bar{a}}\tilde{g}^{b\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} \\ &\quad - \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{a}}\tilde{g}^{b\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}k}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}ik}. \end{aligned} \tag{3.5}$$

Using (3.2), (3.4) and (3.5), we get

$$\begin{aligned}\widetilde{\triangle}S \cong\ & 2\tilde{g}^{\alpha\bar{\beta}}\Big(\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ & +\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}}\Big) \\ & - 2\tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\Big(F_{q\bar{p}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{t\bar{\beta}}\tilde{g}^{\alpha\bar{n}}\varphi_{t\bar{n}\bar{p}}\varphi_{\alpha\bar{\beta}q}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ & +\tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}st} + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}st\alpha}\Big) \\ & + \tilde{g}^{\alpha\bar{\beta}}\Big(\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{q\bar{s}}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ & +\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{t}}\varphi_{\bar{a}b\alpha}\varphi_{p\bar{q}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}}\Big) \\ & - \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}p}\tilde{g}^{\bar{q}s}\tilde{g}^{k\bar{t}}\Big(F_{q\bar{p}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} + \tilde{g}^{t\bar{\beta}}\tilde{g}^{\alpha\bar{n}}\varphi_{t\bar{n}\bar{p}}\varphi_{\alpha\bar{\beta}q}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}} \\ & +\tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}st} + \tilde{g}^{\alpha\bar{\beta}}\varphi_{q\bar{p}\bar{\beta}}\varphi_{i\bar{j}k}\varphi_{\bar{r}st\alpha}\Big) \\ & - \tilde{g}^{\alpha\bar{\beta}}(2\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha} + \tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}b}\tilde{g}^{\bar{a}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{a}b\alpha})(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}\bar{\beta}}) \\ & + 2\,\mathrm{Re}\Big(\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}\varphi_{\bar{r}s\bar{t}}(\tilde{g}^{\alpha\bar{p}}\tilde{g}^{q\bar{\beta}}\varphi_{q\bar{p}k}\varphi_{\alpha\bar{\beta}ij} + F_{i\bar{j}k} - \tilde{g}^{p\bar{a}}\tilde{g}^{b\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} \\ & -\tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{a}}\tilde{g}^{b\bar{q}}\varphi_{\bar{a}bk}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}k}\varphi_{\alpha\bar{\beta}i} + \tilde{g}^{p\bar{\beta}}\tilde{g}^{\alpha\bar{q}}\varphi_{p\bar{q}\bar{j}}\varphi_{\alpha\bar{\beta}ik})\Big) \\ & + \tilde{g}^{\alpha\bar{\beta}}\tilde{g}^{i\bar{r}}\tilde{g}^{\bar{j}s}\tilde{g}^{k\bar{t}}(\varphi_{i\bar{j}k\bar{\beta}}\varphi_{\bar{r}s\bar{t}\alpha} + \varphi_{i\bar{j}k\alpha}\varphi_{\bar{r}s\bar{t}\bar{\beta}}). \end{aligned}$$

Take a coordinate such that at some point, $g_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j}k} = g_{i\bar{j}\bar{\ell}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}$. We get

$$\begin{aligned}
\widetilde{\triangle} S &\cong \frac{2\,\varphi_{\bar{i}p\alpha}\varphi_{q\overline{p\alpha}}\varphi_{i\bar{j}k}\varphi_{\overline{q}j\bar{k}} + 2\,\varphi_{\bar{k}p\alpha}\varphi_{q\bar{i}\overline{\alpha}}\varphi_{i\bar{j}k}\varphi_{\overline{q}j\overline{p}} + \varphi_{\overline{p}q\alpha}\varphi_{j\overline{q\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{i}p\bar{k}}}{(1+\varphi_{\alpha\overline{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{p\overline{p}})(1+\varphi_{q\overline{q}})} \\
&\quad - 2\,\mathrm{Re}\left(\frac{\varphi_{p\bar{i}\overline{\alpha}}\varphi_{i\bar{j}k\alpha}\varphi_{\overline{p}j\bar{k}} + \varphi_{j\overline{p\alpha}}\varphi_{i\bar{j}k}\varphi_{\bar{i}p\bar{k}\alpha} + \varphi_{\bar{i}p\alpha}\varphi_{i\bar{j}k}\varphi_{\overline{p}j\bar{k}\overline{\alpha}}}{(1+\varphi_{\alpha\overline{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})(1+\varphi_{p\overline{p}})}\right) \\
&\quad + \frac{|\varphi_{i\bar{j}k\alpha}|^2 + |\varphi_{i\bar{j}k\overline{\alpha}}|^2}{(1+\varphi_{\alpha\overline{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})} \\
&= \frac{\left|\varphi_{\bar{i}j\bar{k}\alpha} - \frac{\varphi_{\bar{i}p\bar{k}}\varphi_{\overline{p}j\alpha}}{1+\varphi_{p\overline{p}}}\right|^2 + \left|\varphi_{i\bar{j}k\alpha} - \frac{\varphi_{\overline{p}i\overline{\alpha}}\varphi_{p\bar{j}k} + \varphi_{\overline{p}i\bar{k}}\varphi_{p\bar{j}\alpha}}{1+\varphi_{p\overline{p}}}\right|^2}{(1+\varphi_{\alpha\overline{\alpha}})(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})(1+\varphi_{k\bar{k}})}. \qquad \square
\end{aligned}$$

By (2.5) and Proposition 1,

$$\widetilde{\triangle}(\triangle\varphi) \geq \sum \frac{\left|\varphi_{ki\bar{j}}\right|^2}{(1+\varphi_{k\bar{k}})\,(1+\varphi_{i\bar{i}})} - C_1,$$

where C_1 is a constant that can be estimated. Take C_2 large enough, we get

$$\widetilde{\triangle}\,(S + C_2\triangle\varphi) \geq -C_3(S + \sqrt{S} + 1) + C_2(C_4 S - C_1) \geq C_5 S - C_6,$$

where $C_2 \sim C_6$ are positive constants that can be estimated.

Using maximum principle, we see that

$$C_5\,(S + C_2\triangle\varphi) \leq C_6 + C_5 C_2 \triangle\varphi.$$

The estimate on $\triangle\varphi$ then gives an estimate of $\sup\,(S + C_2\triangle\varphi)$ and hence of $\sup S$. Finally, we get the estimates of $\varphi_{;i\bar{j}k}$ for all i, j, k.

Proposition 2. Let M be a compact Kähler manifold with metric g. Let φ be a real-valued function in $C^5(M)$ such that $\int_M \varphi = 0$ and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\,dz^i \otimes d\overline{z}^j$ defines another metric on M. Suppose

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Then there is an estimate of $\varphi_{;i\bar{j}k}$ in terms of g, $\sup|F|$, $\sup|\nabla F|$, $\sup\left(\sup_{i,j}\left|F_{i\bar{j}}\right|\right)$ and $\sup\left(\sup_{i,j,k}\left|F_{i\bar{j}k}\right|\right)$.

4 Solutions of the Equation

So we are going to solve the equation

$$\det(\tilde{g}_{i\bar{j}}) = e^F \det(g_{i\bar{j}}), \tag{4.1}$$

where F satisfies

$$⨍ e^F = 1. \tag{4.2}$$

With the estimates of Section 2 and Section 3, we shall now prove that if $F \in C^k(M)$ with $k \geq 3$ and F satisfies (4.2), then we can find a solution φ of (4.1) where $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$. ($C^{k+1,\alpha}(M)$ are the functions whose $(k+1)$-derivatives are Hölder continuous with exponent α.) We are going to use the continuity method. Consider the set

$$S = \left\{ t \in [0,1] \;\middle|\; \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{e^{tF}\det(g_{i\bar{j}})} = \left(⨍_M e^{tF} \right)^{-1} \text{ has a solution in } C^{k+1,a}(M) \right\}.$$

Since $0 \in S$, we need only to show that S is both closed and open in $[0,1]$.

S is open: Let

$$U = \left\{ \varphi \in C^{k+1,\alpha}(M) \;\middle|\; \int_M \varphi = 0 \text{ and } (g_{i\bar{j}} + \varphi_{i\bar{j}}) \text{ is positive definite.} \right\}$$

and

$$B = \left\{ f \in C^{k-1,\alpha}(M) \;\middle|\; ⨍_M f = 1 \right\}.$$

Then U is an open subset of a hyperplane in the Banach space $C^{k+1,\alpha}(M)$ and B is a hyperplane in the Banach space $C^{k-1,\alpha}(M)$. We have a map $G : U \to B$:

$$G(\varphi) = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})}.$$

We see that

$$dG_{\varphi_0} = \frac{\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})}{\det(g_{i\bar{j}})} \triangle_0,$$

where $\triangle_0$ is the Laplacian of the metric $(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\, dz^i \otimes d\bar{z}^j$.

It is well-known that the condition for $\triangle_0 \varphi = f$ to have a weak solution on M is that $\int_M f \, d\mathrm{Vol}_{\varphi_0} = 0$. Hence the condition for

$$\frac{\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}})}{\det(g_{i\bar{j}})} \triangle_0 \varphi = f$$

to have a weak solution is that $\int_M f = 0$. The Schauder theory makes sure that $\varphi \in C^{k+1,\alpha}(M)$ when $f \in C^{k-1,\alpha}(M)$, which is exactly the tangent space of B. The solution is unique if we assume that $\int_M \varphi = 0$. Hence dG_{φ_0} is invertible. By the inverse function theorem for Banach spaces, G maps an open neighborhood of φ_0 to an open neighborhood of $G(\varphi_0)$ in B. this proves that S is open.

S is closed: Let $\{t_q\}$ be a sequence in S with limit $t_0 \in [0,1]$. Then we have a sequence $\varphi_q \in C^{k+1,\alpha}(M)$ such that

$$\det\left(g_{i\bar{j}} + \varphi_{q,i\bar{j}}\right) = \left(\fint_M e^{t_q F}\right)^{-1} \cdot e^{t_q F} \det(g_{i\bar{j}}) \quad \text{and} \quad \int_M \varphi_q = 0.$$

Differentiating the above equation (in direction ∂_p), we have

$$\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right) \varphi_{q,p} = \left(\fint_M e^{t_q F}\right)^{-1} \cdot \partial_p(e^{t_q F} \det(g_{i\bar{j}})). \tag{4.3}$$

Proposition 1 and Proposition 2 shows that the operator $\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right)$ is uniformly elliptic and the coefficients are Hölder continuous with exponent α for any $0 \leq \alpha \leq 1$.

Using the Schauder estimate, we get an estimate on the $C^{2,\alpha}$-norm of $\varphi_{q,p}$ (and $\varphi_{q,\overline{p}}$ similarly). So the coefficients of $\left(\det(\tilde{g}_{q,i\bar{j}}) \cdot \tilde{g}_q^{i\bar{j}} \partial_i \overline{\partial}_j\right)$ have better differentiability. The Schauder estimate now gives better differentiability of $\varphi_{q,p}$ and $\varphi_{q,\overline{p}}$.

Iterating the process, we get $C^{k+1,\alpha}$-estimates of φ_q (since $F \in C^k(M)$). So the sequence $\{\varphi_q\}$ converges in the $C^{k+1,\alpha}$-norm for $\alpha \in [0,1)$ (by the compact embedding $C^{k+1,1} \to C^{k+1,\alpha}$) to a solution φ_0 of the equation

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{e^{t_0 F} \det(g_{i\bar{j}})} = \left(\fint_M e^{t_0 F}\right)^{-1}.$$

Hence S is closed.

Theorem 1. Assume that M is a compact Kähler manifold with metric g. Let F be $C^k(M)$ with $k \geq 3$ and $\fint_M e^F = 1$. Then there is a function φ in $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ such that $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric and

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}).$$

Corollary (Calabi conjecture)**.** Let M be a compact Kähler manifold with Kähler metric g. Let $\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \otimes d\overline{z}^\beta$ be a tensor whose associated $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\, dz^\alpha \wedge d\overline{z}^\beta$

represents $\mathrm{c}_1(M)$. Then we can find a Kähler metric $\tilde{g}$ whose Ricci tensor is given by $\tilde{R}_{\alpha\overline{\beta}}\,dz^\alpha \otimes d\overline{z}^\beta$. Furthermore, we can require that this Kähler metric has the same Kähler class as the original one. In this case, the required Kähler metric is unique.

Note that

$$R_{\alpha\overline{\beta}} = -\partial_\alpha \overline{\partial}_\beta \log \det(g_{i\overline{j}}). \tag{4.4}$$

Since we assume that $\frac{\sqrt{-1}}{2\pi}\tilde{R}_{\alpha\overline{\beta}}\,dz^\alpha \wedge d\overline{z}^\beta$ represents $\mathrm{c}_1(M)$, we see that

$$\tilde{R}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - \partial_\alpha \overline{\partial}_\beta f \tag{4.5}$$

for some smooth real-valued function f.

By Theorem 1, we can find a smooth function φ so that $(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}})\,dz^\alpha \otimes d\overline{z}^\beta$ defines a Kähler metric and that

$$\det(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}) = Ce^f \det(g_{\alpha\overline{\beta}}), \tag{4.6}$$

where C is a constant chosen to satisfy the equation

$$⨍_M Ce^f = 1.$$

From (4.4), (4.5) and (4.6), it is easy to see that $\tilde{R}_{\alpha\overline{\beta}}\,dz^\alpha \otimes d\overline{z}^\beta$ is the Ricci tensor of $(g_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}})\,dz^\alpha \otimes d\overline{z}^\beta$. This proves the Calabi conjecture.

Remark. The uniqueness was proved by Calabi and will also be indicated and proved in Theorem 2.

5 Complex Monge-Ampère Equation with Degenerate Right-Hand Side

Let L be a line bundle over M. Let s be a nontrivial holomorphic section of L. Suppose L is equipped with a Hermitian metric. Then we have a globally defined function $|s|^2$ on M.

For $k \geq 0$, we consider the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = |s|^{2k} e^F \det(g_{i\bar{j}}), \tag{5.1}$$

where F is a smooth function such that

$$\fint_M |s|^{2k} e^F = 1.$$

In order to solve (5.1), we approximate the equation by

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon (|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}}), \tag{5.2}$$

where $\varepsilon > 0$ is a small constant and

$$C_\varepsilon = \left(\fint_M (|s|^2 + \varepsilon)^k e^F \right)^{-1} \leq \left(\fint_M |s|^{2k} e^F \right)^{-1} = 1.$$

By Theorem 1, (5.2) has a smooth solution φ_ε such that $\left(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}}\right)$ is positive definite and

$$\int_M \varphi_\varepsilon = 0. \tag{5.3}$$

We are going to prove that when $\varepsilon \to 0^+$, φ_ε tends to a solution of (5.1). So we need some estimates of φ_ε which are independent of ε.

To estimate $\inf \varphi_\varepsilon$ and $\triangle \varphi_\varepsilon$ we notice that, when $s \neq 0$,

$$\triangle \log(|s|^2 + \varepsilon) = \frac{\triangle |s|^2}{|s|^2 + \varepsilon} - \frac{|\nabla |s|^2|^2}{(|s|^2 + \varepsilon)^2} \geq \frac{|s|^2}{|s|^2 + \varepsilon} \cdot \triangle \log |s|^2 \geq - \left| \triangle \log |s|^2 \right|. \tag{5.4}$$

Since $\triangle \log |s|^2$ is the trace of $\mathrm{c}_1(L)$ with respect to g for $s \neq 0$, we see that $\triangle \log(|s|^2 + \varepsilon)$ is uniformly bounded from below. Note that both sides of the above inequality are smooth. By taking limit to the points where $|s|^2$ vanish, we see that the above inequality holds on M.

Let $\triangle_\varepsilon$ be the Laplacian of the metric g_ε. Then according to (2.10), we have

$$e^{C\varphi_\varepsilon}\triangle_\varepsilon(e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)) \geq k\triangle\log(|s|^2+\varepsilon)+\triangle F - m^2\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}} - mC(m+\triangle\varphi_\varepsilon) + C_\varepsilon^{-1/(m-1)}\left(C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{(m+\triangle\varphi_\varepsilon)^{1+1/(m-1)}}{e^{F/(m-1)}(|s|^2+\varepsilon)^{k/(m-1)}}. \quad (5.5)$$

Same as in Section 2, we get

$$m+\triangle\varphi_\varepsilon \lesssim e^{C(\varphi_\varepsilon-\inf\varphi_\varepsilon)}. \quad (5.6)$$

For $s\neq 0$, $\triangle_\varepsilon\log|s|^2$ is dominated from below by the trace of $\mathrm{c}_1(L)$ with respect to g_ε. Hence there is a positive constant C_1 independent of ε such that

$$\triangle_\varepsilon\log|s|^2 \geq -C_1\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}. \quad (5.7)$$

Let p be any non-negative number. Then by Schwarz inequality, when $C>pC_1$,

$$\begin{aligned}
e^{C\varphi_\varepsilon}\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^p\right) &= \triangle_\varepsilon(|s|^2+\varepsilon)^p + 2\left\langle\nabla_\varepsilon(|s|^2+\varepsilon)^p,\nabla_\varepsilon e^{-C\varphi_\varepsilon}\right\rangle \\
&\quad + (|s|^2+\varepsilon)^p\left(\left|\nabla_\varepsilon e^{-C\varphi_\varepsilon}\right|^2 - C\triangle_\varepsilon\varphi_\varepsilon\right) \\
&\geq \triangle_\varepsilon(|s|^2+\varepsilon)^p - \frac{|\nabla_\varepsilon(|s|^2+\varepsilon)^p|^2}{(|s|^2+\varepsilon)^p} - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon \\
&= (|s|^2+\varepsilon)^p\triangle_\varepsilon\log(|s|^2+\varepsilon)^p - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon \\
&\geq -pC_1(|s|^2+\varepsilon)^p\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}} - C(|s|^2+\varepsilon)^p\triangle_\varepsilon\varphi_\varepsilon \\
&= (C-pC_1)(|s|^2+\varepsilon)^p\sum_i\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}} - mC(|s|^2+\varepsilon)^p \\
&\geq m(C-pC_1)\frac{(|s|^2+\varepsilon)^{p-k/m}}{C_\varepsilon^{1/m}e^{F/m}} - mC(|s|^2+\varepsilon)^p,
\end{aligned}$$

where the last inequality is due to the AM-GM inequality. Multiplying the above inequality by $(|s|^2+\varepsilon)^k e^{F-C\varphi_\varepsilon}$ and integrating, we get

$$\begin{aligned}
Ce^{\sup F}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p} &\geq C\int_M e^{F-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p} \\
&\geq (C-pC_1)C_\varepsilon^{-1/m}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{(m-1)k/m+p}e^{(m-1)F/m} \\
&\gtrsim (C-pC_1)\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{(m-1)k/m+p}.
\end{aligned}$$

By the above inequality, we see that, for all $q\in\left[\frac{m-1}{m}k+p,k+p\right]$, there exists a positive constant C_2 such that

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^q \leq C_2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p}.$$

Hence, for $n \in \mathbb{N}$ such that $p - \frac{(n-1)k}{m} \geq 0$,

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{nk}{m}} \lesssim \cdots \lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{k}{m}} \lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p},$$

Let n be the largest integer so that $p - \frac{(n-1)k}{m} \geq 0$. Then we have $k \in \big[k+p-\frac{nk}{m}, k+p-\frac{(n-1)k}{m}\big]$ and hence,

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k \leq C_3' \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p-\frac{nk}{m}} \leq \cdots \leq C_3 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+p}. \quad (5.8)$$

for some $C_3, C_3' >$). By (5.5), we can find positive constants C_4 and C_5 such that

$$e^{C\varphi_\varepsilon} \triangle_\varepsilon \left(e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right) \geq C_4(m+\triangle\varphi_\varepsilon) - C_5.$$

Multiplying the above inequality by $(|s|^2+\varepsilon)^k e^{F-C\varphi_\varepsilon}$ and integrating, we obtain

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k (m+\triangle\varphi_\varepsilon) \leq \frac{C_5\, e^{\sup F - \inf F}}{C_4} \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.$$

Since $m + \triangle\varphi_\varepsilon > 0$, it follows from the above inequality that we can find a positive constant C_6 independent of ε (for ε small) such that

$$\begin{aligned}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\triangle\varphi_\varepsilon &\leq \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}(m+\triangle\varphi_\varepsilon)\\ &\leq C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

Integrating by parts in the above inequality, we derive

$$\begin{aligned}C\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}|\nabla\varphi_\varepsilon|^2 &\leq (k+1)\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k \left\langle \nabla\varphi_\varepsilon, \nabla|s|^2\right\rangle\\ &\quad + C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k\\ &\leq \frac{(k+1)^2}{C}\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2\\ &\quad + \frac{1}{4}C\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}|\nabla\varphi_\varepsilon|^2\\ &\quad + C_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{3}{4}C^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}|\nabla\varphi_\varepsilon|^2 &\leq (k+1)^2\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla|s|^2\right|^2\\ &\quad + CC_6\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^k.\end{aligned}$$

On $|s| \neq 0$,

$$\left|\nabla |s|^2\right|^2 = |s|^2 \triangle |s|^2 - |s|^4 \triangle \log |s|^2.$$

Note that $\triangle |s|^2$ and $|s|^2$ are upper bounded and $\triangle \log |s|^2$ is lower bounded. So we see that

$$\left|\nabla |s|^2\right|^2 \leq \left(\sup \triangle |s|^2 + \max\{\sup |s|^2 \cdot \sup(-\triangle \log |s|^2), 0\}\right) \cdot |s|^2.$$

Since both side are smooth on M, we see that $\left|\nabla |s|^2\right|^2$ is dominated by $|s|^2$ on M. Together with (5.8), we see that

$$\begin{aligned}
&\int_M \left|\nabla \left(e^{-C\varphi_\varepsilon/2}(|s|^2+\varepsilon)^{(k+1)/2}\right)\right|^2 \\
&\leq \frac{1}{2}(k+1)^2 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla |s|^2\right|^2 + \frac{1}{2}C^2 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}\left|\nabla \varphi_\varepsilon\right|^2 \\
&\leq \frac{7}{6}(k+1)^2 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k-1}\left|\nabla |s|^2\right|^2 + \frac{2}{3}CC_6 \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k} \\
&\lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k} \\
&\lesssim \int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}. \qquad (5.9)
\end{aligned}$$

Using the Green's function as before, we get an estimate of $\int_M |\varphi_\varepsilon|$ that is independent of ε, we apply the normalization trick in Section 2 that (5.9) gives an estimate of

$$\int_M e^{-C\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1}$$

independent of ε. (Suppose there is no estimate. Then we can find a sequence $\varepsilon_j \to 0$ such that $\int_M e^{-C\varphi_{\varepsilon_j}}(|s|^2+\varepsilon_{\varepsilon_j})^{k+1}$ tends to infinity. Then we define

$$e^{-C\tilde{\varphi}_j} = e^{-C\varphi_{\varepsilon_j}} \left(\int_M e^{-C\varphi_{\varepsilon_j}} \left(|s|^2+\varepsilon_j\right)^{k+1}\right)^{-1}.$$

By (5.9), $\left(|s|^2+\varepsilon_j\right)^{(k+1)/2} e^{-\frac{1}{2}C\tilde{\varphi}_j}$ converges to some f in $L^2(M)$. Using the L^1-estimate of $|\varphi_\varepsilon|$ on the set $\{x \in M \mid |s| \geqslant 1/n\}$, we see that $f \equiv 0$ a.e. and get a contradiction.)

As in (2.15), inequality (5.6) and the estimate of $\sup \varphi_\varepsilon$ give an estimate of

$$\frac{|\nabla \varphi_\varepsilon|}{e^{-C\inf \varphi_\varepsilon}+1}$$

independent of ε. Now we use the geodesic ball trick. For some geodesic ball B of radius

$$R = \frac{C_7(-\inf \varphi_\varepsilon)}{e^{-C\inf \varphi_\varepsilon}+1},$$

φ_ε is not greater than $\frac{1}{2}\inf\varphi_\varepsilon$. (Here C_7 is a positive constant independent of ε, and R is less than the injectivity radius of M.) We see that

$$\begin{aligned}\int_B e^{-N\inf\varphi_\varepsilon}(|s|^2+\varepsilon)^{k+1} &\geq e^{-N\inf\varphi_\varepsilon/2}\int_B |s|^{2(k+1)}\\ &\gtrsim e^{-N\inf\varphi_\varepsilon/2}\int_0^R r^{a(k+1)}dr\\ &\geq \frac{1}{2a(k+1)}e^{-N\inf\varphi_\varepsilon/2}\left(\frac{C_7(-\inf\varphi_\varepsilon)}{e^{-C\inf\varphi_\varepsilon}+1}\right)^{ak+a+1}.\end{aligned}$$

By choosing $N > 2C(ak+a+1)$, we get an estimate of $-\inf\varphi_\varepsilon$ independent of ε and (5.6) gives an upper estimate of $m+\triangle\varphi_\varepsilon$ independent of ε.

Now we want to find the third-order estimate. Let $\rho \geq 0$ be a smooth function in M with $\operatorname{supp}\rho \subseteq K$. Since $(|s|^2+\varepsilon)^k e^F$ has a uniform lower bound over K, the metric g_ε is uniformly equivalent to g.

As in Section 3, we define

$$S_\varepsilon = g_\varepsilon^{i\bar{r}} g_\varepsilon^{\bar{j}s} g_\varepsilon^{k\bar{t}}\varphi_{\varepsilon;i\bar{j}k}\varphi_{\varepsilon;\bar{r}s\bar{t}}.$$

From (2.6), we can find positive constants C_8 and C_9 independent of ε such that

$$\rho\triangle_\varepsilon(\triangle\varphi_\varepsilon) \geq C_8\rho S_\varepsilon - C_9\rho$$

Integrating the above inequality with respect to the volume form $(|s|^2+\varepsilon)^k e^F\, d\mathrm{Vol}$, we see that

$$C_8\int_M \rho S_\varepsilon(|s|^2+\varepsilon)^k e^F \leq C_9\int_M \rho(|s|^2+\varepsilon)^k e^F + \int_M \triangle_\varepsilon\rho\cdot\triangle\varphi_\varepsilon\cdot(|s|^2+\varepsilon)^k e^F.$$

Note that the RHS can be estimated. Since $\inf|s| > 0$ on K, we can find an estimate of $\int_M \rho S_\varepsilon$ independent of ε.

Since the compact set K and the function ρ are chosen arbitrary, we see that we have found an L^1-estimate of S_ε over any compact subset K of M which is disjoint from the divisor of s. Say

$$\int_K S_\varepsilon < C_K, \tag{5.10}$$

where C_K is independent to ε.

Let

$$B(R) = \left\{(z_1,\cdots,z_m)\,\middle|\,\sum_i |z_i|^2 \leq R\right\} \subseteq K$$

be a coordinate chart. We want to estimate $S_\varepsilon(0)$ by the L_1-norm of S_ε over $B(R)$.

Using the computations of Section 3, we know that there are positive constants C_{10} and C_{11} independent of ε and φ_ε, such that on $B(R)$,

$$\triangle_\varepsilon \left(S_\varepsilon + C_{10}\triangle\varphi_\varepsilon + C_{11}\sum_i |z_i|^2 \right) \geq C_{12}S_\varepsilon - C_{13} + C_{11}\triangle_\varepsilon \left(\sum_i |z_i|^2 \right) > 0.$$

We may also assume that the function $\overline{S}_\varepsilon = S_\varepsilon + C_{10}\triangle\varphi_\varepsilon + C_{11}\left(\sum_i |z_i|^2 + 1\right) > 0$.

The Dirichlet problem

$$\begin{cases} \triangle_\varepsilon \psi = 0 & \text{on } B(R), \\ \psi = \overline{S}_\varepsilon & \text{on } \partial B(R). \end{cases}$$

has a smooth solution $\tilde{S}_\varepsilon$. By the maximum principle, $\tilde{S}_\varepsilon \geq \overline{S}_\varepsilon > 0$ in $B(R)$.

Since g_ε is uniformly equivalent to g on $B(R)$, we know that $\tilde{S}_\varepsilon$ is a solution of a uniform elliptic equation of divergence form whose elliplicity is estimated (this means that the eigenvalues have a uniform bound).

By Moser's Harnack inequality

$$\sup_{B(R)} \tilde{S}_\varepsilon \lesssim \inf_{B(R)} \tilde{S}_\varepsilon,$$

we get

$$\tilde{S}_\varepsilon(0) \lesssim \int_{B(R)} \tilde{S}_\varepsilon. \tag{5.11}$$

Let σ be a non-decreasing C^∞-function defined on $\mathbb{R}$ such that

(i) $\sigma(t) = 0$ for $t \leq 0$,

(ii) $\sigma(t) = 1$ for $t \geq \delta$ and

(iii) $\sigma'(t) \leq \frac{2}{\delta}$ for all t.

For $\tau < R$, we define $\psi_\tau(s) = \int_s^\infty t\sigma(\tau - t)\,dt$. We see that $\psi_\tau(r) = \psi_\tau((\sum_i |z_i|^2)^{1/2})$ vanishes outside a compact subset of the interior of $B(R)$.

By direct computation, we have

$$\begin{aligned} \triangle_\varepsilon \psi_\tau(r) = g_\varepsilon^{i\overline{j}}\partial_i\overline{\partial}_j\psi_\tau(r) &= r\sigma'(\tau - r)g_\varepsilon^{i\overline{j}}(\partial_i r)(\overline{\partial}_j r) - \frac{1}{2}\sigma(\tau - r)g_\varepsilon^{i\overline{j}}\partial_i\overline{\partial}_j r^2 \\ &= r\sigma'(\tau - r)g_\varepsilon^{i\overline{j}}(\partial_i r)(\overline{\partial}_j r) - \frac{1}{2}\sigma(\tau - r)g_\varepsilon^{i\overline{i}}. \end{aligned}$$

Multiplying the above equation by $\tilde{S}_\varepsilon \det(g_{\varepsilon p\bar{q}})$ and integrating with respect to the Euclidean volume form dE, we obtain (by integration by parts)

$$\begin{aligned} 0 &= \int_{B(R)} (\triangle_\varepsilon \tilde{S}_\varepsilon)\psi_\tau(r) \det(g_{\varepsilon p\bar{q}})\, dE = \int_{B(R)} \tilde{S}_\varepsilon \left(\triangle_\varepsilon \psi_\tau(r)\right) \det(g_{\varepsilon p\bar{q}})\, dE \\ &= \int_{B(R)} \tilde{S}_\varepsilon r\sigma'(\tau - r) g_\varepsilon^{i\bar{j}}(\partial_i r)(\overline{\partial}_j r) \det(g_{\varepsilon p\bar{q}})\, dE - \frac{1}{2}\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r) g_\varepsilon^{i\bar{j}} \det(g_{\varepsilon p\bar{q}})\, dE. \end{aligned}$$

Since $\sigma \geq 0$, and $\sigma' \geq 0$, it follows from the above equation that

$$\begin{aligned} &\frac{1}{2}\inf_{B(R)}\left(g_\varepsilon^{i\bar{j}} \det(g_{\varepsilon p\bar{q}})\right)\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r)\, dE \\ &\leq \sup_{B(R)}\left(r g_\varepsilon^{i\bar{j}}(\partial_i r)(\overline{\partial}_j r) \det(g_{\varepsilon p\bar{q}})\right)\int_{B(R)} \tilde{S}_\varepsilon \sigma'(\tau - r)\, dE. \end{aligned}$$

Therefore, by the uniform bound of g_ε, we can find a positive constant C_{14} independent of σ, τ, ε such that

$$\int_{B(R)} \tilde{S}_\varepsilon \sigma(\tau - r)\, dE \leq C_{14}\int_{B(R)} \tilde{S}_\varepsilon \sigma'(\tau - r)\, dE.$$

Letting $\tau \to R^-$, we may replace τ by R in the above inequality. Then

$$\int_{B(R-\delta)} \tilde{S}_\varepsilon dE \leq \frac{2C_{14}}{\delta}\int_{B(R)\setminus B(R-\delta)} \tilde{S}_\varepsilon dE.$$

Letting $\delta \to 0^+$, we see that $\int_{B(R)} \tilde{S}_\varepsilon dE$ can be estimated by $\int_{\partial B(R)} \tilde{S}_\varepsilon$. Since $\overline{S}_\varepsilon|_{\partial B(R)} = \tilde{S}_\varepsilon|_{\partial B(R)}$ and $\tilde{S}_\varepsilon > 0$, we conclude from (5.11) that there is a positive constant C_{15} independent of φ_ε and ε such that

$$\overline{S}_\varepsilon(0) \leq \tilde{S}_\varepsilon(0) \leq C_{15}\int_{\partial B(R)} \overline{S}_\varepsilon.$$

Since C_{15} can be chosen to be independent of R when $B(R)$ lies in K, we can integrate the above inequality (over R) to find an estimate of $\overline{S}_\varepsilon(0)$ in terms of the L^1-norm of $\overline{S}_\varepsilon$ over K. Together with the L^1-estimate (5.10) of $\overline{S}_\varepsilon$, we get an estimate of $\overline{S}_\varepsilon$ on K.

Using the method in Section 4, we can estimate the higher derivatives of φ_ε. Differentiate

$$\det\left(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}}\right) = C_\varepsilon(|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}})$$

in direction ∂_k. Then

$$\left(g_\varepsilon^{i\bar{j}}\partial_i\overline{\partial}_j\right)\varphi_{\varepsilon,k} = \partial_k\left(\log\left(C_\varepsilon(|s|^2 + \varepsilon)^k e^F \det(g_{i\bar{j}})\right)\right).$$

Since we have Lipschitz estimates (Hölder exponent 1) of these coefficients over K, the Schauder estimate shows that all higher derivatives of φ_ε can be estimated over these sets.

Since K is arbitrary, by letting $\varepsilon \to 0^+$, we can now conclude that $\{\varphi_\varepsilon\}$ has a subsequence converging to a solution φ of (5.1) such that φ is smooth outside of the divisor of s and $\left\{|\varphi_{i\bar{j}}|\right\}$ is bounded for all i, j.

Theorem 2. Let L be a holomorphic line bundle over a compact Kähler manifold M. Let s be a holomorphic section of L. Let g be the Kähler metric of M. Then, for any $k \geq 0$ and any smooth function F with $⨍_M |s|^{2k} e^F = 1$, we can find a solution φ of the equation

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = |s|^{2k} e^F \det(g_{i\bar{j}})$$

with the following properties:

(i) φ is smooth outside the divisor of s, and

(ii) $\triangle\varphi$ is bounded over M

Furthermore, any function ψ satisfying the above properties must be equal to φ plus a constant.

Proof. We only need to prove the last statement. We claim that, if f is a function such that $\left\{|f_{i\bar{j}}|\right\}$ is bounded over M for all i, j, then

$$\int_M (\widetilde{\triangle} f)|s|^{2k} e^F = 0. \tag{5.12}$$

Indeed, if we let $c(g_\varepsilon)_{i\bar{j}}$ be the $(i, \bar{j})$-th cofactor of the matrix $\left(g_{\varepsilon, i\bar{j}}\right)$, we have

$$\int_M (\triangle_\varepsilon f)\,(|s|^2 + \varepsilon)^k e^F = \int_M c(g_\varepsilon)_{i\bar{j}} f_{i\bar{j}}\, dz^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^m = 0. \tag{5.13}$$

Since $c(g_\varepsilon)_{i\bar{j}}$ and $f_{i\bar{j}}$ are bounded independent of ε, we can use the Lebesgue dominated convergence theorem to obtain (5.12) from (5.13).

Now let ψ be another solution of (5.1) satisfying the properties mentioned in the theorem. Then we have

$$\frac{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi - \varphi)_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)} = 1.$$

Using the AM-GM inequality, we have

$$\widetilde{\triangle}(\psi - \varphi) = \frac{1}{m}\left(m + \widetilde{\triangle}(\psi - \varphi)\right) - 1 \geq 0.$$

Since $\left|\psi_{i\bar{j}}\right|$ and $\left|\varphi_{i\bar{j}}\right|$ are both bounded, $\left|(\psi-\varphi)^2_{i\bar{j}}\right|$ is also bounded over M and $\psi-\varphi\in C^1(M)$. We may assume that $\psi-\varphi\geq 0$ by adding a constant to $\psi-\varphi$. Then applying (5.12) to $f=(\psi-\varphi)^2$, we obtain

$$2\int_M(\psi-\varphi)\widetilde{\triangle}(\psi-\varphi)+2\int_M\left|\widetilde{\nabla}(\psi-\varphi)\right|^2=\int_M\widetilde{\triangle}\left((\psi-\varphi)^2\right)=0.$$

Since $(\psi-\varphi)\geq 0$ and $\widetilde{\triangle}(\psi-\varphi)\geq 0$, we conclude that $\widetilde{\nabla}(\psi-\varphi)=0$ and $\psi-\varphi$ is a constant. ■

6 Complex Monge-Ampère Equation with More General Right-Hand Side

Consider the following equation:

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{6.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

If such φ exists, then integrating (6.1), the integral of the RHS is equal to the volume of M. So we assume that there exists a smooth function ψ such that

$$\fint_M e^{F(x,\psi)} = 1.$$

We are going to use an iteration method to solve (6.1).

Lemma 1 (Uniqueness of the solution of (6.1))**.** Let φ and ψ be two smooth solutions of (6.1) such that both $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\overline{z}^j$ and $\left(g_{i\bar{j}} + \psi_{i\bar{j}}\right) dz^i \otimes d\overline{z}^j$ define Kähler metrics on M. Then $\varphi - \psi$ is a constant.

Proof. Note that

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi - \varphi)_{i\bar{j}})}{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})} = e^{F(x,\psi) - F(x,\varphi)}.$$

Let $\triangle_\varphi$ be the normalized metric Laplacian of the metric $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$. Then it follows from the AM-GM inequality and the above equation that we have the inequality

$$m + \triangle_\varphi(\varphi - \psi) \geq m e^{(F(x,\psi) - F(x,\varphi))/m}.$$

By the mean value theorem we have

$$F(x,\psi) - F(x,\varphi) = \int_{\varphi(x)}^{\psi(x)} F_t(x,\tau)\, d\tau = F_t(x, \bar{t}(x))(\psi(x) - \varphi(x)),$$

where $\bar{t}(x)$ is a number between $\inf\{\varphi(x), \psi(x)\}$ and $\sup\{\varphi(x), \psi(x)\}$.

Since $F_t \geq 0$, we can combine the inequality and the equation above to conclude that whenever $\psi(x) - \varphi(x)$ is strictly positive, $\triangle_\varphi(\psi - \varphi)(x)$ is nonnegative.

Suppose $\sup(\psi - \varphi)(x) > 0$. By the maximal principle we see that $\psi - \varphi$ is locally constant on the set $\{x \in M \mid (\psi - \varphi)(x) > 0\}$. Interchanging φ and ψ, we see that $\psi - \varphi$ must be a constant function. ■

We now introduce the iteration method. By Theorem 1, we can find a smooth function φ_0 such that $(g_{i\bar{j}} + \varphi_{0,i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric and

$$\det(g_{i\bar{j}} + \varphi_{0,i\bar{j}}) = e^{F(x,\psi)} \det(g_{i\bar{j}}). \tag{6.2}$$

If we define

$$\varphi_0^{\pm} = \varphi_0 \pm \sup|\varphi_0 - \psi|,$$

then both φ_0^+ and φ_0^- satisfy the equation.

The set $A = \{(x,t) \mid x \in M, \varphi_0^+(x) \geq t \geq \varphi_0^-(x)\}$ is a compact subset of $M \times \mathbb{R}$. Hence we can define

$$k = \sup_{(x,t)\in A} F_t(x,t) + 1 > 0.$$

For each $i \geq 1$, we define φ_i^+ and φ_i^- as the smooth solutions of the following equations:

$$\det\bigl(g_{\alpha\bar{\beta}} + \varphi^{\pm}_{i,\alpha\bar{\beta}}\bigr) = e^{k(\varphi_i^{\pm} - \varphi_{i-1}^{\pm}) + F(x,\varphi_{i-1}^{\pm})} \det(g_{\alpha\bar{\beta}}) \tag{6.3}$$

so that $g_i^{\pm} = \bigl(g_{\alpha\bar{\beta}} + \varphi^{\pm}_{i,\alpha\bar{\beta}}\bigr)\, dz^\alpha \otimes d\bar{z}^\beta$ define Kähler metrics.

Lemma 2 (Existence of $\varphi_i^{\pm}$)**.** Let M be a compact Kähler manifold with Kähler metric g. Let $F(x)$ be any smooth function defined on M. Then, for any constant $\bar{k} > 0$, there exists a unique smooth function φ such that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\bar{k}\varphi + F} \det(g_{i\bar{j}})$$

and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric.

Proof. As in Theorem 1, we can use the continuation method where the one parameter family (with parameter t) of equations is

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\bar{k}\varphi + tF} \det(g_{i\bar{j}}).$$

By maximum principle and AM-GM inequality, when φ achieves its maximum at a point x_0, we must have

$$e^{\bar{k}\varphi(x_0) + tF(x_0)} = \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} \leq 1.$$

This implies immediately $\sup\varphi \leq -(t/\bar{k})F(x_0)$. Similarly one can draw an estimate of $\inf\varphi$. Since $\bar{k} > 0$, the uniqueness part follows from Lemma 1. ∎

Claim. For all $i \geq 0$, $\varphi_i^- \leq \varphi_{i+1}^- \leq \varphi_{i+1}^+ \leq \varphi_i^+$.

Proof of Claim. The proof is almost based on the maximum principle and AM-GM inequality. We induction on i. For $i = 0$, we see that

$$\det\left(g_{\alpha\bar\beta} + \varphi^+_{1,\alpha\bar\beta}\right) = e^{k\varphi_1^+ - k\varphi_0^+} e^{F(x,\varphi_0^+)} \det(g_{\alpha\bar\beta})$$
$$\geq e^{k(\varphi_1^+ - \varphi_0^+)} e^{F(x,\psi)} \det(g_{\alpha\bar\beta}) = e^{k(\varphi_1^+ - \varphi_0^+)} \det\left(g_{\alpha\bar\beta} + \varphi^+_{0,\alpha\bar\beta}\right).$$

At the point where $\varphi_1^+ - \varphi_0^+$ achieves its maximum, by AM-GM inequlity,

$$\det\left(g_{\alpha\bar\beta} + \varphi^+_{1,\alpha\bar\beta}\right) \leq \det\left(g_{\alpha\bar\beta} + \varphi^+_{0,\alpha\bar\beta}\right).$$

Hence $\sup(\varphi_1^+ - \varphi_0^+) \leq 0$. Similarly, $\sup(\varphi_0^- - \varphi_1^-) \leq 0$.

To show that $\varphi_1^- \leq \varphi_1^+$, by (6.3) we see that

$$\frac{\det\left(g_{\alpha\bar\beta} + \varphi^+_{1,\alpha\bar\beta}\right)}{\det\left(g_{\alpha\bar\beta} + \varphi^-_{1,\alpha\bar\beta}\right)} = e^{k(\varphi_1^+ - \varphi_1^-) + F(x,\varphi_0^+) - F(x,\varphi_0^-) - k(\varphi_0^+ - \varphi_0^-)}.$$

Since $\varphi_0^+ \geq \varphi_0^-$, by mean value theorem we get

$$F(x, \varphi_0^+) - F(x, \varphi_0^-) - k(\varphi_0^+ - \varphi_0^-) \leq 0.$$

Therefore

$$\frac{\det\left(g_{\alpha\bar\beta} + \varphi^+_{1,\alpha\bar\beta}\right)}{\det\left(g_{\alpha\bar\beta} + \varphi^-_{1,\alpha\bar\beta}\right)} \leq e^{k(\varphi_1^+ - \varphi_1^-)}.$$

At the point where $\varphi_1^+ - \varphi_1^-$ achieves its minimum, (by maximum principle and AM-GM inequality,) the RHS of the above inequality is greater than or equal to 1 and hence $\varphi_1^+ \geq \varphi_1^-$.

For general i. Applying (6.3) twice, we have

$$\frac{\det\left(g_{\alpha\bar\beta} + \varphi^+_{i+1,\alpha\bar\beta}\right)}{\det\left(g_{\alpha\bar\beta} + \varphi^+_{i,\alpha\bar\beta}\right)} = e^{k(\varphi_{i+1}^+ - \varphi_i^+) + F(x,\varphi_i^+) - F(x,\varphi_{i-1}^+) - k(\varphi_i^+ - \varphi_{i-1}^+)} \geq e^{k(\varphi_{i+1}^+ - \varphi_i^+)},$$

where the inequality is due to MVT. Hence the maximal principle shows that $\varphi_i^+ \geq \varphi_{i+1}^+$. Similarly one can show that $\varphi_i^- \leq \varphi_{i+1}^-$.

To prove that $\varphi_{i+1}^+ \geq \varphi_{i+1}^-$, by (6.3) we see that

$$\frac{\det\left(g_{\alpha\bar\beta} + \varphi^+_{i+1,\alpha\bar\beta}\right)}{\det\left(g_{\alpha\bar\beta} + \varphi^-_{i+1,\alpha\bar\beta}\right)} = e^{k(\varphi_{i+1}^+ - \varphi_{i+1}^-) + F(x,\varphi_i^+) - F(x,\varphi_i^-) - k(\varphi_i^+ - \varphi_i^-)}.$$

Using $\varphi_i^+ \geq \varphi_i^-$, one can repeat the above argument to show that $\varphi_i^+ \geq \varphi_i^-$. ☐

Therefore both φ_i^+ and φ_i^- are uniformly bounded. Again, we want to find a uniform estimate of $\varphi^+_{i,\alpha\bar\beta}$. As in Section 2, it suffices to estimate $\triangle\varphi_i^+$.

Let $\triangle_i^+$ be the Laplacian operator associated with the metric g_i^+. Let C be any positive constant such that $C + \inf_{i \neq \ell} R_{i\bar{i}\ell\bar{\ell}} > 1$. Then by same computation as in (2.8), we have

$$\begin{aligned}
e^{C\varphi_i^+} \triangle_i^+ (e^{-C\varphi_i^+}(m + \triangle\varphi_i^+)) = & \; k(\triangle\varphi_i^+ - \triangle\varphi_{i-1}^+) + g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}}(x, \varphi_{i-1}^+) \\
& + g^{\alpha\overline{\beta}} F_{t\alpha}(x, \varphi_{i-1}^+)\varphi_{i-1,\overline{\beta}}^+ + g^{\alpha\overline{\beta}} F_{t\overline{\beta}}(x, \varphi_{i-1}^+)\varphi_{i-1,\alpha}^+ \\
& + F_{tt}(x, \varphi_{i-1}^+)|\nabla\varphi_{i-1}^+|^2 + F_t(x, \varphi_{i-1}^+)\triangle\varphi_{i-1}^+ \\
& - Cm(m + \triangle\varphi_i^+) \\
& + \left(C + \inf_{i \neq \ell} R_{i\bar{i}\ell\bar{\ell}}\right)(m + \triangle\varphi_i^+) \sum \frac{1}{1 + \varphi_{i,\alpha\overline{\alpha}}^+}.
\end{aligned}$$

Since $\sup|\varphi_i^+|$ has been estimated, it follows from Schauder's estimate that

$$\sup|\nabla\varphi_i^+| \lesssim \left(\sup|\triangle\varphi_i^+| + 1\right).$$

As in (2.9),

$$\begin{aligned}
\sum \frac{1}{1 + \varphi_{i,\alpha\overline{\alpha}}^+} & \geq (m + \triangle\varphi_i^+)^{1/(m-1)} e^{(-k(\varphi_i^+ - \varphi_{i-1}^+) + F(x, \varphi_{i-1}^+))/(m-1)} \\
& \gtrsim (m + \triangle\varphi_i^+)^{1/(m-1)}.
\end{aligned}$$

Noting again that $\sup|\varphi_i^+|$ has been estimated, it follows from above inequalities that there are positive constants C_1, C_2, independent of i, such that

$$\begin{aligned}
& e^{C\varphi_i^+} \triangle_i^+ (e^{-C\varphi_i^+}(m + \triangle\varphi_i^+)) \\
& \geq C_1(m + \triangle\varphi_i^+)^{1+1/(m-1)} - C_2\left((m + \triangle\varphi_i^+) + (m + \sup\triangle\varphi_{i-1}^+) + 1\right)
\end{aligned}$$

At the point where $e^{-C\varphi_i^+}(m+\triangle\varphi_i^+)$ achieves its maximum, the RHS must be non-positive and so

$$C_1\left(m + \sup\triangle\varphi_i^+\right)^{1+1/(m-1)} \leq e^{\frac{mC}{m-1}\sup\varphi_i^+} C_2\left((m + \sup\triangle\varphi_i^+) + (m + \sup\triangle\varphi_{i-1}^+) + 1\right)$$

Then we can find a positive constant C_3, independent of i, such that

$$\left(m + \sup\triangle\varphi_i^+\right) \leq \frac{1}{2}\left(m + \sup\triangle\varphi_{i-1}^+\right) + C_3.$$

By iteration, this gives

$$m + \sup\triangle\varphi_i^+ \leq \frac{m + \sup\triangle\varphi_0^+}{2^i} + 2C_3.$$

Therefore we have found estimates for $\varphi_{i,\alpha\overline{\beta}}^+$. To find uniform estimate of $\varphi_{i,\alpha\overline{\beta}\gamma}^+$, let

$$S_i = g_i^{+\alpha\overline{\ell}} g_i^{+\overline{\beta}p} g_i^{+\gamma\overline{q}} \varphi_{i,\alpha\overline{\beta}\gamma}^+ \varphi_{i,\overline{\ell}p\overline{q}}^+.$$

By a computation similar to that of (3.1), we have

$$\triangle_i^+(S_i + C_4\triangle\varphi_i^+) \geq C_5 S_i - C_6\sqrt{S_i}\sqrt{S_{i-1}} - C_7, \tag{6.4}$$

where $C_4 \sim C_7$ are positive constants independent of i.

Since $|\triangle\varphi_i^+|$ has been estimated, it follows from the maximum principle that

$$\sup S_i \leq \frac{C_6}{C_5}\sqrt{\sup S_i}\sqrt{\sup S_{i-1}} + \frac{C_7}{C_5} + C_4 \sup|\triangle\varphi_i^+|.$$

It should be noted that in (6.4), we can choose C_5 to be arbitrarily large if we are allowed to increase C_4 and C_7. In particular, we may assume that $2C_6 \leq C_5$. By AM-GM inequality,

$$\frac{C_6}{C_5}\sqrt{\sup S_i}\sqrt{\sup S_{i-1}} \leq \frac{3}{4}\sup S_i + \frac{1}{12}\sup S_{i-1}.$$

Then we get

$$\sup S_i \leq \frac{1}{3}\sup S_{i-1} + \frac{4C_7}{C_5} + 4C_4 \sup|\triangle\varphi_i^+|.$$

By iteration, we can find a uniform estimate of S_i and hence a uniform estimate of $\varphi^+_{i,\alpha\bar{\beta}\gamma}$.

Letting $i \to \infty$, we can then obtain a solution of (6.1). The Schauder estimate guarantees the solution to be smooth.

Theorem 3. Let M be a compact Kähler manifold with Kähler metric g. Let $F(x,t)$ be a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$. Suppose that, for some smooth function ψ defined on M,

$$\fint_M e^{F(x,\psi(x))} = 1.$$

Then there exists a smooth function φ on M such that

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{F(x,\varphi(x))}\det(g_{i\bar{j}})$$

and $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a Kähler metric. Furthermore, any other smooth function satisfying the same property differs from φ by only a constant.

Corollary. Let M be a Kähler manifold with ample canonical line bundle. Then there is a Kähler-Einstein metric whose Ricci tensor is the negative of the metric tensor. Furthermore, a metric of this form is unique and depends only on the complex structure of M.

By hypothesis, $-\,\mathrm{c}_1(M)$ is represented by some positive $(1,1)$-form $\sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$. Take this form as our Kähler form. Since the closed $(1,1)$-form $-\partial\overline{\partial}\log\det(g_{i\bar{j}})$ also represents $\mathrm{c}_1(M)$, we can find a smooth function f such that

$$\partial\overline{\partial}\log\det(g_{i\bar{j}}) = \sqrt{-1}g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j + \partial\overline{\partial} f.$$

Now by Theorem 3, we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{\varphi - f}\det(g_{i\bar{j}})$$

so that $(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j$ defines a Kähler metric. By these equations we have

$$\begin{aligned}-\partial\overline{\partial}\log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) &= -\partial\overline{\partial}\varphi + \partial\overline{\partial} f - \sqrt{-1}g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j - \partial\overline{\partial} f \\ &= -\sqrt{-1}(g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \wedge d\bar{z}^j.\end{aligned}$$

This is indeed the metric we want.

For the uniqueness. Suppose that $\tilde{g}_{i\bar{j}}$ is another such metric. Then its Kähler form must represent $-\,\mathrm{c}_1(M)$. Hence we can find a smooth function ψ defined on M such that $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \psi_{i\bar{j}}$. Together with the fact that $-\tilde{R} = \tilde{g}$, we get

$$\begin{aligned}-\partial\overline{\partial}\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) &= -\sqrt{-1}g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j - \partial\overline{\partial}\psi \\ &= -\partial\overline{\partial}\log\det(g_{i\bar{j}}) + \partial\overline{\partial} f - \partial\overline{\partial}\psi,\end{aligned}$$

which is equivalent to

$$\partial\overline{\partial}\log\left(\frac{\det(g_{i\bar{j}} + \psi_{i\bar{j}})}{\det(g_{i\bar{j}})} e^{f-\psi}\right) = 0$$

Therefore,

$$\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = e^{\psi + c - f}\det(g_{i\bar{j}})$$

for some c. The function $\psi + c$ then satisfies the equation. Lemma 1 shows that $\varphi - \psi$ is a constant. Hence,

$$(g_{i\bar{j}} + \psi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j = (g_{i\bar{j}} + \varphi_{i\bar{j}})\, dz^i \otimes d\bar{z}^j.$$

7 Degenerate Complex Monge-Ampère Equation with General Right-Hand Side

In this section, we combine the main results of the last two sections.

Let L be a line bundle over M. Let s be a nontrivial holomorphic section of L. Suppose L is equipped with a Hermitian metric so that the function $|s|^2$ is globally defined on M. For $k \geq 0$, we consider the equation

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) = |s|^{2k} e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{7.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

As in Section 6, we assume that there is a function ψ whose partial derivatives $\psi_{i\bar{j}}$ are uniformly bounded on M so that

$$⨍_M |s|^{2k} e^{F(x,\psi)} = 1.$$

We approximate (7.1) by

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon(|s|^2 + \varepsilon)^k e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{7.2}$$

where $\varepsilon > 0$ is a smooth constant and

$$C_\varepsilon = \left(⨍_M (|s|^2 + \varepsilon)^k e^{F(x,\psi_\varepsilon)} \right)^{-1}.$$

Consider a sequence of smooth functions $\{\psi_\varepsilon\}$ such that $\psi_\varepsilon \to \psi$ uniformly on M and that $\sup\left|\psi_{\varepsilon,i\bar{j}}\right|$ is uniformly bounded on every coordinate chart.

By Theorem 3, we can find smooth solutions φ_ε of (7.2) such that $(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}})\, dz^i \otimes d\overline{z}^j$ defines a metric. As in the proof of Theorem 3, we get an estimate of $\sup|\varphi_\varepsilon|$ in the following way.

Let φ_ε^+ and φ_ε^- be two smooth solutions of the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon(|s|^2 + \varepsilon)^k e^{F(x,\psi_\varepsilon)} \det(g_{i\bar{j}}) \tag{7.3}$$

such that $\varphi_\varepsilon^+ \geq \psi_\varepsilon \geq \varphi_\varepsilon^-$. Then the arguments of Theorem 3 show that $\varphi_\varepsilon^+ \geq \varphi_\varepsilon \geq \varphi_\varepsilon^-$.

On the other hand, for the unique solution of (7.3) with $\int_M \varphi = 0$, we can find an estimate of $\sup|\varphi|$ which is independent of ε. (This is seen in the proof of Theorem 3. Note that boundedness of $\triangle\psi_\varepsilon$ is needed.) In particular,

$$\sup|\varphi_\varepsilon| \leq \max\{\sup|\varphi_\varepsilon^-|, \sup|\varphi_\varepsilon^+|\} \leq \sup|\varphi| + \sup|\varphi - \psi_\varepsilon| \leq 2\sup|\varphi| + \sup|\psi_\varepsilon|.$$

is bounded from above by a constant independent of ε.

Let us now proceed to estimate $\triangle\varphi_\varepsilon$ from above. Then, as in (5.5), we have

$$\begin{aligned} e^{C\varphi_\varepsilon}\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}\left(m+\triangle\varphi_\varepsilon\right)\right) \geq\ & g^{i\bar{j}}F_{i\bar{j}}+g^{i\bar{j}}F_{it}\varphi_{\varepsilon,\bar{j}}+g^{i\bar{j}}F_{t\bar{j}}\varphi_{\varepsilon,i}+g^{i\bar{j}}F_{tt}\varphi_{\varepsilon,i}\varphi_{\varepsilon,\bar{j}}-mF_t \\ & +k\triangle\log(|s|^2+\varepsilon)-m^2\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}-mC\left(m+\triangle\varphi_\varepsilon\right) \\ & +C_\varepsilon^{-1/(m-1)}\left(C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{\left(m+\triangle\varphi_\varepsilon\right)^{m/(m-1)}}{e^{F/(m-1)}(|s|^2+\varepsilon)^{k/(m-1)}}. \end{aligned}$$

Choose C so that $C+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\geq\frac{1}{2}C\geq 1$. Then noting (5.4) and the fact that $\sup|\varphi_\varepsilon|$ is bounded, we can find positive constants C_1 and C_2 independent of ε such that

$$\triangle_\varepsilon\left(e^{-C\varphi_\varepsilon}\left(m+\triangle\varphi_\varepsilon\right)\right)\geq C_1\left(m+\triangle\varphi_\varepsilon\right)^{m/(m-1)}-C_2\left(\left(m+\triangle\varphi_\varepsilon\right)+|\nabla\varphi_\varepsilon|+1\right). \quad (7.4)$$

On the other hand, by Schauder's estimate and the estimate of $\sup|\varphi_\varepsilon|$, we have

$$\sup|\nabla\varphi_\varepsilon|\lesssim\left(\sup|\triangle\varphi_\varepsilon|+\sup|\varphi_\varepsilon|\right)\lesssim\left(\sup\left(m+\triangle\varphi_\varepsilon\right)+1\right).$$

By the maximum principle, we get an upper estimate of $m+\triangle\varphi_\varepsilon$. Therefore, we have uniform estimates of $\left|\varphi_{\varepsilon,i\bar{j}}\right|$ on every coordinate chart of M.

Using the uniform estimate of $\varphi_{\varepsilon,i\bar{j}}$, we follow the arguments of Section 5 to provide higher derivative estimates of φ_ε on compact subsets of the complement of the divisor of s. Letting $\varepsilon\to 0^+$, we have then proved the following theorem.

Theorem 4. Let L be a holomorphic line bundle over a compact Kähler manifold M whose Kähler metric is given by g. Let s be a holomorphic section of L. Let $F(x,t)$ be a smooth function defined on $M\times\mathbb{R}$ such that $F_t\geq 0$. Suppose, for some function ψ with $\left|\psi_{i\bar{j}}\right|$ bounded on every coordinate chart of M, we have $⨍_M|s|^{2k}e^{F(x,\psi(x))}=1$. Then we can find a solution φ of the equation

$$\det\left(g_{i\bar{j}}+\varphi_{i\bar{j}}\right)=|s|^{2k}e^{F(x,\varphi(x))}\det(g_{i\bar{j}})$$

with the following properties:

(i) φ is smooth outside the divisor of s, and

(ii) $\triangle\varphi$ is bounded over M.

Furthermore, any solution satisfying the above properties must be equal to φ plus a constant.

Proof. We have only to prove the last statement. Let $\widetilde{\triangle}$ be the normalized Laplacian of the metric $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$. Then we claim that if f is a C^1-function on M such that, for all i, j, $\left|f_{i\bar{j}}\right|$ is bounded on every coordinate chart of M, then

$$\int_{\{x|f(x)>0\}} \widetilde{\triangle}(f^2)|s|^{2k} e^{F(x,\varphi(x))} = 0. \tag{7.5}$$

Approximating f by a sequence of smooth functions, we may assume that f is smooth.

For all $\delta > 0$ such that the boundary of $\{x \mid f(x) \geq \delta\}$ is a C^1-manifold (which is true for $\delta \notin E$, where E the set of critical values, whose measure is zero by Sard's theorem), we know that by Stoke's theorem,

$$\int_{\{x|f(x)\geq\delta\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))}$$

can be expressed in terms of the boundary integral of $2f\partial_n f$. Here ∂_n is the normal of the sets $\{x \mid f(x) = \delta\}$ taken with respect to our metric $(g_{i\bar{j}} + \varphi_{\varepsilon,i\bar{j}})\, dz^i \otimes d\bar{z}^j$. It is clear that

$$\int_{\{x|\delta>f(x)>0\}} (|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))}\, d\mathrm{Vol} \to 0 \qquad \text{as } \delta \to 0^+$$

So we can find a sequence $\delta_i \to 0^+$ such that

$$\delta_i \cdot \mathrm{Vol}(\{x \mid f(x) = \delta_i\})$$

tends to zero as δ_i tends to zero. Otherwise, for some $c > 0$,

$$\int_{[0,\delta]\setminus E} \mathrm{Vol}(\{x \mid f(x) = \eta\})\, d\eta \geq \int_0^\delta \frac{c}{\eta}\, d\eta = \infty,$$

a contradiction.

Combining this with the boundary integral, we conclude that

$$\int_{\{x|f(x)\geq\delta_i\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))} \to 0 \qquad \text{as } i \to \infty.$$

Hence we have

$$\int_{\{x|f(x)>0\}} \triangle_\varepsilon(f^2)(|s|^2+\varepsilon)^k e^{F(x,\varphi_\varepsilon(x))} = 0.$$

Letting $\varepsilon \to 0^+$ as in Theorem 3, we see that (7.5) follows from the above formula.

Suppose now that ψ is another solution of (7.1) with all the properties described in the theorem. Then

$$\frac{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\psi-\varphi)_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)} = e^{F(x,\psi)-F(x,\varphi)}.$$

Consider the set $\Omega = \{x \in M \mid \psi(x) - \varphi(x) > 0\}$; if it is nonempty, then the AM-GM inequality shows that the inequality

$$\widetilde{\triangle}(\psi - \varphi) \geq m \cdot e^{(F(x,\psi(x)) - F(x,\varphi(x)))/m} - m \geq 0 \tag{7.6}$$

holds on Ω. (Note that $F_t \geq 0$ is used here.)

Applying (7.5) to $f = \psi - \varphi$, we get

$$\begin{aligned} &2\int_\Omega (\psi - \varphi)\widetilde{\triangle}(\psi - \varphi)|s|^{2k} e^{F(x,\varphi(x))} + 2\int_\Omega \left|\widetilde{\nabla}(\psi - \varphi)\right|^2 |s|^{2k} e^{F(x,\varphi(x))} \\ &= \int_\Omega \widetilde{\triangle}(\psi - \varphi)^2 |s|^{2k} e^{F(x,\varphi(x))} = 0. \end{aligned}$$

Combining (7.6) and the above equality, we see that $\widetilde{\nabla}(\psi - \varphi) = 0$ on Ω and $\psi - \varphi$ is a constant on each component of $\Omega = \{x \mid \psi(x) - \varphi(x) > 0\}$. Since $\psi - \varphi$ is continuous, this is possible only if Ω is empty or $\Omega = M$. In the first case, $\psi(x) \leq \varphi(x)$ for all $x \in M$. In the second case, $\psi - \varphi$ is a constant. Interchanging ψ and φ, we conclude easily that, in any case, $\psi - \varphi$ is a constant. ∎

8 Complex Monge-Ampère Equations with Meromorphic Right-Hand Side

Let L_1 and L_2 be two holomorphic line bundles over a compact Kähler manifold M. Let s_1 and s_2 be two (non-trivial) holomorphic sections of L_1 and L_2 that are equipped with Hermitian metrics so that we have globally defined functions $|s_1|^2$ and $|s_2|^2$ on M. Then, for $k_1 \geq 0$ and $k_2 \geq 0$, we shall study equations of the form

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F \det(g_{i\bar{j}}),$$

where F is a smooth function such that

$$⨍_M \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F = 1. \tag{8.1}$$

As before we approximate the PDE by the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = C_\varepsilon \frac{(|s_1|^2 + \varepsilon)^{k_1}}{(|s_2|^2 + \varepsilon)^{k_2}} e^F \det(g_{i\bar{j}})$$

where

$$C_\varepsilon = \left(⨍_M \frac{(|s_1|^2 + \varepsilon)^{k_1}}{(|s_2|^2 + \varepsilon)^{k_2}} e^F \right)^{-1}.$$

In order to prove that the normalized solutions φ_ε of the above equation converge on the complement of the divisors of s_1 and s_2, we consider the expression $(|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}(m + \triangle\varphi_\varepsilon)$ with $p \geq 0$.

We compute the Laplacian of the above expression as follows:

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon \left((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}(m + \triangle\varphi_\varepsilon) \right) \\
&= \frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon ((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon})(m + \triangle\varphi_\varepsilon) + \triangle_\varepsilon(\triangle\varphi_\varepsilon) \\
&\quad + \frac{2e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \left\langle \nabla_\varepsilon((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon}), \nabla_\varepsilon(\triangle\varphi_\varepsilon) \right\rangle \\
&\geq \frac{e^{C\varphi_\varepsilon}}{(|s_2|^2 + \varepsilon)^p} \triangle_\varepsilon ((|s_2|^2 + \varepsilon)^p e^{-C\varphi_\varepsilon})(m + \triangle\varphi_\varepsilon) + \triangle_\varepsilon(\triangle\varphi_\varepsilon) \\
&\quad - |\nabla_\varepsilon(p \log(|s_2|^2 + \varepsilon) - C\varphi_\varepsilon)|^2 (m + \triangle\varphi_\varepsilon) - \frac{|\nabla_\varepsilon(\triangle\varphi_\varepsilon)|^2}{m + \triangle\varphi_\varepsilon} \\
&\geq (m + \triangle\varphi_\varepsilon)(p\triangle_\varepsilon \log(|s_2|^2 + \varepsilon) - C\triangle_\varepsilon\varphi_\varepsilon) - \frac{|\nabla_\varepsilon(\triangle\varphi_\varepsilon)|^2}{m + \triangle\varphi_\varepsilon} + \triangle_\varepsilon(\triangle\varphi_\varepsilon).
\end{aligned}$$

By applying the same reasoning as in (2.5), (2.6) and (2.7), we have

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2+\varepsilon)^p}\triangle_\varepsilon\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right)\\
&\geq (m+\triangle\varphi_\varepsilon)(p\triangle_\varepsilon\log(|s_2|^2+\varepsilon)-C\triangle_\varepsilon\varphi_\varepsilon)+\triangle F\\
&\quad + k_1\triangle\log(|s_1|^2+\varepsilon)-k_2\triangle\log(|s_2|^2+\varepsilon)+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\left(\sum\frac{1+\varphi_{\varepsilon,i\bar{i}}}{1+\varphi_{\varepsilon,\ell\bar{\ell}}}-m^2\right).
\end{aligned}$$

As in (5.7), we have a positive constant C_1 which is independent of ε such that

$$p\triangle_\varepsilon\log(|s_2|^2+\varepsilon)\geq -pC_1\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}.$$

Note that

$$\triangle_\varepsilon\varphi_\varepsilon = m-\sum\frac{1}{1+\varphi_{i\bar{i}}}.$$

Combining the above inequalities and equation and computing as before, we can find positive constant C_2 and C_3 which are independent of ε such that

$$\begin{aligned}
&\frac{e^{C\varphi_\varepsilon}}{(|s_2|^2+\varepsilon)^p}\triangle_\varepsilon\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}\,(m+\triangle\varphi_\varepsilon)\right)\\
&\geq\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)(m+\triangle\varphi_\varepsilon)\sum\frac{1}{1+\varphi_{\varepsilon,i\bar{i}}}\\
&\quad -C_2-mC\,(m+\triangle\varphi_\varepsilon)-k_2\triangle\log(|s_2|^2+\varepsilon)\\
&\geq C_3\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\frac{(|s_2|^2+\varepsilon)^{k_2/(m-1)}}{(|s_1|^2+\varepsilon)^{k_1/(m-1)}}\,(m+\triangle\varphi_\varepsilon)^{m/(m-1)}\\
&\quad -C_2-mC\,(m+\triangle\varphi_\varepsilon)-k_2\triangle\log(|s_2|^2+\varepsilon). \qquad (8.2)
\end{aligned}$$

Clearly, for any fixed p, we can choose C large enough so that

$$C_3\left(C-pC_1+\inf_{i\neq\ell}R_{i\bar{i}\ell\bar{\ell}}\right)\left(|s_1|^2+\varepsilon\right)^{-k_1/(m-1)}\geq 1$$

With this choice of C, we consider the point where $(|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)$ achieves its maximum. At this point,

$$(|s_2|^2+\varepsilon)^{k_2/(m-1)}(m+\triangle\varphi_\varepsilon)^{m/(m-1)}\lesssim\max\left\{C_2, mC(m+\triangle\varphi_\varepsilon), k_2\triangle\log(|s_2|^2+\varepsilon)\right\}.$$

It follows easily from the above inequality and

$$\triangle\log(|s_2|^2+\varepsilon)\leq\frac{\triangle|s_2|^2}{|s_2|^2+\varepsilon}$$

that

$$\begin{aligned}
\sup\left((|s_2|^2+\varepsilon)^p e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right)\lesssim (C^{m-1}+1)\max\Big\{&\sup\left((|s_2|^2+\varepsilon)^{p-k_2/m}e^{-C\varphi_\varepsilon}\right),\\
&\sup\left((|s_2|^2+\varepsilon)^{p-k_2}e^{-C\varphi_\varepsilon}\right),\\
&\sup\left((|s_2|^2+\varepsilon)^{p-(m-1)/m-k_2/m}e^{-C\varphi_\varepsilon}\right)\Big\}.
\end{aligned}$$

From (8.1), $k_2 < 1$. Hence the third term in the RHS of the above inequality will be the dominating term. If we choose $p = \frac{m-1+k_2}{m} + Cq$ with $q \geq 0$, we see that

$$\sup\left((|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+Cq} e^{-C\varphi_\varepsilon}(m+\triangle\varphi_\varepsilon)\right) \lesssim (C^{m-1}+1)\left(\sup(|s_2|^2+\varepsilon)^q e^{-\varphi_\varepsilon}\right)^C. \quad (8.3)$$

We are going to estimate $\sup|\varphi_\varepsilon|$. As in (2.12), we have an estimate of $\sup\varphi_\varepsilon$. Hence it remains to found an estimate of $\inf\varphi$. Integrating (8.2) with respect to the volume form $\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^{F-C\varphi_\varepsilon}\, d\mathrm{Vol}$, we have

$$\begin{aligned}
&C_3\left(C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) e^{\inf F}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{\frac{(m-2)k_1}{m-1}}}{(|s_2|^2+\varepsilon)^{\frac{(m-2)k_2}{m-1}-p}}(m+\triangle\varphi_\varepsilon)^{m/(m-1)}\\
&- k_2\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F\triangle\log(|s_2|^2+\varepsilon)\\
&\leq C_2 e^{\sup F}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\\
&\quad + mCe^{\sup F}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}(m+\triangle\varphi_\varepsilon).
\end{aligned} \quad (8.4)$$

We can find a positive constant C_4 which is independent of ε such that

$$\triangle\log(|s_2|^2+\varepsilon) \geq \frac{|s|^2}{|s|^2+\varepsilon}\triangle\log|s|^2 \geq -C_4.$$

Hence,

$$\begin{aligned}
&\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F\triangle\log(|s_2|^2+\varepsilon)\\
&\leq \int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} e^F(\triangle\log(|s_2|^2+\varepsilon)+C_4)\\
&\lesssim \int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\triangle\log(|s_2|^2+\varepsilon) + \int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}.
\end{aligned} \quad (8.5)$$

By AM-GM inequality, we know that, for any $\delta > 0$,

$$\begin{aligned}
&m\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}(m+\triangle\varphi_\varepsilon)\\
&\leq (m-1)\delta^{\frac{m}{m-1}}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{\frac{(m-2)k_1}{m-1}}}{(|s_2|^2+\varepsilon)^{\frac{(m-2)k_2}{m-1}-p}}(m+\triangle\varphi_\varepsilon)^{\frac{m}{m-1}}\\
&\quad + \delta^{-m}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}.
\end{aligned} \quad (8.6)$$

For any $p \geq 0$, we choose C large enough so that $C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}} \geq \frac{1}{2}C \geq 1$. Then we choose δ so that

$$\left((m-1)\delta^{\frac{m}{m-1}}\right)Ce^{\sup F} = \frac{1}{2}C_3\left(C - pC_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right)e^{\inf F}.$$

Substituting (8.6) into (8.4) and keeping (8.5) in mind, we see that we can find positive constant C_5 and C_6 which are independent of ε and C for which

$$\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}(m+\triangle\varphi_\varepsilon)-\frac{C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\triangle\log(|s_2|^2+\varepsilon)$$
$$\lesssim C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}+C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}. \tag{8.7}$$

In order to make use of the above inequality to derive an integral estimate $e^{-C\varphi_\varepsilon}$, we shall assume that the integral $\int_M |s_2|^{-2mk_2}$ is finite. Choose $p=C_5+k_2$.

Claim. We have

$$\int_M\left|\nabla\left(e^{-C\varphi_\varepsilon/2}\frac{(|s_1|^2+\varepsilon)^{k_1/2}}{(|s_1|^2+\varepsilon)^{(k_2-p)/2}}\right)\right|^2$$
$$\lesssim C\left(\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}. \tag{8.8}$$

Proof of Claim. Integrating by parts in (8.7), we have

$$\begin{aligned}
&-k_1\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla|s_1|^2,\nabla\varphi_\varepsilon\rangle\\
&-(p-k_2)\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_2|^2,\nabla\varphi_\varepsilon\rangle\\
&+C\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}|\nabla\varphi_\varepsilon|^2-C_5\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla\varphi_\varepsilon,\nabla|s_2|^2\rangle\\
&+\frac{k_1C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle\\
&+\frac{(p-k_2)C_5}{C}\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}}\left|\nabla|s_2|^2\right|^2\\
&\leq C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}+C_6\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}.
\end{aligned}$$

By (5.4), we have

$$\begin{aligned}
C_7\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{p-k_2}}&\geq\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{p-k_2}}\triangle\log(|s_1|^2+\varepsilon)\\
&\geq\int_M e^{-C\varphi_\varepsilon}\Bigg((p-k_2)\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle\\
&\qquad+k_1\frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}}\left|\nabla|s_1|^2\right|^2\\
&\qquad-C\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla\varphi_\varepsilon,\nabla|s_1|^2\rangle\Bigg).
\end{aligned}$$

Hence, using the above inequalities and the assumption that $p = C_5 + k_2$, we get

$$\begin{aligned}
&\int_M \left|\nabla\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{1/2}\right|^2 \\
&= \int_M \frac{e^{-C\varphi_\varepsilon}}{4}\left((k_2-p)^2\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}}\left|\nabla|s_2|^2\right|^2 + k_1^2\frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}}\left|\nabla|s_1|^2\right|^2\right. \\
&\quad + C^2\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}|\nabla\varphi_\varepsilon|^2 + 2k_1(p-k_2)\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle \\
&\quad \left. -2C(p-k_2)\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla\varphi_\varepsilon,\nabla|s_2|^2\rangle - 2k_1C\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla\varphi_\varepsilon,\nabla|s_1|^2\rangle\right) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4}\left(C_5^2\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}}\left|\nabla|s_2|^2\right|^2 + k_1^2\frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}}\left|\nabla|s_1|^2\right|^2\right. \\
&\quad + C^2\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}|\nabla\varphi_\varepsilon|^2 + 2k_1C_5\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle \\
&\quad + \frac{2C(p-k_2)}{(p-k_2+C_5)}\left[k_1\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla|s_1|^2,\nabla\varphi_\varepsilon\rangle - C\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}|\nabla\varphi_\varepsilon|^2\right. \\
&\quad - \frac{k_1C_5}{C}\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle - \frac{C_5^2}{C}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p+2}}\left|\nabla|s_2|^2\right|^2 \\
&\quad \left.\left. + C_6\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + C_6\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}\right] - 2k_1C\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla\varphi_\varepsilon,\nabla|s_1|^2\rangle\right) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4}\left(-k_1C\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\langle\nabla\varphi_\varepsilon,\nabla|s_1|^2\rangle + k_1^2\frac{(|s_1|^2+\varepsilon)^{k_1-2}}{(|s_2|^2+\varepsilon)^{k_2-p}}\left|\nabla|s_1|^2\right|^2\right. \\
&\qquad\qquad + k_1(p-k_2)\frac{(|s_1|^2+\varepsilon)^{k_1-1}}{(|s_2|^2+\varepsilon)^{k_2-p+1}}\langle\nabla|s_1|^2,\nabla|s_2|^2\rangle \\
&\qquad\qquad \left. + CC_6\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + CC_6\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}\right) \\
&\le \int_M \frac{e^{-C\varphi_\varepsilon}}{4}\left((CC_6 + k_1C_7)\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} + CC_6\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}}\right) \\
&\lesssim C\left(\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}},
\end{aligned}$$

where the last inequality is due to the Hölder inequalities

$$\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \le \left(\int_M 1\right)^{\frac{1}{m}}\left(\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}},$$

$$\int_M e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{2k_1}}{(|s_2|^2+\varepsilon)^{2k_2-p}} \le \left(\int_M \frac{(|s_1|^2+\varepsilon)^{mk_1}}{(|s_2|^2+\varepsilon)^{mk_2}}\right)^{\frac{1}{m}}\left(\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}$$

and the assumption $\int_M |s_2|^{-2mk_2} < \infty$. ■

Since we have an estimate of $\int_M |\varphi_\varepsilon|$, we can use the method of Section 5 to find an estimate of

$$\int_M\left(e^{-C\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}}\right)^{\frac{m}{m-1}}$$

which is independent of ε.

From the above inequality, we conclude that when $p = C_5 + k_2$ and N is a large constant, we can find a positive constant C_8 independent of ε such that

$$\int_M e^{-N\varphi_\varepsilon}(|s_1|^2+\varepsilon)^{k_1}(|s_2|^2+\varepsilon)^{C_5} = \int_M e^{-N\varphi_\varepsilon}\frac{(|s_1|^2+\varepsilon)^{k_1}}{(|s_2|^2+\varepsilon)^{k_2-p}} \le C_8. \tag{8.9}$$

From (8.3) and the estimate of $\sup\varphi_\varepsilon$, we derive that, for any $q \ge 0$,

$$\sup\left((|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+Cq}(m+\triangle\varphi_\varepsilon)\right) \lesssim (C^{m-1}+1)e^{C\sup\varphi_\varepsilon}(\sup e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q)^C, \tag{8.10}$$

where C is any positive constant so that

$$C_3\left(C - \left(\tfrac{m-1+k_2}{m}+Cq\right)C_1 + \inf_{i\neq\ell} R_{i\bar{i}\ell\bar{\ell}}\right) \ge \sup(|s_1|^2+\varepsilon)^{\frac{k_1}{m-1}}.$$

Using (8.9) and (8.10) we shall show that, for any $q>0$, $e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q$ has an upper bound which is independent of ε. Note that we may assume q is small enough.

Suppose not, we could find $\varepsilon_i \to 0^+$ and $x_i \to x_0$ in M such that

$$e^{-\varphi_{\varepsilon_i}(x_i)}(|s_2|^2(x_i)+\varepsilon_i)^q = \sup\left(e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right) \to \infty.$$

Suppose the sequence $\{\varepsilon_i^{-1}|s_2|^2(x_i)\}$ is bounded. Then $\varepsilon_i^q e^{-\varphi_{\varepsilon_i}(x_i)} \to \infty$. On the other hand, using (8.10) and the L^1-estimate of φ_ε, we can apply the Schauder estimate to get

$$\sup|\nabla\varphi_{\varepsilon_i}| \lesssim (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+1.$$

It follows from the above estimate, $|\nabla|s_2|^2| \lesssim |s_2|$ and AM-GM inequality that

$$\begin{aligned}\sup\left|\nabla\left(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i}\right)\right| &\lesssim \frac{|\nabla|s_2|^2|}{|s_2|^2+\varepsilon} + (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+1\\ &\lesssim (C^{m-1}+1)\frac{\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^C}{\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}}+\varepsilon_i^{-1/2}.\end{aligned}$$

Clearly we may assume that $\sup(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i}) \ge 0$. Then proceeding geodesic ball trick as in Section 2, we can now conclude that

$$\begin{aligned}C_8 &\ge \int_M e^{-N\varphi_{\varepsilon_i}}(|s_1|^2+\varepsilon_i)^{k_1}(|s_2|^2+\varepsilon_i)^{C_5} \ge \int_M e^{N(\log(\varepsilon_i(|s_2|^2+\varepsilon_i))^q-\varphi_{\varepsilon_i})}\varepsilon_i^{k_1+C_5}(\varepsilon_i(|s_2|^2+\varepsilon_i))^{-qN}\\ &\gtrsim \left(\varepsilon_i^{\frac{m-1+k_2}{m}+Cq}\frac{\sup(\log(|s_2|^2+\varepsilon_i)^q-\varphi_{\varepsilon_i})}{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q)^C+1}\right)^{2m}\cdot\varepsilon_i^{k_1+C_5-qN}\cdot\left(\varepsilon_i^q\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q\right)^{N/2}.\end{aligned}$$

Taking $N > 4mC$, the above inequality shows that $\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2+\varepsilon_i)^q$ is bounded.

So we may assume that $\varepsilon_i^{-1}|s_2|^2(x_i) \to \infty$. For each x_i, let $B_i = B(x_i, \delta_i)$ be a geodesic ball around x_i such that, for each $x \in B_i$,

$$\frac{3}{2}|s_2|^2(x_i) \geq |s_2|^2(x) \geq \frac{1}{2}|s_2|^2(x_i). \tag{8.11}$$

Let $C_9 \geq \sup |\nabla|s_2|^2|$ sufficiently large enough. Then we may assume

$$\delta_i = \frac{1}{2C_9}|s_2|^2(x_i),$$

and is smaller than the injectivity radius of M. It is easy to derive from (8.10) that, over the ball B_i,

$$0 < m + \triangle\varphi_{\varepsilon_i} \lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q)^C}{\left(\frac{1}{2}|s_2|^2(x_i)\right)^{(m-1+k_2)/m+Cq}}.$$

By applying the Schauder estimate on the balls B_i and $B_i' = B(x_i, \frac{\delta_i}{2})$, we get

$$\begin{aligned}\sup_{x \in B_i'} |\nabla\varphi_{\varepsilon_i}|(x) &\lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{\int_{B_i} |\varphi_{\varepsilon_i}|}{\delta_i^{2m+1}} \\ &\lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{\int_{B_i} |\varphi_{\varepsilon_i}|}{(|s_2|^2(x_i))^{2m+1}}.\end{aligned}$$

Since we have an estimate of $\int_M |\varphi_{\varepsilon_i}|$, it follows from the above inequality that

$$\sup_{x \in B_i'} |\nabla(\log(|s_2|^2 + \varepsilon_i)^q - \varphi_{\varepsilon_i})| \lesssim \frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{(m-1+k_2)/m+Cq}} + \frac{1}{(|s_2|^2(x_i))^{2m+1}}. \tag{8.12}$$

Since s_1 is holomorphic, one can find positive constant a such that, for any small $r > 0$ and $x \in M$,

$$\int_{B(x,r)} |s_1|^{2k_1} \gtrsim r^a.$$

As before, we may assume that $\sup(q\log(|s_2|^2 + \varepsilon_i) - \varphi_{\varepsilon_i}) \geq 0$. Then proceeding the geodesic ball trick as above, we can now conclude from (8.11), (8.12), the above inequality and $\varepsilon^{-1}|s_2|^2(x_i) \to \infty$ that

$$\begin{aligned}C_8 &\geq \int_M e^{-N\varphi_{\varepsilon_i}}(|s_1|^2 + \varepsilon_i)^{k_1}(|s_2|^2 + \varepsilon_i)^{C_5} \gtrsim \int_{B'} e^{N(\log(|s_2|^2+\varepsilon)^q - \varphi_{\varepsilon_i})}|s_1|^{2k_1}(|s_2|^2(x_i))^{C_5 - qN} \\ &\gtrsim \left(\left(\frac{(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q)^C}{(|s_2|^2(x_i))^{\frac{m-1+k_2}{m}+Cq}} + \frac{1}{(|s_2|^2(x_i))^{2m+1}}\right)^{-1} \cdot \sup(\log(|s_2|^2 + \varepsilon)^q - \varphi_{\varepsilon_i})\right)^a \\ &\quad \cdot (|s_2|^2(x_i))^{C_5 - qN}\left(\sup e^{-\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q\right)^{N/2}\end{aligned}$$

Take N large enough, we see that the quantity $\sup e^{-C\varphi_{\varepsilon_i}}(|s_2|^2 + \varepsilon_i)^q$ can be estimated by a constant independent of i.

In conclusion we have proved that, for any $q > 0$, $\log(|s_2|^2 + \varepsilon)^q - \varphi_\varepsilon$ is bounded from above by a constant independent of ε. In particular, $-\varphi_\varepsilon$ is uniformly bounded over any compact subset K of the complement of the divisor of s_2. From (8.10) and the estimate of $\sup \varphi_\varepsilon$, we see that both $|\varphi_\varepsilon|$ and $|\triangle\varphi_\varepsilon|$ are uniformly bounded over K. The arguments of Section 5 now show that one can find uniform estimates of $\{\varphi_{\varepsilon;i\bar{j}k}\}$ over K.

Theorem 5. Let L_1 and L_2 be two holomorphic line bundles over a compact Kähler manifold M whose Kähler metric is given by g. Let s_1 and s_2 be two holomorphic sections of L_1 and L_2, respectively, and let F be a smooth function defined on M such that

$$\fint_M \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F = 1 \quad \text{and} \quad \int_M |s_2|^{2mk_2} < \infty.$$

where k_1 and k_2 are two non-negative integers. Then we can solve the equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} e^F \det(g_{i\bar{j}})$$

so that

(i) φ is smooth outside the divisors of s_1 and s_2 with $\sup \varphi < \infty$,

(ii) $(\varphi_{i\bar{j}})$ is a bounded matrix outside the divisor of s_2 and, for any $q > 0$,

$$|s_2|^{2(m-1+k_2)/m+q} \triangle\varphi$$

is bounded on M,

(iii) for any $q > 0$, the function $\varphi - q \log |s_2|^2$ is bounded from below,

(iv) the matrix $(g_{i\bar{j}} + \varphi_{i\bar{j}})$ is positive definite outside the complement of the divisors of s_1 and s_2.

Furthermore, if we assume that

$$\int_M \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} < \infty$$

for some $q > 0$, the any two solutions of the equation which has the above properties (i), (ii) and (iv) must differ from each other by a constant. If we also know that $(|s_2|^{2(m-1+k_2)/m+q})^{-1}$ is integrable over every analytic disc of M, then the unique solution φ is bounded from below on M.

Proof. We have only to prove the last part. Suppose ψ is another solution of the equation with (i), (ii) and (iv). Then the AM-GM inequality shows that

$$\triangle_\varepsilon(\psi - \varphi_\varepsilon) \geq m \left(C_\varepsilon^{-1} \cdot \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} \cdot \frac{(|s_2|^2 + \varepsilon)^{k_2}}{(|s_1|^2 + \varepsilon)^{k_1}} \right)^{1/m} - m. \tag{8.13}$$

Let k be any constant. We claim that, over $\Omega_{\varepsilon,k} = \{x \in M \mid (\psi - \varphi_\varepsilon)(x) \geq k\}$,

$$\int_{\Omega_{\varepsilon,k}} (\psi - \varphi_\varepsilon - k)\triangle_\varepsilon(\psi - \varphi_\varepsilon) \leq 0. \tag{8.14}$$

In fact, for $\delta > 0$, $\Omega_{\varepsilon,k,\delta} = \{x \in M \mid (\psi - \varphi_\varepsilon)(x) \geq k - \delta \log |s_2|^2\}$ is disjoint from the divisor of s_2 by property (i). Hence both $(\psi_{i\bar{j}})$ and $(\varphi_{\varepsilon;i\bar{j}})$ are bounded on $\Omega_{\varepsilon,k,\delta}$ and we can integrate by parts on $\Omega_{\varepsilon,k,\delta}$ to find

$$\begin{aligned} &\int_{\Omega_{\varepsilon,k,\delta}} (\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2)\triangle_\varepsilon(\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2) \\ &= -\int_{\Omega_{\varepsilon,k,\delta}} |\nabla_\varepsilon(\psi - \varphi_\varepsilon - k + \delta \log |s_2|^2)|^2. \end{aligned} \tag{8.15}$$

Using property (ii) and the assumption $\int_M \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} < \infty$, we can find a constant $C(\varepsilon)$ independent of δ such that

$$\begin{aligned} \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle_\varepsilon(\psi - \varphi_\varepsilon)| &\lesssim \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle\psi| + \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle\varphi_\varepsilon| \\ &\leq \left(\int_{\Omega_{\varepsilon,k,\delta}} |s_2|^{2(m-1+k_2)/m+q} (\log |s_2|^2 |\triangle\psi|)^2 \right)^{1/2} \\ &\quad \cdot \left(\int_{\Omega_{\varepsilon,k,\delta}} \frac{1}{|s_2|^{2(m-1+k_2)/m+q}} \right)^{1/2} + \int_{\Omega_{\varepsilon,k,\delta}} \log |s_2|^2 |\triangle\varphi_\varepsilon| \\ &\leq C(\varepsilon). \end{aligned} \tag{8.16}$$

It follows easily from (8.15), (8.16) and the boundedness of $|\triangle_\varepsilon \log |s_2|^2|$ that

$$\lim_{\delta \to 0^+} \int_{\Omega_{\varepsilon,k,\delta}} (\psi - \varphi_\varepsilon - k)\triangle_\varepsilon(\psi - \varphi_\varepsilon) \leq 0.$$

Using the definition of $\Omega_{\varepsilon,k,\delta}$, we see that, over $\Omega_{\varepsilon,k,\delta}$, $(\psi - \varphi_\varepsilon - k)$ is bounded by a constant independent of δ when δ is small. The function $(\psi - \varphi_\varepsilon - k)\triangle_\varepsilon(\psi - \varphi_\varepsilon)$ is therefore uniformly integrable and we can apply Lebesgue's dominated convergence theorem to prove (8.14).

Applying (8.13) and (8.14), we can now prove the following inequality:

$$\int_{\Omega_{\varepsilon,k}} (\psi - \varphi_\varepsilon - k) \left(m - m \left(C_\varepsilon^{-1} \cdot \frac{|s_1|^{2k_1}}{|s_2|^{2k_2}} \cdot \frac{(|s_2|^2 + \varepsilon)^{k_2}}{(|s_1|^2 + \varepsilon)^{k_1}} \right)^{1/m} \right) \geq 0. \tag{8.17}$$

When $\varepsilon \to 0^+$, the integral on the LHS tends to zero. Let K be a compact subset of the complement of the divisor of s_2. By (8.13) and the above inequality, we have

$$\lim_{\varepsilon\to 0^+}\int_{K\cap\Omega_{\varepsilon,k}}(\psi-\varphi_\varepsilon-k)\left(m-m\left(C_\varepsilon^{-1}\cdot\frac{|s_1|^{2k_1}}{|s_2|^{2k_2}}\cdot\frac{(|s_2|^2+\varepsilon)^{k_2}}{(|s_1|^2+\varepsilon)^{k_1}}\right)^{1/m}+\triangle_\varepsilon(\psi-\varphi_\varepsilon)\right)=0.$$

Let $\Omega_k=\{x\in M \mid (\psi-\varphi)(x)\geq k\}$. Then the above equation gives

$$\int_{K\cap\Omega_k}(\psi-\varphi-k)\widetilde{\triangle}(\psi-\varphi)=0.$$

As in (8.13), we know that $\widetilde{\triangle}(\psi-\varphi)\geq 0$ and hence, $\widetilde{\triangle}(\psi-\varphi)=0$ on Ω_k. The AM-GM inequality now becomes equality, so $\psi_{i\bar{j}}=\varphi_{i\bar{j}}$ on $K\cap\Omega_k$. Since k and K are arbitrary, $\psi_{i\bar{j}}=\varphi_{i\bar{j}}$ on the complement of the divisor of s_2. Letting first $\delta\to 0^+$ and then $\varepsilon\to 0^+$ in (8.15). We get

$$\int_{K\cap\Omega_k}|\widetilde{\nabla}(\psi-\varphi)|^2\leq-\lim_{\varepsilon\to 0^+}\int_{\Omega_{\varepsilon,k}}(\psi-\varphi_\varepsilon-k)\triangle_\varepsilon(\psi-\varphi_\varepsilon),$$

which is equal to zero by (8.17). Hence, $\psi-\varphi$ is constant.

It remains to prove that $-\inf\varphi<\infty$. From (8.3) and the estimate of $e^{-\varphi_\varepsilon}(|s_2|^2+\varepsilon)^q$ for any $q>0$, we know that, for any $q>0$,

$$\sup\left((m+\triangle\varphi_\varepsilon)(|s_2|^2+\varepsilon)^{\frac{m-1+k_2}{m}+\frac{q}{2}}\right)\lesssim 1 \tag{8.18}$$

Let x be any point on the divisor of s_2. Let D_x be an analytic disc passing through x such that s_2 is not zero on ∂D_x. Then $|\varphi_\varepsilon|$ is uniformly bounded on ∂D_x when $\varepsilon\to 0^+$. It follows from (8.18) that when we restrict φ_ε to D_x, the absolute value of its Laplacian is estimated by $(|s_2|^2+\varepsilon)^{-(\frac{m-1+k_2}{m}+\frac{q}{2})}$ over D_x. Cauchy integral formula gives

$$2\pi\sqrt{-1}\partial\varphi_\varepsilon(p)=\int_{\partial D_x}\frac{\partial\varphi_\varepsilon(z)}{z-p}dz+\int_{D_x}\frac{\triangle\varphi}{z-p}$$

Integrate over the curve $\gamma(t)=tp+(1-t)\overline{p}$, where $\overline{p}=\frac{p}{|p|}$, we get

$$\begin{aligned}|\varphi_\varepsilon(p)|\lesssim|\varphi_\varepsilon(\overline{p})|&+\int_{\partial D_x}|\partial\varphi(z)|\cdot|\log(z-p)-\log(z-\overline{p})|\,dz\\&+\int_{D_x}\frac{1}{|s|^{2(\frac{m-1+k_2}{m})+q}}\cdot|\log(z-p)-\log(z-\overline{p})|\end{aligned}$$

Using Hölder inequality and taking a smaller q, we obtain an estimate of $|\varphi_\varepsilon|$ on D_x. Since x is arbitrary, we can conclude the boundedness of φ. ∎

9 The General Case

Let $t_1,\ t_2,\ \ldots,\ t_{n_1+n_2}$ be non-zero non-negative functions defined on M such that $t_i = \sum_{j=1}^{\ell} |s_j|^{2k_j}$, where $k_j \geq 0$ for each j and $s_1,\ s_2,\ \ldots,\ s_\ell$ are holomorphic sections of some holomorphic line bundle.

Then we consider

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\varphi)} \det(g_{i\bar{j}}), \tag{9.1}$$

where $F(x,t)$ is a smooth function defined on $M \times \mathbb{R}$ with $F_t \geq 0$.

Then we assume t_i's satisfying the following properties:

- there exists a smooth function ψ defined on M such that

$$⨍_M \frac{t_1 \cdots t_{n_1}}{t_{n_1+1} \cdots t_{n_1+n_2}} e^{F(x,\psi)} = 1.$$

- $(t_{n_1+1} \cdots t_{n_1+n_2})^{-m}$ is integrable over M.

- for some $q > 0$,

$$\frac{|\triangle \log(t_{n_1+1} \cdots t_{n_1+n_2})|^{(m-1)/m}}{(t_{n_1+1} \cdots t_{n_1+n_2})^{q/m}}$$

  is integrable over M and over every analytic disk of M.

As before, we have

Theorem 6. Let M be a compact Kähler manifold. Suppose that, in equation (9.1), the t_i are functions satisfying the above mentioned properties. Then we can find a solution φ of (9.1) such that

(i) φ is smooth outside the divisors of the t_i's and $\sup|\varphi| < \infty$,

(ii) $\sup \dfrac{(t_{n_1+1} \cdots t_{n_1+n_2})^{q+1/m} (\triangle \varphi)}{(|\triangle \log t_{n_1+1} \cdots t_{n_1+n_2}| + 1)^{(m-1)/m}} < \infty$, and

(iii) $\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \otimes d\bar{z}^j$ defines a Kähler metric outside the divisors of the t_i's.

Furthermore, any solution of (9.1) satisfying the above three properties differs from φ by a constant.

Corollary 1. Let M be a compact Kähler variety with log terminal singularity such that the canonical line bundle is ample. Then there is a Kähler-Einstein metric whose Ricci tensor is the negative of the metric tensor on the smooth part of M.

Take a resolution of singularities $\pi : \widetilde{M} \to M$ so that

$$K_{\widetilde{M}} = \pi^* K_M + \sum_{E \in \mathcal{E}} a_E E$$

and $a_E > -1$ for all $E \in \mathcal{E}$. We know that there exists $c_E \in \mathbb{Q}^+$ such that

$$L = \pi^* K_M - \sum_{E \in \mathcal{E}} c_E E$$

is ample. Then

$$K_{\widetilde{M}} = L + \sum_{E \in \mathcal{E}} (a_E + c_E) E$$

gives

$$- \mathrm{c}_1(\widetilde{M}) = \mathrm{c}_1(L) + \sum (a_E + c_E) \, \mathrm{c}_1(E)$$

Since $\mathrm{c}_1(L)$ is represented by some positive $(1,1)$-form $\sqrt{-1} g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j$. Take this form as our Kähler form on $\widetilde{M}$. Then $- \mathrm{c}_1(\widetilde{M})$ is represented by

$$\sqrt{-1} h_{i\bar{j}} \, dz^i \wedge d\bar{z}^j - \sum (a_E + c_E) \partial\overline{\partial} \log |s_E|^2.$$

Since the closed $(1,1)$-form $-\partial\overline{\partial} \log \det(g_{i\bar{j}})$ also represents $\mathrm{c}_1(\widetilde{M})$, we can find a smooth function f such that

$$\partial\overline{\partial} \log \det(g_{i\bar{j}}) = \sqrt{-1} g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j - \sum (a_E + c_E) \partial\overline{\partial} \log |s_E|^2 + \partial\overline{\partial} f.$$

Now by Theorem 6, we can solve the equation (since $a_E + c_E > -1$)

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \prod_E |s_E|^{2(a_E + c_E)} \cdot e^{\varphi - f} \det(g_{i\bar{j}})$$

so that $(g_{i\bar{j}} + \varphi_{i\bar{j}}) \, dz^i \otimes d\bar{z}^j$ defines a Kähler metric outside $\bigcup_{E \in \mathcal{E}} E$. By these equations we have

$$\begin{aligned} -\partial\overline{\partial} \log \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) &= -\partial\overline{\partial}\varphi - \sqrt{-1} g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j \\ &= -\sqrt{-1} (g_{i\bar{j}} + \varphi_{i\bar{j}}) \, dz^i \wedge d\bar{z}^j \end{aligned}$$

on the $\widetilde{M}$. Since the smooth part of M is isomorphic to some open subset of $\widetilde{M}$, we get the metric we want.

DIFFERENTIAL GEOMETRY FINAL REPORT

MING-WEI KUO, CHEN-KUAN LEE

ABSTRACT. This survey is mainly based on S. Brendle, *Ricci flow with surgery on manifolds with positive isotropic curvature* [6].

He established a fabulous generalization of Hamilton-Ivey pinched estimate for dimension ≥ 12 case, but we will focus on his work of Ricci flow with surgery. First, we try to explain how he extended G. Perelman's Canonical Neighborhood property to dimension ≥ 12 case, which is the key step for surgery. Then we demonstrate his definition of Ricci flow with surgery.

CONTENTS

Work distribution

Kuo, Ming-Wei: Section 1, Section 2, Section 3, Section 4.

Lee, Chen-Kuan: Section 1, Section 2, Section 5, Section 6, Section 7.

1. Preliminary

Note that we denote scalar curvature by S throughout the article.

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Definition. (Isotropic curvature conditions)

(i) We denote by strictly PIC the set of all algebraic curvature tensors that have positive isotropic curvature in the sense that $R(\varphi, \bar{\varphi}) > 0$ for all complex two-forms of the form $\varphi = (e_1 + ie_2) \wedge (e_3 + ie_4)$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame.
 Note that $R(\varphi, \bar{\varphi}) > 0$ if and only if $R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$.
 Moreover, by summing all the indices, PIC implies positive scalar curvature.

(ii) We denote by uniformly PIC the set of all algebraic curvature tensors that have uniformly positive isotropic curvature in the sense that there exists a constant $\theta > 0$ such that $R(\varphi, \bar{\varphi}) \geq 4\theta S > 0$ for all complex two-forms of the form $\varphi = (e_1 + ie_2) \wedge (e_3 + ie_4)$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame.

(iii) We denote by (weakly) PIC the set of all algebraic curvature tensors that have non-negative isotropic curvature in the sense that $R(\varphi, \bar{\varphi}) \geq 0$ for all complex two-forms of the form $\varphi = (e_1 + ie_2) \wedge (e_3 + ie_4)$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame.

(iv) We denote by (weakly) PIC1 the set of all algebraic curvature tensors satisfying $R(\varphi, \bar{\varphi}) \geq 0$ for all complex two-forms of the form $\varphi = (e_1 + ie_2) \wedge (e_3 + i\lambda e_4)$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in [0, 1]$.

(v) We denote by (weakly) PIC2 the set of all algebraic curvature tensors satisfying $R(\varphi, \bar{\varphi}) \geq 0$ for all complex two-forms of the form $\varphi = (e_1 + i\mu e_2) \wedge (e_3 + i\lambda e_4)$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda, \mu \in [0, 1]$.

Note that PIC2 $\subset$ PIC1 $\subset$ PIC.

Definition (ε-close)**.** If $U \subseteq M$ is an open subset and g, g_0 are two Riemannian metrics on U, we say g is ε-close to g_0 on U if

$$\|g - g_0\|_{\lceil \varepsilon^{-1} \rceil, U, g} < \epsilon.$$

Here the norm is, for a tensor T

$$\|T\|^2_{N,U,g} := \sup_{x \in U} \sum_{k=0}^{N} |\nabla_g^k T(x)|_g^2,$$

where the pointwise norm is the Euclidean one.

Before giving the next definition, we need some notations.

- $B(x, t, r)$ denotes the open metric ball of radius r, which respect to the metric $g(t)$ at time t, center at x.
- $P(x, t, r, \Delta t)$ denotes a parabolic neighborhood, that is the set $\{(x', t') : x' \in B(x, t', r), t' \in [t, t + \Delta t]$ or $t' \in [t + \Delta t, t]$ depend on the sign of $\Delta t\}$

Definition (neck)**.** There are two kinds of necks.

(1) A ball $B(x,t,\varepsilon^{-1}r)$ is called an ε-neck if, after rescaling with the factor r^{-2}, it is ε-close to the standard neck $\mathbb{S}^{n-1}\times\mathbb{I}$, with product metric, where $\mathbb{S}^{n-1}$ has constant curvature 1, and $\mathbb{I}$ is an interval with length $2\varepsilon^{-1}$.

(2) A parabolic neighborhood $P(x,t,\varepsilon^{-1}r,r^2)$ is called a strong (or a evolving) ε-neck if, after rescaling r^{-2}, it is ε-close to the evolving ε-neck, which at each time $t'\in[-1,0]$ has length $2\varepsilon^{-1}$ and scalar curvature $(1-t')^{-1}$.

Definition (cap)**.** Let us fix a small number $\varepsilon_0=\varepsilon_0(n)$ and let $0<\varepsilon<\frac{\varepsilon_0}{4}$. We say that a compact domain $\Omega\subseteq(M,g)$ is an ε-cap if the following holds:

- The domain Ω is diffeomorphic to unit ball B^n, and the boundary $\partial\Omega$ is a cross-sectional sphere of an ε-neck.
- If $\bar\Omega\subseteq\Omega$ is a compact domain diffeomorphic to B^n and the boundary $\partial\bar\Omega$ is a cross-sectional sphere of an $(\varepsilon_0-\varepsilon)$-neck, then there exist a diffeomorphism $F:\bar\Omega\to B^n$ and an $(\varepsilon_0+\varepsilon)$-isometry $f:\partial\bar\Omega\to\mathbb{S}^{n-1}$ with the property that $F|_{\partial\bar\Omega}:\partial\bar\Omega\to\mathbb{S}^{n-1}$ is isotopic to f.

Definition. In the Ricci flow, we can categorize necks and caps depending on whether the scalar curvature blows up or not.

(1) An ε-neck is called
- an ε-tube if the scalar curvature stays bounded on both ends.
- an ε-horn if the scalar curvature stays bounded on one end tends to infinite on the other.
- an double ε-horn if the scalar curvature tends to infinite on both ends.

(2) An ε-cap is called
- an ε-cap if the scalar curvature stays bounded on the end.
- an capped ε-horn if the scalar curvature tends to infinite on the end.

Theorem 1.1 (cf. S. Brendle [6, Th. 1.2])**.** *Assume that $n\geq 12$. Let $\mathcal{K}$ be a compact set of algebraic curvature tensors in dimension n that is contained in the interior of the PIC cone, and let $T>0$ be given. Then there exist a small positive real number θ, a large positive real number N, an increasing concave function $f>0$ satisfying $\lim_{s\to\infty}\frac{f(s)}{s}=0$, and a continuous family of closed, convex, $O(n)$-invariant sets $\{\mathcal{F}_t : t\in[0,T]\}$ such that the family $\{\mathcal{F}_t : t\in[0,T]\}$ is invariant under the Hamilton ODE (cf. [6], [7]) $\frac{d}{dt}R=Q(R)$; $\mathcal{K}\subset\mathcal{F}_0$; and*

$$\begin{aligned}\mathcal{F}_t\subset\{R\,\big|\,R-\theta S\,\mathrm{id}\owedge\mathrm{id}\in\mathrm{PIC}\}\\ \cap\{R\,\big|\,Ric_{11}+Ric_{22}-\theta S+N\geq 0\}\\ \cap\{R\,\big|\,R+f(S)\mathrm{id}\owedge\mathrm{id}\in\mathrm{PIC2}\}\end{aligned}$$

for all $t\in[0,T]$.

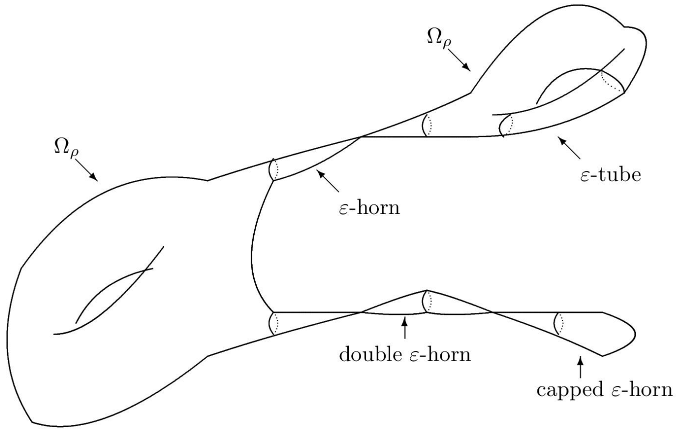

FIGURE 1. Kinds of necks and caps (cf. [9, P.415])

Here, $\owedge$ denotes the Kulkarni-Nomizu product. Instead of stating the precise defnition, we use it in our speacial case: if A and B are symmetric bilinear forms, then $(A \owedge B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}$.

Via Hamilton's PDE-ODE principle (cf. [15, Th. 3] or [14, Th. 10.16]), Theorem 1.1 gives curvature pinching estimates for solutions to the Ricci flow starting from initial metrics with positive isotropic curvature:

Corollary 1.2 (cf. S. Brendle [6, Th. 1.3]). *Let (M, g_0) be a compact manifold of dimension $n \geq 12$ with positive isotropic curvature, and let $g(t)$ denote the solution to the Ricci flow with initial metric g_0. Then there exist a small positive real number θ, a large positive real number N, and an increasing concave function f satisfying $\lim_{s\to\infty} \frac{f(s)}{s} = 0$ such that the curvature tensor of $(M, g(t))$ satisfies*

(i) $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$.
(ii) $Ric_{11} + Ric_{22} - \theta S + N \geq 0$.
(iii) $R + f(S)\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC2}$.

for all $t \geq 0$.

Property (iii) can be viewed as a higher dimensional version of Hamilton-Ivey pinching estimate in dimension 3 (cf. [19], [21]). It ensures that blow-up limits are weakly PIC2. And Property (i) ensures that blow-up limits are uniformly PIC.

When studying ancient solutions to the Ricci flow that are weakly PIC2 and uniformly PIC, an important ingredient is the Harnack inequality for the curvature tensor. In [18], Hamilton first proved the version for solutions to the Ricci flow with nonnegative curvature operator, and Brendle generalized to any solution to the Ricci flow that is weakly PIC2 in [4].

Theorem 1.3 (cf. R. Hamilton [18]; S. Brendle [3]). *Assume that $(M, g(t))$, $t \in (0, T)$, is a solution to the Ricci flow that is complete with bounded curvature and is weakly PIC2. Then*

$$\partial_t S + 2\langle \nabla S, v \rangle + 2Ric(v, v) + \frac{1}{t} S \geq 0$$

for every tangent vector v. In particular, the product $t \cdot S$ is monotone increasing at each point in space.

Integrating the differential Harnack inequality along paths in space-time, we obtain the following:

Corollary 1.4. *Assume that $(M, g(t))$ is an ancient solution to the Ricci flow that is complete with bounded curvature and is weakly PIC2. Then*

$$S(x_1, t_1) \leq \exp\left(\frac{d_{g(t_1)}(x_1, x_2)^2}{2(t_2 - t_1)}\right) S(x_2, t_2)$$

whenever $t_1 < t_2$.

Also, we state some splitting theorems, which are based on the strict maximum principle (cf. [17], [4]).

Proposition 1.5. *Let $(M, g(t))$, $t \in (0, T]$, be a (possibly incomplete) solution to the Ricci flow that is weakly PIC2 and strictly PIC. Moreover, suppose that there exist a point (x_0, t_0) in space-time and a unit vector $v \in T_{x_0} M$ with the property that $Ric(v, v) = 0$. Then, for each $t \leq t_0$, the flow $(M, g(t))$ locally splits as a product of an $(n-1)$-dimensional manifold with an interval.*

Proof. Suppose not, then $\exists \tau \in (0, t_0)$ such that $(M, g(\tau))$ does not locally split as a product of an $(n-1)$-dimensional manifold with an interval. Since $(M, g(\tau))$ is strictly PIC, we conclude that $(M, g(\tau))$ is locally irreducible. The Ricci tensor of (M, g(t)) satisfies the evolution equation

$$\partial_t Ric = \Delta Ric + 2R * Ric,$$

where $(R * Ric)_{ik} = \sum_{p,q=1}^{n} R_{ipkq} Ric_{pq}$. Since R is weakly PIC2, the term $R * Ric$ is weakly positive definite. The strict maximum principle (cf. S. Brendle [4, §9] or R. S. Hamilton [17]) shows that the null space of $Ric_{g(\tau)}$ defines a parallel subbundle of the tangent bundle of $(M, g(\tau))$. Since$(M, g(\tau))$ is locally irreducible, this subbundle must have rank 0. Consequently, the Ricci curvature of $(M, g(\tau))$ is strictly positive.

Let Ω be a bounded open neighborhood of the point p_0 with smooth boundary. Choose a smooth function $f : \bar{\Omega} \to \mathbb{R}$ such that $f > 0$ in Ω, $f = 0$ on $\partial\Omega$, and $Ric_{g(\tau)} - f\mathrm{id}$ is weakly positive definite. Let $F : \bar{\Omega} \times [\tau, t_0] \to \mathbb{R}$ denote the solution of the linear heat equation with respect to the evolving metric $g(t)$ with initial data $F(\cdot, \tau) = f$ and Dirichlet boundary condition $F = 0$ on $\partial\Omega \times [\tau, t_0]$. The maximum principle shows that $Ric_{g(t)} - F(\cdot, t)\mathrm{id}$ is weakly positive definite at each point in $\Omega \times [\tau, t_0]$. Since $F(p_0, t_0) > 0$, the Ricci curvature at (p_0, t_0) is strictly positive, contrary to our assumption. □

Proposition 1.6 (cf. S. Brendle [6, Prop. 6.6]). *Let $(M, g(t))$, $t \in (0, T]$, be a complete solution to the Ricci flow, which possibly has unbounded curvature. Assume that $(M, g(t))$ is weakly PIC2 and strictly PIC. Suppose that there exists a point (p_0, t_0) in space-time such that the curvature tensor at (p_0, t_0) lies on the boundary of the PIC2 cone. Then, for each $t \le t_0$, the universal cover of $(M, g(t))$ splits off a line.*

Corollary 1.7 (cf. S. Brendle [6, Cor. 6.7]). *Let $(M, g(t)), t \in (-\infty, T]$, be a complete, nonflat ancient solution to the Ricci flow with bounded curvature. Moreover, we assume that $(M, g(t))$ is weakly PIC2 and satisfies $R - \theta S \ id \owedge id \in PIC$ for some uniform constant $\theta > 0$. Suppose that there exists a point (p_0, t_0) in space-time with the property that the curvature tensor at (p_0, t_0) lies on the boundary of the PIC2 cone. Then, for each $t \le t_0$, the universal cover of $(M, g(t))$ is isometric to a family of shrinking cylinders $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Let us recall some results due to Perelman:

Proposition 1.8 (cf. S. Brendle [6, Prop. 6.8]). *Assume that (M, g) is a complete noncompact manifold that is weakly PIC2. Fix a point $p \in M$ and let p_j be a sequence of points such that $d(p, p_j) \to \infty$. Moreover, suppose that λ_j is a sequence of positive real numbers satisfying $\lambda_j d(p, p_j)^2 \to \infty$. If the rescaled manifolds $(M, \lambda_j g, p_j)$ converge in the Cheeger-Gromov sense (cf. [10]) to a smooth limit, then the limit splits off a line.*

Proposition 1.9 (cf. S. Brendle [6, Prop. 6.9]). *Let (M, g) be a complete noncompact Riemannian manifold that is weakly PIC2. Then (M, g) does not contain a sequence of necks with radii converging to 0.*

As a consequence, we have the following property:

Proposition 1.10. *Suppose that $(M, g(t)), t \in (-\infty, 0]$, is a complete ancient solution to the Ricci flow that is κ-noncollapsed on all scales, is weakly PIC2, and satisfies $R - \theta S \ id \owedge id \in PIC$ for some uniform constant $\theta > 0$. Moreover, suppose that $(M, g(t))$ satisfies the Harnack inequality*

$$\partial_t S + 2\langle \nabla S, v \rangle + 2Ric(v, v) \ge 0$$

for every tangent vector v. Then $(M, g(t))$ has bounded curvature.

Proof. Since $(M, g(t))$ satisfies the Harnack inequality, it suffices to show that $(M, g(0))$ has bounded curvature. We argue by contradiction, say that $(M, g(0))$ has unbounded curvature. By the strict maximum principle, $\exists$ a real number $\delta > 0$ such that the scalar curvature $S(t)$ of $(M, g(t))$ is strictly positive for $t \in (-\delta, 0]$. Consequently, $(M, g(t))$ is strictly PIC for $t \in (-\delta, 0]$. We distinguish the following two cases:

Case 1: Suppose that $(M, g(0))$ is strictly PIC2. In this case, M is diffeomorphic to $\mathbb{R}^n$ by the soul theorem (cf. [11]). By a standard point-picking argument, there exists a sequence of points $x_j \in M$ such that $Q_j := S(x_j, 0) \geq j^4$ and

$$\sup_{x \in B_{g(0)}(x_j, 2jQ_j^{-\frac{1}{2}})} S(x, 0) \leq 4Q_j.$$

Since $(M, g(t))$ satisfies the Harnack inequality, we derive

$$\sup_{(x,t) \in B_{g(0)}(x_j, 2jQ_j^{-\frac{1}{2}}) \times [-4j^2 Q_j^{-1}, 0]} S(x, t) \leq 4Q_j.$$

Now, Shi's interior derivative estimate (cf. [26]) implies that there are bounds for all the derivatives of curvature on $B_{g(0)}(x_j, jQ_j^{-\frac{1}{2}}) \times [-j^2 Q_j^{-1}, 0]$. Dilating the flow $(M, g(t))$ around the point $(x_j, 0)$ by the factor Q_j, the noncollapsing assumption and the curvature derivative estimates show that, after passing to a subsequence, the rescaled flows converge in the Cheeger-Gromov sense (cf. [10]) to a smooth non-flat ancient solution $(M^\infty, g^\infty(t)), t \in (-\infty, 0]$. The limit $(M^\infty, g^\infty(t))$ is complete, has bounded curvature, is weakly PIC2, and satisfies $R - \theta S \text{ id} \owedge \text{id} \in \text{PIC}$. By Proposition 1.8, the manifold $(M^\infty, g^\infty(0))$ splits off a line. By Corollary 1.7, universal cover of $(M^\infty, g^\infty(t))$ is isometric to a family of shrinking cylinders $\mathbb{S}^{n-1} \times \mathbb{R}$. Therefore, $(M^\infty, g^\infty(0))$ is isometric to a quotient $(\mathbb{S}^{n-1}/\Gamma) \times \mathbb{R}$. If Γ is nontrivial, then a result of Hamilton implies that M contains a nontrivial incompressible $(n-1)$ -dimensional space form $\mathbb{S}^{n-1}/\Gamma$ (cf. [5, Th. A.2]), which is impossible since M is diffeomorphic to $\mathbb{R}^n$. Hence, Γ is trivial, and $(M^\infty, g^\infty(0))$ is isometric to a standard cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Consequently, $(M, g(0))$ contains a sequence of necks with radii converging to 0, which contradicts Proposition 1.9.

Case 2: Suppose finally that $(M, g(0))$ is not strictly PIC2. By Proposition 1.6, the universal cover of $(M, g(t))$ is isometric to a product $(X, g_X(t)) \times \mathbb{R}$ for each $t \in (-\delta, 0]$. Note that $(X, g_X(t)), t \in (-\delta, 0]$, is a complete solution to the Ricci flow of dimension $n-1$, which is κ-noncollapsed on all scales, weakly PIC2, and uniformly PIC1. By assumption, $(X, g_X(0))$ has unbounded curvature. Again, point-picking argument leads to a sequence of points $x_j \in X$ such that $Q_j := S(x_j, 0) \geq j^4$ and

$$\sup_{x \in B_{g_X(0)}(x_j, 2jQ_j^{-\frac{1}{2}})} S(x, 0) \leq 4Q_j.$$

As in Case 1, the Harnack inequality and Shi's interior derivative estimate (cf. [26]) give bounds for all the derivatives of curvature on $B_{g_X(0)}(x_j, jQ_j^{-\frac{1}{2}}) \times [-j^2 Q_j^{-1}, 0]$. Dilating the manifold $(X, g_X(0))$ around the point x_j by the factor Q_j and passing to the limit as $j \to \infty$, we obtain a smooth nonflat limit that is uniformly PIC1 and that must split off a line by Proposition 1.8. By the similar argument in Case 1, this is a contradiction.

$\square$

Theorem 1.11 (cf. S. Brendle [6, Th. 6.18]). *Given $\varepsilon > 0$ and $\theta > 0$, we can find large positive constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ with the following property: Suppose that $(M, g(t))$ is a noncompact ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC that is not locally isometric to a family of shrinking cylinders. Then, for each point (x_0, t_0) in space-time, we can find a neighborhood B of x_0 satisfying*

$$B_{g(t_0)}(x_0, C_1^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subseteq B \subseteq B_{g(t_0)}(x_0, C_1 S(x_0, t_0)^{-\frac{1}{2}})$$

and

$$C_2^{-1} S(x_0, t_0) \leq S(x, t_0) \leq C_2 S(x_0, t_0) \quad \forall\, x \in B.$$

Moreover, B satisfies one of the following conditions:

- *B is an ε-neck with center at x_0.*
- *B is an ε-cap in the sense of Definition mentioned above.*

In particular, $(M, g(t_0))$ is κ_0-noncollapsed for some universal constant $\kappa_0 = \kappa_0(n, \theta)$.

A key point is that the constants C_1 and C_2 in Theorem 1.11 do not depend on κ.

Theorem 1.12 (cf. G. Perelman [23]; B. L. Chen, X. P. Zhu [12]; S. Brendle [6, Th. 6.19]). *Fix $\theta > 0$. We can find a constant $\kappa_0 = \kappa_0(n, \theta)$ such that the following holds: Suppose that $(M, g(t))$ is an ancient κ-solution for some $\kappa > 0$, which in addition satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Then either $(M, g(t))$ is κ_0-noncollapsed for all t; or $(M, g(t))$ is a metric quotient of the round sphere $\mathbb{S}^n$; or $(M, g(t))$ is a noncompact metric quotient of the standard cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Theorem 1.13 (cf. G. Perelman [25, §1.5]). *Given $\varepsilon > 0$ and $\theta > 0$, there exist positive constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ such that the following holds: Assume that $(M, g(t))$ is an ancient κ-solution satisfying $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Then, for each point (x_0, t_0) in space-time, there exists a neighborhood B of x_0 such that $B_{g(t_0)}(x_0, C_1^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 S(x_0, t_0)^{-\frac{1}{2}})$ and $C_2^{-1} S(x_0, t_0) \leq S(x, t_0) \leq C_2 S(x_0, t_0)$ for all $x \in B$. Finally, B satisfies one of the following conditions:*

- *B is an ε-neck with center at x_0.*
- *B is an ε-cap.*
- *B is a closed manifold diffeomorphic to $\mathbb{S}^n / \Gamma$.*
- *B is an ε-quotient neck of the form $(\mathbb{S}^{n-1} \times [-L, L]) / \Gamma$.*

Proof. If M is noncompact, the assertion follows from Theorem 1.11. Hence, it suffices to consider the case when M is compact. As usual, it is enough to consider the case $t_0 = 0$. Suppose that the assertion is false. Then we can find a sequence of compact ancient κ_j-solutions $(M^{(j)}, g^{(j)}(t))$ satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC and a sequence of points $x_j \in M^{(j)}$ with the following property: There does not exist a neighborhood B of x_j with the property that $B_{g^{(j)}(0)}(x_j, j^{-1}S(x_j,0)^{-\frac{1}{2}}) \subseteq B \subseteq B_{g^{(j)}(0)}(x_j, jS(x_j,0)^{-\frac{1}{2}})$, $j^{-1}S(x_j,0) \leq S(x,0) \leq jS(x_j,0)$ for all $x \in B$, and such that B is either an ε-neck with center at x_j; or an ε-cap; or a closed manifold diffeomorphic to S^n/Γ; or an ε-quotient neck. By scaling, we may assume $S(x_j,0) = 1$ for each j.

The noncollapsing assumption implies that $(M^{(j)}, g^{(j)}(t))$ cannot be isometric to a compact quotient of the standard cylinder. By Corollary 1.7, $(M^{(j)}, g^{(j)}(t))$ is strictly PIC2. Clearly, $(M^{(j)}, g^{(j)}(t))$ cannot be isometric to a quotient of a round sphere. By Theorem 1.12, $(M^{(j)}, g^{(j)}(t))$ is κ_0-noncollapsed for some uniform constant κ_0 that is independent of j.

We now apply the compactness theorem for ancient κ_0-solutions to the sequence $(M^{(j)}, g^{(j)}(t), x_j)$. Consequently, after passing to a subsequence if necessary, the sequence $(M^{(j)}, g^{(j)}(t), x_j)$ will converge in the Cheeger-Gromov sense (cf. [10]) to an ancient κ_0-solution satisfying $R - \theta S \, \mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Let us denote this limiting ancient κ_0-solution by $(M^\infty, g^\infty(t))$, and let x_∞ denote the limit of the sequence x_j. There are two possibilities:

Case 1: We first consider the case that M^∞ is compact. In this case, the diameter of $(M^{(j)}, g^{(j)}(0))$ has a uniform upper bound independent of j. Therefore, if j is sufficiently large, then $B^{(j)} := M^{(j)}$ is a neighborhood of the point x_j satisfying $B_{g^{(j)}(0)}(x_j, j^{-1}) \subseteq B^{(j)} \subseteq B_{g^{(j)}(0)}(x_j, j)$ and $j^{-1} \leq S(x,0) \leq j$ for all $x \in B^{(j)}$. Since $(M^{(j)}, g^{(j)}(t))$ is strictly PIC2, results in [8] imply that $B^{(j)} = M^{(j)}$ is diffeomorphic to S^n/Γ. This contradicts our choice of x_j.

Case 2: We now consider the case that M^∞ is noncompact. If $(M^\infty, g^\infty(t))$ is isometric to a noncompact quotient of the standard cylinder, then, for j large enough, the point x_j lies at the center of an ε-neck or it lies on an ε-quotient neck. This contradicts our choice of x_j. Consequently, $(M^\infty, g^\infty(t))$ is not isometric to a quotient of the standard cylinder. At this point, we apply Theorem 1.11 to $(M^\infty, g^\infty(t))$. and with ε replaced by $\frac{\varepsilon}{2}$). Therefore, we can find a neighborhood $B^\infty \subseteq M^\infty$ of the point x_∞ satisfying $B_{g^\infty(0)}(x_\infty, C_1^{-1}) \subseteq B^\infty \subseteq B_{g^\infty(0)}(x_\infty, C_1)$ and $C_2^{-1} \leq S(x,0) \leq C_2$ for all $x \in B^\infty$. Furthermore, B^∞ is either an $\frac{\varepsilon}{2}$-neck with center at x_∞ or an $\frac{\varepsilon}{2}$-cap. Hence, if we choose j sufficiently large, then we can find a neighborhood $B^{(j)} \subseteq M^{(j)}$ of the point x_j satisfying $B_{g^{(j)}(0)}(x_j, (2C_1)^{-1}) \subseteq B^{(j)} \subseteq B_{g^{(j)}(0)}(x_j, 2C_1)$ and $(2C_2)^{-1} \leq S(x,0) \leq 2C_2$ for all $x \in B^{(j)}$. Furthermore, $B^{(j)}$ is either an ε-neck with center at x_j or an ε-cap. This contradicts our choice of x_j.

□

For the purpose of the surgery construction, we will need the following refinement of Theorem 1.13:

Corollary 1.14 (cf. G. Perelman [25, §1.5]). *Given $\varepsilon > 0$ and $\theta > 0$, there exist positive constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ such that the following holds: Assume that $(M, g(t))$ is an ancient κ-solution satisfying $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Then, for each point (x_0, t_0) in space-time, there exists a neighborhood B of x_0 such that $B_{g(t_0)}(x_0, C_1^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 S(x_0, t_0)^{-\frac{1}{2}})$ and $C_2^{-1} S(x_0, t_0) \leq S(x, t_0) \leq C_2 S(x_0, t_0)$ for all $x \in B$. Finally, B satisfies one of the following conditions:*

- *B is a strong ε-neck with center at x_0.*
- *B is an ε-cap.*
- *B is a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$.*
- *B is an ε-quotient neck of the form $(\mathbb{S}^{n-1} \times [-L, L])/\Gamma$.*

Proof. Given $\varepsilon > 0$, $\exists$ a positive real number $\tilde{\varepsilon} < \varepsilon$, depending only on n, θ, ε such that, if (x_0, t_0) lies at the center of an $\tilde{\varepsilon}$-neck, then (x_0, t_0) lies at the center of a strong ε-neck. Hence, the assertion follows from Theorem 1.13. □

2. **The elliptic type estimate**

In this section, we establish a crucial elliptic type estimate of scalar curvature, which plays a role in the Canonical Neighborhood and the surgery procedure. Starting from a key result of Perelman's first paper:

Theorem 2.1 (cf. G. Perelman [23, Cor. 11.6]; S. Brendle [6, Th. 6.12]). *Given a positive real number $w > 0$, we can find positive constants $B = B(w, n)$ and $C = C(w, n)$ such that the following holds: Let $(M, g(t)), t \in [0, T]$, be a solution to the Ricci flow that is weakly PIC2. Suppose that the ball $B_{g(T)}(x_0, r_0)$ is compactly contained in M, and $r_0^{-n} vol_{g(t)}(B_{g(t)}(x_0, r_0)) \geq w$ for each $t \in [0, T]$. Then $S(x, t) \leq C r_0^{-2} + B t^{-1}$ for all $t \in (0, T]$ and all $x \in B_{g(t)}(x_0, \frac{1}{4} r_0)$.*

Note that Perelman imposes the stronger assumption $(M, g(t))$ has nonnegative curvature operator. However, his proof works under the weaker assumption that $(M, g(t))$ is weakly PIC2. One main ingredient in Perelman's work is the trace Harnack inequality (see Theorem 1.3). The proof also relies on the fact that a solution to the Ricci flow that has evolved for some positive time cannot contain an open set that is isometric to a piece of a nonflat cone. This argument relies on the strict maximum principle and works if the solution is weakly PIC2 (see Proposition 1.5).

One of the main tools in Perelman's theory is the long-range curvature estimate for ancient κ-solutions in dimension 3. In the next step, we verify that this estimate holds in our situation.

Theorem 2.2 (G. Perelman [23, §11.7]; H. D. Cao, X. P. Zhu [13, Th. 6.4.3]). *Given $\kappa > 0$, there exists a positive function $\omega : [0, \infty) \to (0, \infty)$, depending on n and κ with the following* *property: Let $(M, g(t))$ be a an ancient κ-solution. Then*

$$S(x,t) \leq S(y,t)\omega(S(y,t)d_{g(t)}(x,y)^2)$$

for all points $x, y \in M$ and all t.

Proof. The proof is essentially the same as in Section 11.7 in Perelman's paper [23] (see also [12]). We sketch the argument for the convenience of the reader. Let us fix a point $y \in M$. By rescaling, we can arrange that $S(y,0) = 1$. For abbreviation, let $A = \{x \in M \big| S(x,0)d_{g(0)}(y,x)^2 \geq 1\}$ We distinguish two cases:

Case 1: Suppose that $A = \emptyset$. In this case, we can find a point $z \in M$ such that $\sup_{x\in M} S(x,0) = S(z,0)$. Using the Harnack inequality (cf. Theorem 1.3), we obtain

$$\sup_{x\in M} S(x,t) \leq S(z,0) \quad \forall\, t \in (-\infty, 0].$$

Shi's derivative estimates (cf. [26]) leads to $\partial_t S(z,t) \leq C(n)S(z,0)^2 \quad \forall\, t \in [-S(z,0)^{-1}, 0]$. Moreover, $d_{g(0)}(y,z) \leq S(z,0)^{-\frac{1}{2}}$ since $A = \emptyset$. Hence, we can find a small positive constant β, depending only on n, such that for all $t \in [-\beta S(z,0)^{-1}, 0]$,

$$S(z,t) \geq \frac{1}{2}S(z,0) \quad \text{ and } \quad d_{g(t)}(y,z) \leq 2S(z,0)^{-\frac{1}{2}}.$$

If we apply the Harnack inequality (cf. Corollary 1.4) with $t = -\beta S(z,0)^{-1}$, then we obtain

$$\begin{aligned}\frac{1}{2}S(z,0) &\leq S(z,t)\\ &\leq \exp(\frac{d_{g(t)}(y,z)^2}{(-2t)})S(y,0)\\ &\leq \exp(\frac{2}{(-t)S(z,0)})S(y,0)\\ &= \exp(\frac{2}{\beta})\end{aligned}$$

Putting these facts together, we conclude that $\sup_{x\in M} S(x,0) \leq 2\exp(\frac{2}{\beta})$.

Case 2: Suppose now that $A \neq \emptyset$. In this case, we choose a point $z \in A$ that has minimal distance from y with respect to the metric $g(0)$. Notice that $S(z,0) = d_{g(0)}(y,z)^{-2}$ since z lies on the boundary of A. Let p be the mid-point of the minimizing geodesic in $(M, g(0))$ joining y and z. Note that $B_{g(0)}(p, \frac{1}{4}d_{g(0)}(y,z)) \cap A = \emptyset$, then

$$\sup_{x\in B_{g(0)}(p,\frac{1}{4}d_{g(0)}(y,z))} S(x,0) \leq 16 d_{g(0)}(y,z)^{-2}.$$

By the Harnack inequality (cf. Theorem 1.3),

$$\sup_{x\in B_{g(t)}(p,\frac{1}{4}d_{g(0)}(y,z))} S(x,t) \leq 16 d_{g(0)}(y,z)^{-2} \quad \forall\, t \in (-\infty, 0].$$

The noncollapsing assumption gives

$$vol_{g(t)}(B_{g(t)}(p, \frac{1}{4}d_{g(0)}(y,z))) \geq \kappa(\frac{1}{4}d_{g(0)}(y,z))^n \quad \forall\, t \in (-\infty, 0],$$

which implies

$$(4r)^{-n}\,\mathrm{vol}_{g(t)}(B_{g(t)}(p,4r)) \geq \kappa(\frac{1}{16}r^{-1}d_{g(0)}(y,z))^n \quad \forall\, t \in (-\infty,0] \text{ and } r \geq d_{g(0)}(y,z).$$

Applying Theorem 2.1 with $x_0 := p$, $r_0 := 4r$, and $w := \kappa(\frac{1}{16}r^{-1}d_{g(0)}(y,z))^n$, we obtain

$$\sup_{x \in B_{g(0)}(p,r)} S(x,0) \leq d_{g(0)}(y,z)^{-2}\omega(d_{g(0)}(y,z)^{-1}r) \quad \forall\, r \geq d_{g(0)}(y,z),$$

where $\omega : [1,\infty) \to [0,\infty)$ is a positive and increasing function that may depend on n and κ. In particular, if we put $r = d_{g(0)}(y,z)$ and apply the Harnack inequality (cf. Theorem 1.3), then

$$\sup_{x \in B_{g(0)}(p, d_{g(0)}(y,z))} S(x,t) \leq d_{g(0)}(y,z)^{-2}\omega(1) \quad \forall\, t \in (-\infty, 0].$$

By Shi's derivative estimates (cf. [26]),

$$\partial_t S(z,t) \leq C(n,\kappa) d_{g(0)}(y,z)^{-4} \quad \forall\, t \in [-d_{g(0)}(y,z)^2, 0].$$

Moreover, $S(z,0) = d_{g(0)}(y,z)^{-2}$ by our choice of z. Therefore, there exists a small positive constant β, depending only on n and κ, such that $S(z,t) \geq \frac{1}{2}d_{g(0)}(y,z)^{-2}$ and $d_{g(t)}(y,z) \leq 2d_{g(0)}(y,z)$ for all $t \in [-\beta d_{g(0)}(y,z)^2, 0]$. If we apply the Harnack inequality (cf. Corollary 1.4) with $t = -\beta d_{g(0)}(y,z)^2$, then

$$\begin{aligned}
\frac{1}{2}d_{g(0)}(y,z)^{-2} &\leq S(z,t) \\
&\leq \exp(\frac{d_{g(t)}(y,z)^2}{(-2t)})S(y,0) \\
&\leq \exp(\frac{2d_{g(0)}(y,z)^2}{(-t)})S(y,0) \\
&= \exp(\frac{2}{\beta}).
\end{aligned}$$

It finally leads to $\forall\, r \geq 0$,

$$\begin{aligned}
\sup_{x \in B_{g(0)}(y,r)} S(x,0) &\leq \sup_{x \in B_{g(0)}(p, r + d_{g(0)}(y,z))} S(x,0) \\
&\leq d_{g(0)}(y,z)^{-2}\omega(d_{g(0)}(y,z)^{-1}r + 1) \\
&\leq 2e^{\frac{2}{\beta}}\omega(\sqrt{2}e^{\frac{1}{\beta}}r + 1)
\end{aligned}$$

□

The following immediately results from Theorem 2.2, the Harnack inequality (cf. Theorem 1.3) and the local derivative estimate of Shi (cf. [26]). In fact, Shi's derivative estimates shows that we can bound the m-th covariant derivative of the Riemann curvature tensor by a constant times $S^{\frac{m+2}{2}}$ at each point in space-time.

Corollary 2.3. *Given $\kappa > 0$, we can find a large positive constant $\eta = \eta(n, \kappa)$ with the following property: Let $(M, g(t))$ be an ancient κ-solution. Then $|\nabla S| \leq \eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq \eta S^2$ at each point in space-time.*

Now, we state the main result of this section:

Corollary 2.4. *Fix $\theta > 0$. We can find a constant $\eta = \eta(n, \theta)$ such that the following holds: Suppose that $(M, g(t))$ is an ancient κ-solution for some $\kappa > 0$, which in addition satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Then $|\nabla S| \leq \eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq \eta S^2$ at each point in space-time.*

Proof. If $(M, g(t))$ is a metric quotient of S^n or $S^{n-1} \times \mathbb{R}$, the assertion is trivial. Otherwise, Theorem 1.12 implies that $(M, g(t))$ is an ancient κ_0-solution, where κ_0 depends only on n and θ. Hence, the assertion follows from Corollary 2.3. □

3. A Canonical Neighborhood Theorem for Ricci flows starting from initial metrics with positive isotropic curvature

In this section, we consider a solution of the Ricci flow starting from a compact manifold of dimension $n \geq 12$ with positive isotropic curvature. Our goal is to establish an analogue of Perelman's Canonical Neighborhood Theorem. We begin with a definition:

Definition. Assume that $f : [0, \infty) \to [0, \infty)$ is a concave and increasing function satisfying $\lim_{s \to \infty} \frac{f(s)}{s} = 0$, and θ is a positive real number. We say that a Riemannian manifold has (f, θ)-pinched curvature if $R + f(S) \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC2}$ and $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$.

If (M, g_0) is a compact manifold of dimension $n \geq 12$ with positive isotropic curvature, then Corollary 1.2 implies that the subsequent solution to the Ricci flow has (f, θ)-pinched curvature for some suitable choice of f and θ.

Theorem 3.1. *Let (M, g_0) be a compact manifold with positive isotropic curvature of dimension $n \geq 12$, which does not contain any nontrivial incompressible $(n-1)$-dimensional space forms. Let $g(t), t \in [0, T)$, denote the solution to the Ricci flow with initial metric g_0. Given a small number $\tilde{\varepsilon} > 0$ and a large number A_0, we can find a positive number $\hat{r}$ with the following property: If (x_0, t_0) is a point in space-time with $Q := S(x_0, t_0) \geq \hat{r}^{-2}$, then the parabolic neighborhood $B_{g(t_0)}(x_0, A_0 Q^{-\frac{1}{2}}) \times [t_0 - A_0 Q^{-1}, t_0]$ is, after scaling by the factor Q, $\tilde{\varepsilon}$-close to the corresponding subset of an ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC.*

Proof. By Corollary 1.2, the flow $(M, g(t))$ has (f, θ)-pinched curvature for some function f satisfying $\lim_{s\to\infty} \frac{f(s)}{s} = 0$ and some constant $\theta > 0$. Let us fix a small number $\varepsilon > 0$, and let $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ denote the constants in Corollary 1.14. It suffices to prove the assertion when $A_0 \geq 8C_1$ and $\tilde{\varepsilon}$ is much smaller than ε. To do that, we argue by contradiction. If the assertion is false, then we can find a sequence of points (x_j, t_j) in space-time with the following properties:

(i) $Q_j := S(x_j, t_j) \geq j^2$

(ii) After dilating by the factor Q_j, the parabolic neighborhood

$$B_{g(t_j)}(x_j, A_0 Q_j^{-\frac{1}{2}}) \times [t_j - A_0 Q_j^{-1}, t_j]$$

is not $\tilde{\varepsilon}$-close to the corresponding subset of any ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC.

By a point-picking argument, we can arrange that (x_j, t_j) satisfies the following condition:

(iii) If $(\tilde{x}, \tilde{t})$ is a point in space-time such that $\tilde{t} \leq t_j$ and $\tilde{Q} := S(\tilde{x}, \tilde{t}) \geq 4Q_j$, then the parabolic neighborhood $B_{g(\tilde{t})}(\tilde{x}, A_0 \tilde{Q}^{-\frac{1}{2}}) \times [\tilde{t} - A_0 \tilde{Q}^{-1}, \tilde{t}]$ is, after dilating by the factor $\tilde{Q}$, $\tilde{\varepsilon}$-close to the corresponding subset of an ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC.

The detail construction is followed below. Suppose NOT, set $(x_{j_1}, t_{j_1}) = (x_j, t_j)$, then we can pick $(x_{j_l}, t_{j_l}) \in M \times \left[t_{j_{(l-1)}} - A_0\, S(x_{j_{(l-1)}}, t_{j_{(l-1)}})^{-1}, t_{j_{(l-1)}}\right]$ s.t. $\tilde{Q} = S(x_{j_l}, t_{j_l}) \geq 4Q_j$, but the parabolic neighborhood is not $\tilde{\varepsilon}$-close to the corresponding subset of ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC and we have $S(x_{j_l}, t_{j_l}) \geq 4\, S(x_{j_{(l-1)}}, t_{j_{(l-1)}})$, for each $l \in \mathbb{N}$. Since the solution is smooth, but

$$S(x_{j_l}, t_{j_l}) \geq 4\, S(x_{j_{(l-1)}}, t_{j_{(l-1)}}) \geq \cdots \geq 4^{l-1}\, S(x_j, t_j),$$

and

$$\begin{aligned} t_j \geq t_{j_{(l-1)}} \geq t_{j_l} &\geq t_{j_{(l-1)}} - A_0\, S(x_{j_{(l-1)}}, t_{j_{(l-1)}})^{-1} \\ &\geq t_k - A_0 \sum_{i=1}^{l-1} \frac{1}{2^{i-1}}\, S(x_j, t_j)^{-1} \geq 0 \end{aligned}$$

So, to avoid the scalar curvature blowing up, this process must terminate after finite number of steps and the last one fits.

Our strategy is to rescale the flow $(M, g(t))$ around the point (x_j, t_j) by the factor Q_j. We will show that the rescaled flows converge to an ancient κ-solution satisfying $R - \theta S$ id $\owedge$ id $\in$ PIC. To that end, we proceed in several steps:

Step 1: We first establish a pointwise curvature derivative estimate. By Corollary 2.4, we can find a large constant η, depending only on n and θ, such that $|\nabla S| \leq \eta\, S^{\frac{3}{2}}$ and $|\frac{\partial}{\partial t} S| \leq \eta\, S^2$ on every ancient κ-solution. Using property (iii) above, we conclude that $|\nabla S| \leq 2\eta\, S^{\frac{3}{2}}$ and $|\frac{\partial}{\partial t} S| \leq 2\eta\, S^2$ for each point (x,t) in space-time satisfying $t \leq t_j$ and $S(x,t) \geq 4Q_j$.

Step 2: We next prove bounds for the higher order covariant derivatives of the curvature tensor. Suppose that (x,t) is a point in space-time satisfying $S(x,t) + Q_j \leq r_0^{-2}$. The pointwise curvature derivative estimate in Step 1 implies that $S \leq 8r_0^{-2}$ in the parabolic neighborhood $P(x, t, \frac{r_0}{100\eta}, -\frac{r_0^2}{100\eta})$. Using Shi's interior derivative estimates (cf. [26]), we conclude $|\nabla^m R| \leq C(n, m, \eta) r_0^{-m-2}$ at the point (x,t).

Step 3: We next prove a long-range curvature estimate. Given any $\rho > 0$, we define

$$\mathbb{M}(\rho) = \limsup_{j\to\infty} \sup_{x \in B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})} Q_j^{-1} S(x, t_j)$$

For now, we have $\mathbb{M}(\rho) \leq 8$ for $0 < \rho < \frac{1}{100\eta}$, because we can integrate the scalar curvature over the segment from (x_j, t_j) and use the result of Step 1. We claim that $\mathbb{M}(\rho) < \infty$ for all $\rho > 0$. Suppose this is false. Let

$$\rho^* = \sup\{\rho \geq 0 : \mathbb{M}(\rho) < \infty\} < \infty$$

Clearly, $\rho^* \geq \frac{1}{100\eta}$. By definition of ρ^*, we have an upper bound for the curvature in the geodesic ball $B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})$ for each $\rho < \rho^*$. By Step 2, we obtain bounds for the covariant derivatives of the curvature tensor in the geodesic ball $B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})$ for each $\rho < \rho^*$. Moreover, Perelman's noncollapsing estimate gives a lower bound for the volume. We rescale around (x_j, t_j) by the factor Q_j and pass to the limit as $j \to \infty$. In the limit, we obtain an incomplete manifold (B^∞, g^∞) that is weakly PIC2. Clearly, $\rho^* \geq \frac{1}{100\eta}$, so $B^\infty \neq \emptyset$. Note that S is smooth, so $\mathbb{M}(\rho^*) = \infty$. By definition of ρ^*, we can find a sequence of points y_j such that

$$\rho_j := Q_j^{\frac{1}{2}} d_{g(t_j)}(x_j, y_j) \to \rho^* \quad \text{and} \quad Q_j^{-1} S(y_j, t_j) \to \infty$$

For each j, we can find a unit speed geodesic $\gamma_j : [0, \rho_j Q_j^{-\frac{1}{2}}] \to (M, g(t_j))$ such that $\gamma_j(0) = x_j$ and $\gamma_j(\rho_j Q_j^{-\frac{1}{2}}) = y_j$. Let $\gamma_\infty : [0, \rho^*) \to (B^\infty, g^\infty)$ denote the limit of γ_j. Using the pointwise curvature derivative estimate in Step 1, we obtain

$$S_{g^\infty}(\gamma_\infty(s)) = \lim_{j\to\infty} Q_j^{-1} S(\gamma_j(s Q_j^{-\frac{1}{2}}), t_j) \geq (\eta(\rho^* - s))^{-2} \geq 100$$

if $s \in [\rho^* - \frac{1}{100\eta}, \rho^*)$.

Let us consider a real number $\bar{s} \in [\rho^* - \frac{1}{100\eta}, \rho^*)$ such that $8C_1\eta(\rho^* - \bar{s}) \leq \bar{s}$.

We claim that $\gamma_j(\bar{s} Q_j^{-\frac{1}{2}})$ lies at the center of a 2ε-neck if j is sufficiently large (depending on $\bar{s}$). Observe that if j is sufficiently large, it follows from property (iii) and Corollary 1.14 that the point $(\gamma_j(\bar{s} Q_j^{-\frac{1}{2}}), t_j)$ has a Canonical Neighborhood that is either a 2ε-neck; or a 2ε-cap; or a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$; or a 2ε-quotient neck. Thus we want to rule out the last three candidates. Before that, we need two more observations. Recall that (Corollary 1.14) the Canonical Neighborhood is contained in a geodesic ball around $\gamma_j(\bar{s}Q_j^{-\frac{1}{2}})$ of radius $2C_1S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)^{-\frac{1}{2}}$, and the scalar curvature is at most $2C_2S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$ at each point in the Canonical Neighborhood. Hence, we get:

- The Canonical Neighborhood does not contain the point y_j, provided that j is sufficiently large.: Since $\mathbb{M}(\bar{s}) < \infty$, we can compare two limits

  $$\limsup_{j\to\infty} Q_j^{-1}S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j) < \infty \quad \text{and} \quad \lim_{j\to\infty} Q_j^{-1}S(y_j,t_j) = \infty$$

  and obtain

  $$\lim_{j\to\infty} S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)^{-1}S(y_j,t_j) = \infty;$$

  consequently if j is sufficiently large, $S(y_j,t_j) \geq 4C_2S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$, which is larger than maximum of scalar curvature on the Canonical Neighborhood. Hence, it does not contain the point y_j.
- The Canonical Neighborhood does not contain the point x_j, provided that j is sufficiently large.: First, observe that

  $$8C_1S_{g^\infty}(\gamma_\infty(\bar{s}))^{-\frac{1}{2}} \leq 8C_1\eta(\rho^* - \bar{s}) \leq \bar{s},$$

  and

  $$S_{g^\infty}(\gamma_\infty(\bar{s})) = \lim_{j\to\infty} Q_j^{-1}S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$$

  This implies if j is sufficiently large,

  $$4C_1S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)^{-\frac{1}{2}} \leq 8C_1S_{g^\infty}(\gamma_\infty(\bar{s}))^{-\frac{1}{2}} \leq \bar{s}Q_j^{-\frac{1}{2}}.$$

  It means that $d_{g(t_j)}(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),x_j) = \bar{s}Q_j^{-\frac{1}{2}}$ larger than radius of the Canonical Neighborhood. Hence, it does not contain the point x_j.

In particular, if j is sufficiently large, then the Canonical Neighborhood of $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$ cannot be a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$. Moreover, if the Canonical Neighborhood of $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$ is a 2ε-cap, then the geodesic γ_j must enter and exit this 2ε-cap, but this is impossible since γ_j minimizes length. Finally, if the Canonical Neighborhood of $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$ is a quotient neck, then Theorem A.1 in [5] implies that M contains a nontrivial incompressible $(n-1)$-dimensional space form, contrary to our assumption. To summarize, if j is sufficiently large (depending on $\bar{s}$), then the point $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$ has a Canonical Neighborhood that is a 2ε-neck. In particular, if j is sufficiently large (depending on $\bar{s}$), then we have $|\nabla S| \leq C(n)\varepsilon\, S^{\frac{3}{2}}$ at the point $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}),t_j)$.

Passing to the limit as $j\to\infty$, we conclude that $|\nabla S_{g^\infty}| \leq C(n)\varepsilon\, S_{g^\infty}^{\frac{3}{2}}$ at the point $\gamma_\infty(\bar{s})$. Integrating this estimate along γ_∞ gives $S_{g^\infty}(\gamma_\infty(\bar{s})) \geq (C(n)\varepsilon(\rho^* - \bar{s}))^{-2}$.

Moreover, since $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ lies at the center of a 2ε-neck for j sufficiently large, the point $\gamma_\infty(\bar{s})$ must lie on a $C(n)\varepsilon$-neck in (B^∞, g^∞).

As in the Perelman's paper (cf. [22, §12.1]), there is a sequence of rescalings that converges to a piece of a nonflat metric cone in the limit. Using the pointwise curvature derivative estimate established in Step 1, we can extend the metric backwards in time. This gives a locally defined solution to the Ricci flow that is weakly PIC2 and that, at the final time, is a piece of nonflat metric cone. This contradicts Proposition 1.5.

Step 4: We now rescale the manifold $(M, g(t_j))$ around the point x_j by the factor Q_j. By Step 3, we have uniform bounds for the curvature at bounded distance. Using the curvature derivative estimate in Step 1 together with Shi's interior derivative estimates (cf. [26]), we conclude that the covariant derivatives of the curvature tensor are bounded at bounded distance. Combining this with Perelman's noncollapsing estimate, we conclude that (after passing to a subsequence) the rescaled manifolds converge in the Cheeger-Gromov sense (cf. [10]) to a complete smooth limit manifold (M^∞, g^∞). Since $(M, g(t_j))$ has (f, θ)-pinched curvature, the curvature tensor of (M^∞, g^∞) is weakly PIC2 and satisfies $R - \theta S \text{ id} \owedge \text{id} \in \text{PIC}$. Using property (iii) and Corollary 1.14, we conclude that every point in (M^∞, g^∞) with scalar curvature greater than 4 (since we normalize the curvature) has a Canonical Neighborhood that is either a 2ε-neck; or a 2ε-cap; or a 2ε-quotient neck. Note that the last possibility cannot occur; indeed, if (M^∞, g^∞) contains a quotient neck, then $(M, g(t_j))$ contains a quotient neck for j sufficiently large, and Theorem A.1 in [5] then implies that M contains a nontrivial incompressible $(n-1)$-dimensional space form, contrary to our assumption.

We claim that the limit manifold (M^∞, g^∞) has bounded curvature. Indeed, if there is a sequence of points in (M^∞, g^∞) with curvature going to infinity, then (M^∞, g^∞) contains a sequence of necks with radii converging to 0, contradicting Proposition 1.9. Thus, (M^∞, g^∞) has bounded curvature.

Step 5: We now extend the limit (M^∞, g^∞) backwards in time. By Step 4, the scalar curvature of (M^∞, g^∞) is bounded from above by a constant $\Lambda > 4$. Using the pointwise curvature derivative estimate in Step 1, we conclude that

$$\limsup_{j\to\infty} \sup_{(x,t)\in B_{g(t_j)}(x_j, AQ_j^{-\frac{1}{2}})\times[t_j - \frac{1}{100\eta\Lambda}Q_j^{-1}, t_j]} Q_j^{-1} S(x,t) \leq 2\Lambda$$

for each $A > 1$. Hence, if we put $\tau_1 := -\frac{1}{200\eta\Lambda}$, then we can extend (M^∞, g^∞) backwards in time to a complete solution $(M^\infty, g^\infty(t))$ of the Ricci flow that is defined for $t \in [\tau_1, 0]$ and satisfies $\Lambda_1 := \sup_{t\in[\tau_1,0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda$ In the next step, we put $\tau_2 := \tau_1 - \frac{1}{200\eta\Lambda_1}$. Using the pointwise curvature derivative estimate in Step 1, we can extend the solution $(M^\infty, g^\infty), t \in [\tau_1, 0]$ backwards in time to a solution $(M^\infty, g^\infty(t)), t \in [\tau_2, 0]$. Moreover, $\Lambda_2 := \sup_{t\in[\tau_2,0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda_1$. Continuing this process, we can extend the solution backwards in time to the interval $[\tau_m, 0]$, where $\tau_{m+1} := \tau_m - \frac{1}{200\eta\Lambda_m}$ and $\Lambda_{m+1} := \sup_{t\in[\tau_{m+1},0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda_m$.

Let $\tau^* = \lim_{m\to\infty} \tau_m \leq -\frac{1}{200\eta\Lambda}$. Using a standard diagonal sequence argument, we obtain a complete, smooth limit flow $(M^\infty, g^\infty(t))$ that is defined on the interval $(\tau^*, 0]$ and that has bounded curvature for each $t \in (\tau^*, 0]$. Since $(M, g(t))$ has (f, θ)-pinched curvature, the curvature tensor of the limit flow $(M^\infty, g^\infty(t))$ is weakly PIC2 and satisfies $R - \theta S \text{ id} \owedge \text{id} \in \text{PIC}$.

Step 6: We claim that $\tau^* = -\infty$. To prove this, we argue by contradiction. Suppose $\tau^* > -\infty$. Clearly, $\lim_{m\to\infty}(\tau_m - \tau_{m+1}) = 0$, hence $\lim_{m\to\infty} \Lambda_m = \infty$.

By the Harnack inequality (cf. Theorem 1.3), the function $t \mapsto (t - \tau^*) S_{g^\infty(t)}(x)$ is monotone increasing at each point $x \in M^\infty$. Since $S_{g^\infty(0)}(x) \leq \Lambda$ for all $x \in M^\infty$, we obtain

$$S_{g^\infty(t)}(x) \leq \frac{-\tau^*}{t - \tau^*}\Lambda$$

for all $x \in M^\infty$ and all $t \in (\tau^*, 0]$. Using Lemma 8.3(b) in [23], we conclude that

$$0 \leq -\frac{d}{dt} d_{g^\infty(t)}(x, y) \leq C(n)\sqrt{\frac{-\tau^*}{t - \tau^*}\Lambda}$$

for all $x, y \in M^\infty$ and all $t \in (\tau^*, 0]$. Integrating over t gives

$$d_{g^\infty(0)}(x, y) \leq d_{g^\infty(t)}(x, y) \leq d_{g^\infty(0)}(x, y) + C(n)(-\tau^*)\sqrt{\Lambda}$$

for all $x, y \in M^\infty$ and all $t \in (\tau^*, 0]$.

By the maximum principle,

$$\inf_{M^\infty} S_{g^\infty(t)} \leq \inf_{M^\infty} S_{g^\infty(0)} \leq 1$$

for all $t \in (\tau^*, 0]$. Hence, we can find a point $y_\infty \in M^\infty$ such that $S_{g^\infty(t)}(y_\infty) \leq 4$ for $t = \tau^* + \frac{1}{1000\eta\Lambda} \in (\tau^*, 0]$. Using the pointwise curvature derivative estimate in Step 1, we obtain $S_{g^\infty(t)}(y_\infty) \leq 8$ for all $t \in (\tau^*, \tau^* + \frac{1}{1000\eta\Lambda}]$. In particular, $S_{g^\infty(\tau_m)}(y_\infty) \leq 8$ if m is sufficiently large. Arguing as in Step 3 , we can show that

$$\limsup_{m\to\infty} \sup_{B_{g^\infty(\tau_m)}(y_\infty, A)} S_{g^\infty(\tau_m)} < \infty$$

for every $A > 1$.

Consequently, a subsequence of the manifolds $(M^\infty, g^\infty(\tau_m), y_\infty)$ converges in the Cheeger-Gromov sense (cf. [10]) to a complete, smooth limit. If this limit manifold has unbounded curvature, then (by property (iii) above) it contains a sequence of necks with radii converging to 0, contradicting Proposition 1.9. Therefore, a subsequence of the manifolds $(M^\infty, g^\infty(\tau_m), y_\infty)$ converges in the Cheeger-Gromov (cf.

[10]) sense to a complete, smooth limit with bounded curvature. Consequently, we can find a constant $\Lambda^* > \Lambda$ (independent of A) such that

$$\liminf_{m\to\infty} \sup_{B_{g^\infty(\tau_m)}(y_\infty,A)} S_{g^\infty(\tau_m)} \leq \Lambda^*$$

for every $A > 1$. Using the distance estimate, we obtain $B_{g^\infty(0)}(y_\infty, A) \subset B_{g^\infty(\tau_m)}(y_\infty, A + C(n)(-\tau^*)\sqrt{\Lambda})$. Putting these facts together, we conclude that

$$\liminf_{m\to\infty} \sup_{B_{g^\infty(0)}(y_\infty,A)} S_{g^\infty(\tau_m)} \leq \Lambda^*$$

for every $A > 1$. Hence, for each $A > 1$, we can find a large integer m (depending on A) such that $\tau_m \in (\tau^*, \tau^* + \frac{1}{1000\eta\Lambda^*}]$ and

$$\sup_{B_{g^\infty(0)}(y_\infty,A)} S_{g^\infty(\tau_m)} \leq 2\Lambda^*.$$

Using the pointwise derivative estimate in Step 1, we obtain

$$\sup_{t\in(\tau^*,\tau^*+\frac{1}{1000\eta\Lambda^*}]} \sup_{B_{g^\infty(0)}(y_\infty,A)} S_{g^\infty(t)} \leq 4\Lambda^*$$

for every $A > 1$. Since Λ^* is independent of A, we conclude that

$$\sup_{t\in(\tau^*,\tau^*+\frac{1}{1000\eta\Lambda^*}]} \sup_{M^\infty} S_{g^\infty(t)} \leq 4\Lambda^*.$$

Therefore, the flow $(M^\infty, g^\infty(t)), t \in (\tau^*, 0]$, has bounded curvature. This contradicts the fact that $\lim_{m\to\infty} \Lambda_m = \infty$. Thus, $\tau^* = -\infty$.

To summarize, if we dilate the flow $(M, g(t))$ around the point (x_j, t_j) by the factor Q_j, then (after passing to a subsequence), the rescaled flows converge in the Cheeger-Gromov sense (cf. [10]) to an ancient κ-solution $(M^\infty, g^\infty(t))$, $t \in (-\infty, 0]$, satisfying $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Here, κ depends only on the initial data. This contradicts statement (ii). This completes the proof of Theorem 3.1. □

Finally, by combining Theorem 3.1 with Corollary 1.14, we can draw the following conclusion:

Corollary 3.2. *Let (M, g_0) be a compact manifold with positive isotropic curvature of dimension $n \geq 12$, which does not contain any nontrivial incompressible $(n-1)$-dimensional space forms. Let $g(t), t \in [0, T)$, denote the solution to the Ricci flow with initial metric g_0. Given any $\varepsilon > 0$, there exists a positive number $\hat{r}$ with the following property: If (x_0, t_0) is a point in space-time with $Q := S(x_0, t_0) \geq \hat{r}^{-2}$, then we can find a neighborhood B of x_0 such that*

$$B_{g(t_0)}(x_0, (2C_1)^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1 S(x_0, t_0)^{-\frac{1}{2}})$$

and

$$(2C_2)^{-1} S(x_0, t_0) \leq S(x, t_0) \leq 2C_2 S(x_0, t_0)$$

for all $x \in B$. Furthermore, B satisfies one of the following conditions:

(1) B is a strong 2ε-neck with center at x_0.
(2) B is a 2ε-cap.
(3) B is a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$.

Here, $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ are the constants appearing in Corollary 1.14. Finally, we have $|\nabla S| \leq 2\eta\, S^{\frac{3}{2}}$ and $|\frac{\partial}{\partial t} S| \leq 2\eta\, S^2$ at the point (x_0, t_0), where η is a constant that depends only on n and θ.

4. **The behavior of the flow at the first singular time**

Throughout this section, we fix a compact initial manifold (M, g_0) of dimension $n \geq 12$ that has positive isotropic curvature and does not contain any nontrivial incompressible $(n-1)$-dimensional space forms. Let $(M, g(t))$ be the solution of the Ricci flow with initial metric g_0, and let $[0, T)$ denote the maximal time interval on which the solution is defined. Note that $T \leq \frac{n}{2 \inf_{x \in M} S(x,0)}$. By Theorem 1.1, we can find a continuous family of closed, convex, $O(n)$-invariant sets $\{\mathcal{F}_t : t \in [0, T]\}$ such that the family $\{\mathcal{F}_t : t \in [0, T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$; the curvature tensor of (M, g_0) lies in the set $\mathcal{F}_0$; and

$$\mathcal{F}_t \subset \{R : R - \theta S \text{ id} \owedge \text{id} \in \text{PIC}\} \cap \{R : R + f(S)\text{id} \owedge \text{id} \in \text{PIC2}\}$$

for all $t \in [0, T]$. Here, f is a concave and increasing function satisfying $\lim_{s \to \infty} \frac{f(s)}{s} = 0$, and θ and N are positive numbers. Note that f, θ, and N depend only on the initial data. By Hamilton's PDE-ODE principle, the curvature tensor of $(M, g(t))$ lies in the set $\mathcal{F}_t$ for each $t \in [0, T)$.

By Corollary 3.2, every point in space-time where the scalar curvature is sufficiently large admits a Canonical Neighborhood that is either a 2ε-neck; or a 2ε-cap; or a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$. Let ρ be a small positive number with the property that every point with $S \geq \frac{1}{4}\rho^{-2}$ satisfies the conclusion of the Canonical Neighborhood Theorem. In particular, we have $|\nabla S| \leq 2\eta\, S^{\frac{3}{2}}$ and $|\frac{\partial}{\partial t} S| \leq 2\eta\, S^2$ whenever $S \geq \frac{1}{4}\rho^{-2}$. We define

$$\Omega := \{x \in M : \limsup_{t \to T} S(x, t) < \infty\}.$$

The pointwise curvature derivative estimate implies that Ω is an open subset of M. Using the pointwise curvature derivative estimate together with Shi's interior estimates (cf. [26]), we conclude the metrics $g(t)$ converge to a smooth metric $g(T)$ on Ω. Following Perelman's paper, we consider the set

$$\Omega_\rho := \{x \in M : \limsup_{t \to T} S(x, t) \leq \rho^{-2}\} = \{x \in \Omega : S(x, T) \leq \rho^{-2}\}$$

We distinguish two cases:

Case 1: Suppose that $\Omega_\rho = \emptyset$. Using the inequality $|\frac{\partial}{\partial t} S| \leq 2\eta\, S^2$, we obtain $\inf_{x\in M} S(x,t) \geq \frac{1}{2}\rho^{-2}$ if t is sufficiently close to T. Hence, if t is sufficiently close to T, then every point in $(M, g(t))$ admits a Canonical Neighborhood that is either a 2ε-neck; or a 2ε-cap; or a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$.

Case 2: Suppose now that $\Omega_\rho \neq \emptyset$. Pick any 2ε-neck $\subseteq \mathrm{cl}(\Omega \setminus \Omega_\rho)$. Choose a point x on one side of the boundary of the neck. Suppose that $x \in \Omega \setminus \Omega_\rho$, then there exists another 2ε-cap, or 2ε-neck adjacent to the initial neck. In the latter case, we can take the point on the second boundary of the second 2ε-neck and continue. We will stop this process only when we get a 2ε-cap, or we get a point in Ω_ρ. Otherwise, we get an infinite many 2ε-necks, which produces a 2ε-horn. The same procedure can be repeated on the other boundary of the initial 2ε-neck.

The upshot is that every point in $\Omega \setminus \Omega_\rho$ lies either

(i) on 2ε-tube with boundary in Ω_ρ (from adjoining finite many necks on both side); or
(ii) on a 2ε-cap with boundary in Ω_ρ (from adjoining finite many necks on one side and a cap on one side at the end); or
(iii) on a closed manifold diffeomorphic to S^n/Γ (from adjoining a cap on both side at the end); or
(iv) on a 2ε-horn (from adjoining infinite many necks on one side and finite many on the other side)
(v) on a double 2ε-horn (from adjoining infinite many necks on both sides)
(vi) on a capped 2ε-horn (from adjoining infinite many necks on one side and a cap on the other side at the end).

Following Perelman, we perform surgery on (iv). We leave unchanged all the (i), and all (ii). We discard all the others, i.e. (iii), (v), and (vi), because we still can capture those information when we want to know the diffeomorphic type of M. The reason for (v) and (vi) is that, slightly before the surgery time T, double 2ε-horns and 2ε-horns are still 2ε-tubes and capped 2ε-tubes.

Proposition 4.1. *The pre-surgery manifold M is diffeomorphic to a connected sum of the post-surgery manifold with a finite collection of standard spaces, each of which is a quotient of $\mathbb{S}^n$ or $\mathbb{S}^{n-1}\times\mathbb{R}$ by standard isometries.*

Proof. Suppose first that $\Omega_\rho = \emptyset$. In this case, M is diffeomorphic to either a quotient of $\mathbb{S}^n$ by standard isometries; or a tube with caps attached on both sides; or an $\mathbb{S}^{n-1}$-bundle over S^1 with a fiberwise round metric. In the second case, Definition of caps ensures that M is diffeomorphic to $\mathbb{S}^n$. To handle the third case, we note that that there are two $\mathbb{S}^{n-1}$-bundles over S^1 with a fiberwise round metric. One of them is orientable, the other one is not. Both are diffeomorphic to quotients of $\mathbb{S}^{n-1}\times\mathbb{R}$ by standard isometries. To summarize, M is diffeomorphic to a quotient of $\mathbb{S}^n$ or a quotient of $\mathbb{S}^{n-1}\times\mathbb{R}$ by standard isometries.

Suppose next that $\Omega_\rho \neq \emptyset$. In this case, we can recover the pre-surgery manifold M from the post-surgery manifold as follows. We first reinstate the components that were discarded after surgery. More precisely, we form a disjoint union of the post-surgery manifold and a finite collection of standard spaces, each of which is a quotient of $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$ by standard isometries. In the next step, we reverse the surgery by gluing in finitely many handles of the form $\mathbb{S}^{n-1} \times I$. Note that, as we glue in these handles, the attaching maps are nearly isometric. Thus, the pre-surgery manifold is diffeomorphic to a connected sum of the post-surgery manifold with finitely many quotients of $\mathbb{S}^n$ and $\mathbb{S}^{n-1} \times \mathbb{R}$. This completes the proof of Proposition 4.1. □

In the remainder of this section, we show that the surgery procedure preserves our curvature pinching estimates, provided that the surgery parameters are sufficiently fine.

Proposition 4.2. *Suppose that the curvature tensor of a δ-neck lies in the set $\mathcal{F}_t$ prior to surgery. If δ is sufficiently small and the curvature of the neck is sufficiently large, then the curvature tensor of the surgically modified manifold lies in the set $\mathcal{F}_t$. Moreover, the scalar curvature is pointwise increasing under surgery.*

Proof. Suppose that the scalar curvature of the neck is close to h^{-2}, where h is small. Let us rescale by the factor h^{-1} so that the scalar curvature of the neck is close to 1 after rescaling. Let us, therefore, assume that g is a Riemannian metric on $\mathbb{S}^{n-1} \times [-10, 10]$ that is close to the round metric with scalar curvature 1, and that has curvature in the set $h^2\mathcal{F}_t$. We first recall the definition of the surgically modified metric $\tilde{g}$. To that end, let z denote the height function on $\mathbb{S}^{n-1} \times [-10, 10]$, and let $\varphi(z) = e^{-\frac{1}{z}}$ for $z \in (0, \frac{1}{10}]$. In the region $\mathbb{S}^{n-1} \times [-10, 0]$, the metric is unchanged under surgery, i.e., $\tilde{g} = g$. In the region $\mathbb{S}^{n-1} \times (0, \frac{1}{20}]$, we change the metric conformally by $\tilde{g} = e^{-2\varphi} g$. In the region $\mathbb{S}^{n-1} \times (\frac{1}{20}, \frac{1}{10}]$, we define $\tilde{g} = e^{-2\varphi}(\chi(z) g + (1 - \chi(z))\bar{g})$, where $\bar{g}$ denotes the standard metric on the cylinder and $\chi : (\frac{1}{20}, \frac{1}{10}] \to [0, 1]$ is a smooth cutoff function satisfying $\chi(z) = 1$ for $z \in [\frac{1}{20}, \frac{1}{18}]$ and $\chi(z) = 0$ for $z \in [\frac{1}{12}, \frac{1}{10}]$. In particular, the surgically modified metric $\tilde{g}$ is rotationally symmetric for $z \in [\frac{1}{12}, \frac{1}{10}]$. Hence, we may extend $\tilde{g}$ by gluing in a rotationally symmetric cap.

We now analyze the curvature of the surgically modified metric $\tilde{g}$. It suffices to consider the case when $z > 0$ is small. In this region, $\tilde{g} = e^{-2\varphi} g$. Let $\{e_1, \ldots, e_n\}$ denote a local orthonormal frame with respect to the metric g. If we put $\tilde{e}_i = e^{\varphi} e_i$, then $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ is an orthonormal frame with respect to the metric $\tilde{g}$. We will express geometric quantities associated with the metric g relative to the frame $\{e_1, \ldots, e_n\}$, while geometric quantities associated with $\tilde{g}$ will be expressed in terms of $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. With this understood, the curvature tensor after surgery is related to the curvature tensor before surgery by the formula

$$\tilde{R} = e^{2\varphi} R + e^{2\varphi}(\nabla^2 \varphi + d\varphi \otimes d\varphi - \frac{1}{2}|d\varphi|^2 \mathrm{id}) \owedge \mathrm{id}.$$

This implies

$$|\tilde{R} - R - z^{-4}e^{-\frac{1}{z}}(dz \otimes dz) \owedge \mathrm{id}| \ll z^{-4}e^{-\frac{1}{z}}$$

for $z > 0$ sufficiently small. Consequently, $S(\tilde{R}) > S(R)$ if $z > 0$ is sufficiently small. Since the metric g is close to the cylindrical metric, we obtain

$$|R - \frac{1}{2}(\mathrm{id} - 2z \otimes z) \owedge \mathrm{id}| \ll 1,$$

hence

$$\begin{aligned}
&|\tilde{R} - (1 - z^{-4}e^{-\frac{1}{z}})R - \frac{1}{2}z^{-4}e^{-\frac{1}{z}}\mathrm{id} \owedge \mathrm{id}| \\
&\leq |\tilde{R} - R - z^{-4}e^{-\frac{1}{z}}(dz \otimes dz) \owedge \mathrm{id}| + z^{-4}e^{-\frac{1}{z}}|R - \frac{1}{2}(\mathrm{id} - 2z \otimes z) \owedge \mathrm{id}| \\
&\ll z^{-4}e^{-\frac{1}{z}}
\end{aligned}$$

for $z > 0$ sufficiently small. Therefore, we may write

$$\tilde{R} = (1 - z^{-4}e^{-\frac{1}{z}})R + z^{-4}e^{-\frac{1}{z}}S,$$

where $|S - \frac{1}{2}\mathrm{id} \owedge \mathrm{id}| \ll 1$ for $z > 0$ sufficiently small. Consequently, $S \in h^2\mathcal{F}_t$ if $z > 0$ is sufficiently small. Moreover, $R \in h^2\mathcal{F}_t$ in view of our assumption. Since $\mathcal{F}_t$ is a convex set, we conclude that $\tilde{R} \in h^2\mathcal{F}_t$ if $z > 0$ is sufficiently small. This easily implies that $\tilde{R} \in h^2\mathcal{F}_t$ for all $z \in (0, 10)$. □

5. **The standard solution**

We recall some basic facts concerning the so-called standard solution in this section. The standard solution is used to model the evolution of a cap that is glued in during a surgery procedure. More precisely, suppose that $(\mathbb{S}^{n-1} \times \mathbb{R}, g(t)), t < 0$, is a family of shrinking cylinders, normalized so that $S_{g(t)} = \frac{1}{1-\frac{2t}{n-1}}$ for $t < 0$. Suppose that we perform surgery at time $t = 0$, i.e., we remove a half-cylinder and glue in a cap that is rotationally symmetric and has positive curvature. This gives a rotationally symmetric metric $g(0)$ on $\mathbb{R}^n$. The standard solution is obtained by evolving the manifold $(\mathbb{R}^n, g(0))$ under the Ricci flow.

The following results were proved by Perelman [25] in dimension 3 and were extended to higher dimensions in [12].

Theorem 5.1 (cf. G. Perelman [25, §2]; B. L. Chen, X. P. Zhu [12, Th. A.1])**.** *There exists a complete solution $(\mathbb{R}^n, g(t)), t \in [0, \frac{n-1}{2})$, to the Ricci flow with the following properties:*

(i) The initial manifold $(\mathbb{R}^n, g(0))$ is isometric to a standard cylinder with scalar curvature 1 outside of a compact set, and this compact set is isometric to the cap used in the surgery procedure.

(ii) For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is rotationally symmetric.

(iii) For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is asymptotic to a cylinder with scalar curvature $\frac{1}{1-\frac{2t}{n-1}}$ at infinity.

(iv) The scalar curvature is bounded from below by $\frac{1}{K_{stad}(1-\frac{2t}{n-1})}$, where K_{stnd} is a positive constant that depends only on n.

(v) For each $t \in [0, \frac{n-1}{2})$, the manifold $(\mathbb{R}^n, g(t))$ is weakly PIC2 and satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$ for some constant $\theta > 0$ that depends only on n.

(vi) The flow $(\mathbb{R}^n, g(t))$ is κ-noncollapsed for some constant $\kappa > 0$ that depends only on n.

(vii) There exists a function $\omega : [0, \infty) \to (0, \infty)$ such that

$$S(x,t) \leq S(y,t)\omega(S(y,t)d_{g(t)}(x,y)^2)$$

for all points x, y and all $t \in [0, \frac{n-1}{2})$.

Proof. The statements (i), (ii), (iii), (iv), (vi) and (vii) are established in [12, App. A]. Moreover, it is shown in [12] that $(\mathbb{R}^n, g(t))$ has nonnegative curvature operator. Hence, it remains to show that $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$.

Observe that the initial manifold $(\mathbb{R}^n, g(0))$ is uniformly PIC. Moreover, on the initial manifold $(\mathbb{R}^n, g(0))$, the sum of the two smallest eigenvalues of the Ricci tensor has a lower bound, a small multiple of the scalar curvature. Hence, $\exists$ a cone $\mathcal{C}$ such that the curvature tensor of $(\mathbb{R}^n, g(0))$ lies in it (cf. S. Brendle [6. Sec.5]). By Hamilton's PDE-ODE principle ([14, Th. 12.34]), the curvature tensor of $(\mathbb{R}^n, g(t))$ lies in $\mathcal{C}$ for each $t \geq 0$. Consequently, the curvature tensor of $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. □

It turns out that the standard solution satisfies a Canonical Neighborhood Property:

Theorem 5.2 (cf. G. Perelman [25]; B. L. Chen, X. P. Zhu [12, Cor. A.2])**.** *Given a small number $\tilde{\varepsilon} > 0$ and a large number $A_0 > 0$, $\exists \alpha \in [0, \frac{n-1}{2})$ with the following property: If (x_0, t_0) is a point on the standard solution such that $t_0 \in [\alpha, \frac{n-1}{2})$, then the parabolic neighborhood $P(x_0, t_0, A_0 S(x_0, t_0)^{-\frac{1}{2}}, -A_0 S(x_0, t_0)^{-1})$ is, after scaling by the factor $S(x_0, t_0)$, $\tilde{\varepsilon}$-close to the corresponding subset of a noncompact ancient κ_0-solution satisfying $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$.*

Proof. Suppose not. Then $\exists$ a sequence of points (x_j, t_j) on the standard solution such that $t_j \to \frac{n-1}{2}$ and the parabolic neighborhood $P(x_0, t_0, A_0 S(x_0, t_0)^{-\frac{1}{2}}, -A_0 S(x_0, t_0)^{-1})$ is not $\tilde{\varepsilon}$-close to the corresponding subset of a noncompact ancient κ_0-solution satisfying $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$.

Dilating the solution around the point (x_j, t_j) by the factor $S(x_j, t_j)$, property (vii) in Theorem 5.1 and the Harnack inequality (cf. Theorem 1.3) shows that the rescaled flows converge to a complete, noncompact ancient solution $(M^\infty, g^\infty(t))$. The limiting ancient solution $(M^\infty, g^\infty(t))$ is weakly PIC2 and satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. Moreover, the limiting ancient solution is κ_0-noncollapsed.

By Theorem 1.3, the standard solution satisfies the Harnack inequality

$$\partial_t S + 2\langle \nabla S, v\rangle + 2Ric(v,v) + \frac{1}{t} S \geq 0$$

for $t \in (0, \frac{n-1}{2})$. Consequently, the limiting ancient solution $(M^\infty, g^\infty(t))$ satisfies

$$\partial_t S + 2\langle \nabla S, v\rangle + 2Ric(v,v) \geq 0.$$

By Brendle's previous work (cf. S. Brendle [6, Prop 6.11]), $(M^\infty, g^\infty(t))$ has bounded curvature. Thus, $(M^\infty, g^\infty(t))$ is a noncompact ancient κ_0-solution satisfying $R - \theta S\, \mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$, which contradicts the assumption. $\square$

Corollary 5.3 (cf. G. Perelman [25]; B. L. Chen, X. P. Zhu [12, Cor. A.2])**.** *Given $\varepsilon > 0$, $\exists$ positive constants $C_1 = C_1(n,\varepsilon)$ and $C_2 = C_2(n,\varepsilon)$ such that the following holds: For each point (x_0, t_0) on the standard solution, there exists a neighborhood B of x_0 such that*

$$B_{g(t_0)}(x_0, C_1^{-1} S(x_0,t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 S(x_0,t_0)^{-\frac{1}{2}})$$

and

$$C_2^{-1} S(x_0,t_0) \leq S(x,t_0) \leq C_2 S(x_0,t_0) \quad \forall\, x \in B.$$

Moreover, B is either a strong ε-neck with center at x_0 or a 4ε-cap. Finally, $|\nabla S| \leq \eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq \eta S^2$ in B.

Proof. If t_0 is sufficiently close to $\frac{n-1}{2}$ (depending on ε), this follows from Theorems 5.2 together with 1.12. If t_0 is bounded away from $\frac{n-1}{2}$, this follows from the fact that the standard solution is asymptotic to a cylinder at infinity. $\square$

Finally, we state a lemma that will be needed later.

Lemma 5.4. *Given $\alpha \in [0, \frac{n-1}{2})$ and $l > 0$, there exists a large number $A = A(\alpha, l)$ with the following property: Suppose that $t_1 \in [0,\alpha]$ and γ is a space-time curve on the standard solution (parametrized by the interval $[0,t_1]$) such that $\gamma(0)$ lies on the cap at time 0, and $\int_0^{t_1} |\gamma'(t)|^2_{g(t)} dt \leq l$. Then the curve γ is contained in the parabolic neighborhood $P(\gamma(0), 0, \frac{A}{2}, t_1)$.*

Proof. By $\int_0^{t_1} |\gamma'(t)|^2_{g(t)} dt \leq l$ and Hölder's inequality, we obtain $\int_0^{t_1} |\gamma'(t)|_{g(t)} dt \leq \alpha^{\frac{1}{2}} l^{\frac{1}{2}}$. Then $A = 100\alpha^{\frac{1}{2}} l^{\frac{1}{2}}$ is what we want. $\square$

6. A priori estimates for Ricci flow with surgery

We give the definition of Ricci flow with surgery in this section. Moreover, we discuss how Perelman's noncollapsing estimate and the Canonical Neighborhood Theorem can be extended to Ricci flow with surgery. In the follwing, we fix a compact initial manifold of dimension $n \geq 12$ that has positive isotropic curvature and does not contain any nontrivial incompressible $(n-1)$-dimensional space forms. Let $\{\mathcal{F}_t | t \in [0,T]\}$ be a family of convex, closed, $O(n)$-invariant sets such that the family $\{\mathcal{F}_t | t \in [0,T]\}$ is invariant under the Hamilton ODE $\frac{d}{dt}S = Q(S)$; the curvature tensor of (M, g_0) lies in $\mathcal{F}_0$; and

$$\begin{aligned}\mathcal{F}_t \subset \{R | R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}\} \\ \cap \{R | R + f(S)\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC2}\}\end{aligned}$$

for all $t \in [0,T]$, where f is a concave and increasing function satisfying $\lim_{s\to\infty} \frac{f(s)}{s} = 0$, and θ and N are positive numbers.

The key is that having fixed θ, we can find a universal constant κ_0 such that the conclusion of Theorem 1.12 holds. Moreover, we fix a constant η such that the conclusions of Corollary 2.4 and Corollary 5.3 hold. In other words, we have $|\nabla S| \leq \eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq \eta S^2$ on any ancient κ-solution, and the same inequalities hold on the standard solution.

Let us choose a small positive number $\varepsilon > 0$. Then we fix constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ such that the conclusions of Corollaries 1.14 and 5.3 hold.

Definition. A Ricci flow with surgery on the intcrval $[0, T)$ consists of the following data:

- A decomposition of $[0,T)$ into a disjoint union of finitely many subintervals $[t_k^-, t_k^+), 0 \leq k \leq l$. That is to say, $t_0^- = 0, t_l^+ = T$ and $t_k^- = t_{k-1}^+$ for $1 \leq k \leq l$
- A collection of smooth Ricci flows $(M^{(k)}, g^{(k)}(t))$, defined on $t \in [t_k^-, t_k^+)$ and become singular as $t \to t_k^+$ for $0 \leq k \leq l-1$.
- Positive numbers $\varepsilon, r, \delta, h$ such that $\delta \leq \varepsilon$ and $h \leq \delta r$. These are referred to as the surgery parameters.

For each $0 \leq k \leq l-1$, let $\Omega^{(k)} = \{x \in M^{(k)} | \limsup_{t \to t_k^+} S(x,t) < \infty\}$. We assume that the following conditions are satisfied:

- The manifold $(M^{(0)}, g^{(0)}(0))$ is isometric to the given initial manifold (M, g_0).
- The manifold $(M^{(k)}, g^{(k)}(t_k^-))$ is obtained from $(M^{(k-1)}, g^{(k-1)}(t_{k-1}^+))$ by performing surgery on finitely many necks. For each neck on which we perform surgery, we can find a point (x_0, t_0) at the center of that neck such that $S(x_0, t_0) = h^{-2}$; moreover, the parabolic neighborhood $P(x_0, t_0, \delta^{-1}h, -\delta^{-1}h^2)$ is surgery-free and is a strong δ-neck.
- After each surgery, we discard all double 4ε-horns and all capped 4ε-horns. Moreover, we remove all connected components that are diffeomorphic to $\mathbb{S}^n/\Gamma$.
- Each flow $(M^{(k)}, g^{(k)}(t))$ satisfies the Canonical Neighborhood Property with accuracy 4ε on all scales less than r. In other words, if (x_0, t_0) is an arbitrary point in space-time satisfying $S(x_0, t_0) \geq r^{-2}$, then $\exists$ a neighborhood B of x_0 such that

$$B_{g(t_0)}(x_0, (8C_1)^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 8C_1 S(x_0, t_0)^{-\frac{1}{2}})$$

and

$$(8C_2)^{-1} S(x_0, t_0) \leq S(x, t_0) \leq 8C_2 S(x_0, t_0) \quad \forall x \in B.$$

Moreover, B is either a strong 4ε-neck with center at x_0 or a 4ε-cap.

- If (x_0, t_0) is an arbitrary point in space-time satisfying $S(x_0, t_0) \geq r^{-2}$, then $|\nabla S| \leq 4\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 4\eta S^2$ at (x_0, t_0).

Note that the manifold $M^{(k)}$ may have multiple connected components. In the following, we will write the surgically modified solution simply as $g(t)$. However, it is important to remember that the underlying manifold changes across surgery times.

In the first step, we prove an upper bound for the length of the time interval on which the solution is defined.

Proposition 6.1. *Suppose that we have a Ricci flow with surgery starting from (M, g_0) that is defined on $[0, T)$. Then $T \leq \frac{n}{2 \inf_{x \in M} S(x,0)}$.*

Proof. Note that the function

$$t \mapsto \frac{n}{2 \inf_{x \in M} S(x,t)} + t$$

is monotone decreasing under smooth Ricci flow by the maximum principle. Also, this function is monotone decreasing across surgery times by Proposition 4.2. Therefore, this function is monotone decreasing under Ricci flow with surgery. □

Proposition 6.2. *Let f, θ be as above, and let $g(t)$ be a Ricci flow with surgery starting from (M, g_0). Then $(M, g(t))$ has (f, θ)-pinched curvature.*

Proof. By Theorem 1.1 and Hamilton's PDE-ODE principle (cf. [15, Th. 3] or [14, Th. 10.16]), the property that the curvature tensor of $g(t)$ lies in $\mathcal{F}_t$ is preserved by the Ricci flow. By Proposition 4.2, the property that the curvature tensor lies in $\mathcal{F}_t$ is preserved under surgery. Therefore, the property that the curvature tensor of $g(t)$ lies in $\mathcal{F}_t$ is preserved under Ricci flow with surgery. □

Proposition 6.3. *Let $g(t)$ be a Ricci flow with surgery, and let $\varepsilon, r, \delta, h$ denote the surgery parameters. Choose (x_0, t_0) to be a point in space-time and let r_0 be a positive real number such that $t_0 \geq r_0^2$ and $S(x,t) \leq r_0^{-2}\, \forall\, (x,t) \in P(x_0, t_0, r_0, -r_0^2)$. Then $|\nabla^m R| \leq C(n,m) r_0^{-m-2}$ at the point (x_0, t_0).*

Proof. If the parabolic neighborhood $P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$ is surgery-free, this follows from the classical Shi estimate (cf. [26]). Then suppose that the parabolic neighborhood $P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$ does contain surgeries. At each point modified by surgery, the scalar curvature is at least $\frac{1}{4}h^{-2}$. Consequently, $\frac{1}{4}h^{-2} \leq r_0^{-2}$. The classical Shi estimate (cf. [26]) implies $|\nabla^m R| \leq C(n,m) h^{-m-2} \leq C(n,m) r_0^{-m-2}$ on each strong neck on which we perform surgery. Moreover, $|\nabla^m R| \leq C(n,m) h^{-m-2} \leq C(n,m) r_0^{-m-2}$ at each point modified by surgery. The assertion now follows from Theorem 3.29 in [22]. □

Proposition 6.4 (cf. G. Perelman [25, Lemma 4.5]). *Fix $\varepsilon > 0$ small, $\alpha \in [0, \frac{n-1}{2})$, and $A > 1$. Then $\exists\, \bar{\delta}(\alpha, A) > 0$ with the following property: Suppose that we have a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$, where $\delta \leq \bar{\delta}$. Let $T_0 \in [0, T)$ be a surgery time, and let x_0 be a point that lies on a gluing cap at time T_0. Let $T_1 = \min\{T, T_0 + \alpha h^2\}$. Then one of the following statements holds:*

(i) The flow is defined on the parabolic neighborhood $P(x_0, T_0, Ah, T_1 - T_0)$. Furthermore, after dilating the flow by h^{-2} and shifting time T_0 to 0, $P(x_0, T_0, Ah, T_1 - T_0)$ is A^{-1}-close to the corresponding subset of the standard solution.

(ii) There exists a surgery time $t^+ \in (T_0, T_1)$ such that the flow is defined on the parabolic neighborhood $P(x_0, T_0, Ah, t^+ - T_0)$. Furthermore, after dilating by the factor h^{-2}, $P(x_0, T_0, Ah, t^+ - T_0)$ is A^{-1}-close to the corresponding subset of the standard solution. Finally, for each point $x \in B_{g(T_0)}(x_0, Ah)$, the flow exists exactly until time t^+.

Proof. Same as the proof of Lemma 4.5 in Perelman's paper [25]. We omit the details. □

As in Perelman's work [25], it is crucial to establish a noncollapsing estimate in the presence of surgeries.

Definition. Given a Ricci flow with surgery, the flow is said to be κ-noncollapsed on scales less than ρ if the following holds: If (x_0, t_0) is a point in space-time, r_0 is a positive number such that $r_0 \leq \rho$ and $S(x, t) \leq r_0^{-2}\, \forall\, (x, t) \in P(x_0, t_0, r_0, -r_0^2)$ for which the flow is defined, then $vol_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n$.

As in Perelman's work [25], the noncollapsing estimate for Ricci flow with surgery will follow from the monotonicity formula for the reduced volume.

Definition. Given a Ricci flow with surgery, a curve in space-time is said to be admissible if it stays in the region unaffected by surgery. A curve in space-time is called barely admissible if it is on the boundary of the set of admissible curves.

Lemma 6.5 (cf. G. Perelman [25, Lemma 5.3]). *Fix ε, r, L. Then there exists a real number $\bar{\delta}(r, L) > 0$ such that the following holds: Given a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$, where $\delta \leq \bar{\delta}$, let (x_0, t_0) be a point in space-time such that $S(x_0, t_0) \leq r^{-2}$, and let $T_0 < t_0$ be a surgery time. Fix γ to be a barely admissible curve, which is parametrized by the interval $[T_0, t_0]$, such that $\gamma(t_0) = x_0$, and $\gamma(T_0)$ lies on the boundary of a surgical cap at time T_0. Then*

$$\int_{T_0}^{t_0} \sqrt{t_0 - t}\,(S(\gamma(t), t) + |\gamma'(t)|^2_{g(t)})\,dt \geq L.$$

Proof. We argue by contradiction. Recall that we have the key estimate $|\nabla S| \leq 4\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 4\eta S^2$ whenever $S \geq r^{-2}$. Since $S(x_0, t_0) \leq r^{-2}$, it follows that $S \leq 4r^{-2}$

in $P(x_0, t_0, \frac{r}{100\eta}, -\frac{r^2}{100\eta})$. Let γ be a barely admissible curve in space-time satisfying the assumptions we need and suppose that

$$\int_{T_0}^{t_0} \sqrt{t_0 - t}(S(\gamma(t), t) + |\gamma'(t)|^2_{g(t)})dt < L.$$

Then $\int_{t_0-\tau}^{t_0} |\gamma'|_{g(t)}dt < (2L)^{\frac{1}{2}}\tau^{\frac{1}{4}}$ for $\tau > 0$ follows from Hölder's inequality and the positivity of the scalar curvature. Hence, we can find a real number $\tau(r, L) \in (0, \frac{r^2}{100\eta})$ such that $\gamma\big|_{[t_0-\tau,t_0]}$ is contained in the parabolic neighborhood $P(x_0, t_0, \frac{r}{100\eta}, -\frac{r^2}{100\eta})$, which implies

$$S(\gamma(t), t) \leq 4r^{-2} \quad \forall\, t \in [t_0 - \tau, t_0].$$

Having fixed τ, we define real numbers $\alpha \in [0, \frac{n-1}{2})$ and $l > 0$ by the relations

$$\begin{cases} \frac{(n-1)\sqrt{\tau}}{4K_{\text{stnd}}}|\log(1 - \frac{2\alpha}{n-1})| = L \\ \frac{l}{2}\sqrt{\tau} = L \end{cases},$$

where K_{stnd} is a positive constant appering in the standard solution, which depends only on dimension n. Having chosen α and l, we choose a large constant A so that the conclusion of Lemma 5.4 holds. Having fixed α and A, we choose $\bar{\delta}$ so that the conclusion of Proposition 6.4 holds. Moreover, by choosing $\bar{\delta}$ small enough, we can settle that $K_{\text{stnd}}\bar{\delta}^2 \leq \frac{1}{16}$.

Assume that $\delta \leq \bar{\delta}$ in the following. Let $T_1 \in [T_0, T_0 + \alpha h^2]$ denote the largest number with the property that $\gamma\big|_{[T_0,T_1]}$ is contained in the parabolic neighborhood $P(\gamma(T_0), T_0, Ah, \alpha h^2)$. By Proposition 6.4, the parabolic neighborhood $P(\gamma(T_0), T_0, Ah, T_1 - T_0)$ is close to the corresponding subset of the standard solution. Since $h \leq \delta r$, we conclude that

$$S(\gamma(t), t) \geq \frac{1}{2K_{\text{stnd}}(h^2 - \frac{2(t-T_0)}{n-1})} \geq \frac{1}{2K_{\text{stnd}}\delta^2 r^2} \geq 8r^{-2} \quad \forall\, t \in [T_0, T_1].$$

Since $S(\gamma(t), t) \leq 4r^{-2}\, \forall\, t \in [t_0 - \tau, t_0]$, the intervals $[T_0, T_1]$ and $[t_0 - \tau, t_0]$ are disjoint. That is to say, $T_1 \leq t_0 - \tau$. We distinguish two cases:

Case 1: Suppose that $T_1 < T_0 + \alpha h^2$. Then $\gamma\big|_{[T_0,T_1]}$ lies in the the parabolic neighborhood $P(\gamma(T_0), T_0, Ah, \alpha h^2)$. Since $P(\gamma(T_0), T_0, Ah, T_1 - T_0)$ is close to the corresponding subset of the standard solution, combining Lemma 5.4 and the fact that $\int |\gamma'(t)|^2_{g(t)}dt$ is invariant under scaling shows that

$$\int_{T_0}^{T_1} |\gamma'(t)|^2_{g(t)}dt \geq \frac{l}{2}.$$

It follows that

$$\begin{aligned} L &> \int_{T_0}^{T_1} \sqrt{t_0 - t}(S(\gamma(t), t) + |\gamma'(t)|^2_{g(t)})dt \\ &\geq \sqrt{\tau} \int_{T_0}^{T_1} |\gamma'(t)|^2_{g(t)} dt \\ &\geq \frac{l}{2}\sqrt{\tau} \end{aligned},$$

which contradicts to the choice of l.

Case 2: Suppose that $T_1 = T_0 + \alpha h^2$. Then

$$\begin{aligned} L &> \int_{T_0}^{T_1} \sqrt{t_0 - t}(S(\gamma(t), t) + |\gamma'(t)|^2_{g(t)})dt \\ &\geq \sqrt{\tau} \int_{T_0}^{T_1} S(\gamma(t), t) dt \\ &\geq \sqrt{\tau} \int_{T_0}^{T_1} \frac{1}{2K_{\text{stnd}}(h^2 - \frac{2(t-T_0)}{n-1})} dt \\ &= \frac{(n-1)\sqrt{\tau}}{4K_{\text{stnd}}} |\log(1 - \frac{2\alpha}{n-1})| \end{aligned},$$

which contradicts to the choice of α.

□

Proposition 6.6 (cf. G. Perelman [25, Lemma 5.2]). *Fix a small number $\varepsilon > 0$. Then there exists a positive number κ and a positive function $\tilde{\delta}(\cdot)$ with the following property: Given a Ricci flow with surgery with parameters $\varepsilon, r, \delta, h$, where $\delta \leq \tilde{\delta}(r)$, then the flow is κ-noncollapsed on all scales less than ε.*

Note that the constant κ in the noncollapsing estimate may depend on the initial data, but it is independent of the surgery parameters $\varepsilon, r, \delta, h$.

Proof. Let (x_0, t_0) be a point in space-time and consider a positive number $r_0 \leq \varepsilon$ so that $S(x,t) \leq r_0^{-2} \, \forall \, (x,t) \in P(x_0, t_0, r_0, -r_0^2)$ for which the flow is defined. It suffices to show that $vol_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n$ for some uniform constant $\kappa > 0$. We disuss the following three cases:

Case 1. Suppose that $S(x_0, t_0) \geq r^{-2}$. Then Canonical Neighborhood Assumption leads to the desired result.

Case 2. Suppose that the parabolic neighborhood $P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$ contains points modified by surgery. Say (x,t) to be one of such point. Then $\frac{1}{4}h^{-2} \leq S(x,t) \leq r_0^{-2}$, $r_0 \leq 2h$. It implies that $vol_{g(t_0)}(B_{g(t_0)}(x, \frac{r_0}{100})) \geq \kappa r_0^n$ for some uniform constant $\kappa > 0$. Therefore $vol_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq vol_{g(t_0)}(B_{g(t_0)}(x, \frac{r_0}{4})) \geq vol_{g(t_0)}(B_{g(t_0)}(x, \frac{r_0}{100})) \geq \kappa r_0^n$.

Case 3. Suppose that $S(x_0,t_0) \le r^{-2}$ and the parabolic neighborhood $P(x_0,t_0,\frac{r_0}{2},-\frac{r_0^2}{4})$ is surgery-free. First notice that t_0 has upper bound by Proposition 6.1. Also, by Lemma 6.5, there exists a positive function $\tilde{\delta}(\cdot)$ such that the following holds: Suppose that the surgery parameters satisfy $\delta \le \tilde{\delta}(r)$, $T_0 < t_0$ is a surgery time and γ is a barely admissible curve parametrized by the interval $[T_0,t_0]$ such that $\gamma(t_0) = x_0$, and $\gamma(T_0)$ lies on the boundary of a surgical cap at time T_0. Then

$$\int_{T_0}^{t_0} \sqrt{t_0-t}(S(\gamma(t),t) + |\gamma'(t)|^2_{g(t)})dt \ge 8n\sqrt{t_0}.$$

Thus, if $\delta \le \delta(r)$, then every barely admissible curve has reduced length greater than $2n$.

In the following, we assume that $\delta \le \tilde{\delta}(r)$. For $t < t_0$, we denote by $\ell(x,t)$ the reduced distance from (x_0,t_0), i.e., the infimum of the reduced length over all admissible curves joining (x,t) and (x_0,t_0). Let us claim that $\inf_x \ell(x,t) \le \frac{n}{2}\,\forall\, t < t_0$, which is clearly true if t is sufficiently close to t_0. Now, if $\ell(x,t) < 2n$ for some point (x,t) in space-time, then the reduced length is attained by a strictly admissible curve. Hence, a work of Perelmann (cf. [23, §7]) shows that

$$\partial_t \ell \ge \Delta \ell + \frac{1}{t_0-t}(\ell - \frac{n}{2})$$

whenever $\ell < 2n$. The disired result follows from the maximum principle.

In particular, there exists a point $y \in M$ such that $\ell(y,\varepsilon) \le \frac{n}{2}$. Therefore, we can find a radius $\rho > 0$ such that $\sup_{x \in B_{g(0)}(y,\rho)} \ell(x,0) \le n$. Note that ρ depends only on ε and the initial data (M,g_0), but not on the surgery parameters. Hence, for each point $x \in B_{g(0)}(y,\rho)$, the reduced distance is attained by a strictly admissible curve, and this curve must be an $\mathcal{L}$-geodesic.

Given a tangent vector v at (x_0,t_0), let $\gamma_v(t) = \mathcal{L}_{t,t_0}\exp_{x_0}(v)$ be the $\mathcal{L}$-geodesic satisfying $\lim_{t\to t_0}\sqrt{t_0-t}\gamma_v'(t) = v$. Notice that $\gamma_v(t)$ may not be defined on the entire interval $[0,t_0)$ due to the presence of surgeries.

Let $\mathcal{V} := \{v \in (T_{x_0}M, g(t_0)) \big| \gamma_v \text{ is defined on } [0,t_0), \gamma_v \text{ has minimal } \mathcal{L}\text{-length}, \gamma_v(0) \in B_{g(0)}(y,\rho)\}$. From aforementioned discussion, the map $\mathcal{L}_{0,t_0}\exp_{x_0} : \mathcal{V} \to B_{g(0)}(y,\rho)$ is onto. For each $t \in [0,t_0)$, we define

$$V(t) = \int_{\mathcal{V}} (t_0-t)^{-\frac{n}{2}} e^{-\ell(\gamma_v(t),t)} J_v(t),$$

where $J_v(t) = \det(D\mathcal{L}_{t,t_0}\exp_{x_0})_v$ denotes the Jacobian determinant of the $\mathcal{L}$- exponential map, and the integration is with respect to the standard Lebesgue measure on the tangent space $(T_{x_0}M, g(t_0))$. For each tangent vector $v \in \mathcal{V}$, Perelman's Jacobian comparison theorem (cf. [23, §7]) implies that the function $t \mapsto (t_0 - t)^{-\frac{n}{2}} e^{-\ell(\gamma_v(t),t)} J_v(t)$ is monotone increasing. Moreover, $\lim_{t\to t_0}(t_0-t)^{-\frac{n}{2}} e^{-\ell(\gamma_v(t),t)} J_v(t) =$

$2^n e^{-|v|^2}\,\forall\, v \in \mathcal{V}$. The monotonicity property for the Jacobian determinant implies that the function $t \mapsto V(t)$ is monotone increasing.

First, we estimate the reduced volume from below in terms of the initial data. Since $\ell(x, 0) \leq n$ for all points $x \in B_{g(0)}(y, \rho)$, there exists a uniform lower bound for $V(0)$:

$$\begin{aligned} V(0) &= \int_{\mathcal{V}} t_0^{-\frac{n}{2}} e^{-\ell(\gamma_v(0),0)} J_v(0) \\ &\geq \int_{B_{g(0)}(y,\rho)} t_0^{-\frac{n}{2}} e^{-\ell(x,0)} dvol_{g(0)}(x) \cdot \\ &\geq t_0^{-\frac{n}{2}} e^{-n} vol_{g(0)}(B_{g(0)}(y,\rho)) \end{aligned}$$

Next, we estimate the reduced volume from above. By assumption, $S \leq r_0^{-2}$ in the surgery-free parabolic neighborhood $P(x_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$. Shi's interior derivative (cf. [26]) estimates shows that $|\nabla R| \leq C(n) r_0^{-3}$ and $|\nabla^2 R| \leq C(n) r_0^{-4}$ in $P(x_0, t_0, \frac{r_0}{4}, -\frac{r_0^2}{16})$. Using the $\mathcal{L}$-geodesic equation, we conclude that there exists a small positive constant $\mu(n)$, depending only on dimension, with the following property: if $\bar{t} \in [t_0 - \mu(n) r_0^2, t_0)$ and $|v| \leq \frac{r_0}{32\sqrt{t_0 - t}}$, then $\sqrt{t_0 - t}|\gamma_v'(t)|_{g(t)} \leq \frac{r_0}{16\sqrt{t_0-t}}$ and $\gamma_v(t) \in B_{g(t_0)}(x_0, \frac{r_0\sqrt{t_0-t}}{4\sqrt{t_0-t}}) \subset B_{g(t_0)}(x_0, \frac{r_0}{4})\,\forall\, t \in [\bar{t}, t_0)$, which leads to

$$\begin{aligned} &V(0) \leq V(\bar{t}) \\ &\leq \int_{\{v\in\mathcal{V}\,\big|\,|v|\leq \frac{r_0}{32\sqrt{t_0-t}}\}} (t_0 - \bar{t})^{-\frac{n}{2}} e^{-\ell(\gamma_v(\bar{t}),\bar{t})} J_v(\bar{t}) \\ &+ \int_{\{v\in\mathcal{V}\,\big|\,|v|\geq \frac{r_0}{32\sqrt{t_0-t}}\}} (t_0 - \bar{t})^{-\frac{n}{2}} e^{-\ell(\gamma_v(\bar{t}),\bar{t})} J_v(\bar{t}) \\ &\leq \int_{\{v\in\mathcal{V}\,\big|\,|v|\leq \frac{r_0}{32\sqrt{t_0-t}}\}} (t_0 - \bar{t})^{-\frac{n}{2}} J_v(\bar{t}) + \int_{\{v\in\mathcal{V}\,\big|\,|v|\geq \frac{r_0}{32\sqrt{t_0-t}}\}} 2^n e^{-|v|^2} \\ &\leq (t_0 - \bar{t})^{-\frac{n}{2}} vol_{g(t)}(B_{g(t_0)}(x_0, \frac{r_0}{4})) + \int_{\{v\in\mathcal{V}\,\big|\,|v|\geq \frac{r_0}{32\sqrt{t_0-t}}\}} 2^n e^{-|v|^2} \end{aligned}$$

for all $\bar{t} \in [t_0 - \mu(n) r_0^2, t_0)$.

Now, putting the aforementioned estimates together gives

$$(t_0 - \bar{t})^{-\frac{n}{2}} vol_{g(t)}(B_{g(t_0)}(x_0, \frac{r_0}{4})) \geq t_0^{-\frac{n}{2}} e^{-n} vol_{g(0)}(B_{g(0)}(y,\rho)) - \int_{\{v\in\mathcal{V}\,\big|\,|v|\geq \frac{r_0}{32\sqrt{t_0-t}}\}} 2^n e^{-|v|^2}$$

for all $\bar{t} \in [t_0 - \mu(n) r_0^2, t_0)$. Finally, we select $\bar{t} \in [t_0 - \mu(n) r_0^2, t_0)$ such that $t_0 - \bar{t}$ is a fixed, small multiple of r_0^2, and the quantity is a small, but fixed, multiple of , and the quantity

$$t_0^{-\frac{n}{2}} e^{-n} vol_{g(0)}(B_{g(0)}(y,\rho)) - \int_{\{v\in\mathcal{V}\,\big|\,|v|\geq \frac{r_0}{32\sqrt{t_0-t}}\}} 2^n e^{-|v|^2}$$

has a positive lower bound. Such choice of $\bar{t}$ gives the desired lower bound for $r_0^{-n} vol_{g(t_0)}(B_{g(t_0)}(x, \frac{r_0}{4}))$.

□

We now state the main result of this section. This result guarantees that, for a suitable choice of $\varepsilon, \hat{r}, \hat{\delta}$, every Ricci flow with surgery with parameters $\varepsilon, \hat{r}, \hat{\delta}, h$ will satisfy the Canonical Neighborhood Property with accuracy 2ε on all scales less than $2\hat{r}$.

Theorem 6.7 (cf. G. Perelman [25, §5])**.** *Fix a small number $\varepsilon > 0$. There exists positive numbers $\hat{r}$ and $\hat{\delta}$ with the following property: Given a Ricci flow with surgery with parameters $\varepsilon, \hat{r}, \hat{\delta}, h$ which is defined on some interval $[0, T)$, suppose that (x_0, t_0) is an arbitrary point in space-time satisfying $S(x_0, t_0) \geq (2\hat{r})^{-2}$. Then $\exists$ a neighborhood B of x_0 such that*

$$B_{g(t_0)}(x_0, (2C_1)^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1 S(x_0, t_0)^{-\frac{1}{2}})$$

and

$$(2C_2)^{-1} S(x_0, t_0) \leq S(x, t_0) \leq 2C_2 S(x_0, t_0) \quad \forall\, x \in B.$$

Moreover, B is either a strong 2ε-neck with center at x_0 or a 2ε-cap. Finally, $|\nabla S| \leq 2\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 2\eta S^2$ at (x_0, t_0).

Proof. Assuming the opposite, suppose that there exists a sequence of Ricci flows with surgery $\mathcal{M}^{(j)}$ and a sequence of points (x_j, t_j) in space-time with the following properties:

(i) The flow $\mathcal{M}^{(j)}$ is defined on the time interval $[0, T_j)$ and has surgery parameters $\varepsilon, \hat{r}_j, h_j, \hat{\delta}_j$, where $\hat{r}_j \leq \frac{1}{j}$ and $\hat{\delta}_j \leq \min\{\tilde{\delta}(\hat{r}_j), \frac{1}{j}\}$. Here, $\tilde{\delta}(\cdot)$ is the function introduced in Proposition 6.6.
(ii) $Q_j := S(x_j, t_j) \geq (2\hat{r}_j)^{-2}$.
(iii) The point (x_j, t_j) does not satisfy the conclusion of Theorem 6.7.

Note that the Property (iii) means that at least one of the following statements is true:

(1) There does not exist a neighborhood B of x_0 so that

$$B_{g(t_0)}(x_0, (2C_1)^{-1} S(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1 S(x_0, t_0)^{-\frac{1}{2}})$$

and

$$(2C_2)^{-1} S(x_0, t_0) \leq S(x, t_0) \leq 2C_2 S(x_0, t_0) \quad \forall\, x \in B,$$

and such that B is either a strong 2ε-neck with center at x_0 or a 2ε-cap.
(2) $|\nabla S| > 2\eta S^{\frac{3}{2}}$ at (x_j, t_j).
(3) $|\partial_t S| > 2\eta S^2$ at (x_j, t_j).

The proof will contain several steps:

Step 1. By definition, we have $|\nabla S| \leq 4\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 4\eta S^2$ for each point (x,t) in space-time satisfying $S(x,t) \geq 4Q_j \geq \hat{r}_j^{-2}$. Moreover, by Proposition 6.6, the flow $\mathcal{M}^{(j)}$ is κ-noncollapsed on scales less than ε for some uniform constant κ that may depend on the initial data, but it is independent of j.

Step 2. Suppose that (x_0, t_0) is a point in space-time satisfying $S(x_0,t_0) + Q_j \leq r_0^{-2}$. Then the pointwise curvature derivative estimate show that $S \leq 8r_0^{-2}$ in the parabolic neighborhood $P(x_0, t_0, \frac{r_0}{100\eta}, -\frac{r_0^2}{100\eta})$. And $|\nabla^m R| \leq C(n,m,\eta) r_0^{-m-2}$ at the point (x_0,t_0) follows from Proposition 6.3. Furthermore, $vol_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \kappa r_0^n$ for some uniform constant κ that is independent of j due to Proposition 6.6.

Step 3. The goal is to establish a long-range curvature estimate. Given any $\rho > 0$, we set

$$\mathbb{M}(\rho) = \limsup_{j\to\infty} \sup_{x\in B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})} Q_j^{-1} S(x, t_j).$$

The pointwise curvature derivative estimate implies that $\mathbb{M}(\rho) \leq 16$ for $0 < \rho < \frac{1}{100\eta}$.

We claim that $\mathbb{M}(\rho) < \infty$ for all $\rho > 0$. Suppose not, say

$$\rho^* := \sup\{\rho \geq 0 \big| \mathbb{M}(\rho) < \infty\} < \infty.$$

Note that $\exists$ an upper bound for the curvature in the geodesic ball $B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})$ $\forall\, \rho < \rho^*$ by the definition of ρ^*. As a result of Step 2, we derive upper bounds for all the covariant derivatives of the curvature tensor in the geodesic ball $B_{g(t_j)}(x_j, \rho Q_j^{-\frac{1}{2}})$ $\forall\, \rho < \rho^*$. Also, the noncollapsing estimate in Step 2 gives a lower bound for the volume. Rescaling around (x_j, t_j) by the factor Q_j and passing to the limit as $j \to \infty$, we obtain an incomplete manifold (B^∞, g^∞) which is weakly PIC2 (cf. [22, Th. 5.6]).

According to the definition of ρ^*, $\exists$ a sequence of points y_j such that

$$\rho_j := Q_j^{\frac{1}{2}} d_{g(t_j)}(x_j, y_j) \to \rho^* \quad \text{and} \quad Q_j^{-1} S(y_j, t_j) \to \infty.$$

Let $\gamma_j : [0, \rho_j Q_j^{-\frac{1}{2}}] \to (M, g(t_j))$ be a unit-speed geodesic such that $\gamma_j(0) = x_j$ and $\gamma_j(\rho_j Q_j^{-\frac{1}{2}}) = y_j$, and let $\gamma_\infty : [0, \rho^*) \to (B^\infty, g^\infty)$ denote the limit of γ_j. Since $|\nabla S| \leq 4\eta S^{\frac{3}{2}}$, we have

$$S_{g^\infty}(\gamma_\infty(s)) = \lim_{j\to\infty} Q_j^{-1} S(\gamma_j(s Q_j^{-\frac{1}{2}}), t_j) \geq (2\eta(\rho^* - s))^{-2} \geq 100 \quad \forall\, s \in [\rho^* - \frac{1}{100\eta}, \rho^*).$$

Consider a real number $\bar{s} \in [\rho^* - \frac{1}{100\eta}, \rho^*)$ such that $64 C_1 \eta(\rho^* - \bar{s}) \leq \bar{s}$. We claim that $\gamma_j(\bar{s} Q_j^{-\frac{1}{2}})$ lies at the center of a strong 4ε-neck if j is sufficiently large (depending on $\bar{s}$). Indeed, if j is sufficiently large, then the Canonical Neighborhood Assumption implies that the point $(\gamma_j(\bar{s} Q_j^{-\frac{1}{2}}), t_j)$ has a Canonical Neighborhood that is either a strong 4ε-neck or a 4ε-cap. Moreover, the Canonical Neighborhood is contained in a geodesic ball around $\gamma_j(\bar{s} Q_j^{-\frac{1}{2}})$ of radius $8C_1 S(\gamma_j(\bar{s} Q_j^{-\frac{1}{2}}), t_j)^{-\frac{1}{2}}$, and the scalar curvature is at most $8C_2 S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ at each point in the Canonical Neighborhood. Since $\mathbb{M}(\bar{s}) < \infty$, we derive $\lim_{j\to\infty}(S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j))^{-1}S(y_j, t_j) = \infty$. Consequently, $S(y_j, t_j) \geq 16C_2 S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ if j is sufficiently large. It follows that the Canonical Neighborhood does not contain the point y_j if j is sufficiently large. Next, observe that $32C_1 S_{g^\infty}(\gamma_\infty(\bar{s}))^{-\frac{1}{2}} \leq 64C_1\eta(\rho^* - \bar{s}) \leq \bar{s}$, which implies that $16C_1 S(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)^{-\frac{1}{2}} \leq \bar{s}Q_j^{-\frac{1}{2}}$ if j is sufficiently large. Hence, if j is sufficiently large, then the Canonical Neighborhood does not contain the point x_j. If the Canonical Neighborhood of $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ is a 4ε-cap, then the geodesic γ_j must enter and exit this 4ε-cap, which is impossible since γ_j minimizes length. That is to say, if j is sufficiently large (depending on $\bar{s}$), then the point $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ has a Canonical Neighborhood that is a strong 4ε-neck. In particular, if j is sufficiently large (depending on $\bar{s}$), then

$$|\nabla S| \leq C(n)\varepsilon S^{\frac{3}{2}} \text{ at the point } (\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j).$$

Passing to the limit as $j \to \infty$, we conclude that $|\nabla S_{g^\infty}| \leq C(n)\varepsilon S_{g^\infty}^{\frac{3}{2}}$ at the point $\gamma_\infty(\bar{s})$. Integrating this estimate along γ_∞ gives $S_{g^\infty}(\gamma_\infty(\bar{s})) \geq (C(n)\varepsilon(\rho^* - \bar{s}))^{-2}$. Moreover, since $(\gamma_j(\bar{s}Q_j^{-\frac{1}{2}}), t_j)$ lies at the center of a strong 4ε-neck for sufficiently large j, the point $\gamma_\infty(\bar{s})$ must lie on a strong $C(n)\varepsilon$-neck in (B^∞, g^∞).

As in [23, §12.1], there is a sequence of rescalings that converges to a piece of a nonflat metric cone in the limit. Let us fix a point on this metric cone. In view of the preceding discussion, this point must lie on a strong $C(n)\varepsilon$-neck. This gives a locally defined solution to the Ricci flow that is weakly PIC2 and that, at the final time, is a piece of nonflat metric cone, which contradicts Proposition 1.5.

Step 4. Now, we dilate the manifold $(M, g(t_j))$ around the point x_j by the factor Q_j. By Step 3, we have uniform bounds for the curvature at bounded distance. The result in Step 2 shows that there exists bounds for all the covariant derivatives of the curvature tensor at bounded distance. Using these estimates together with the noncollapsing estimate in Step 2, we conclude that the rescaled manifolds converge in the Cheeger-Gromov sense (cf. [10]) to a complete limit manifold (M^∞, g^∞). Since $(M, g(t_j))$ has (f, θ)-pinched curvature, the curvature tensor of (M^∞, g^∞) is weakly PIC2 and satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. By the Canonical Neighborhood Assumption, we conclude that every point in (M^∞, g^∞) with scalar curvature greater than 4 has a neighborhood that is either a strong 8ε-neck or a 8ε-cap.

We claim that (M^∞, g^∞) has bounded curvature. Indeed, if there is a sequence of points in (M^∞, g^∞) with curvature going to infinity, then (M^∞, g^∞) contains a sequence of necks with radii converging to 0, which contradicts Proposition 1.9. Therefore (M^∞, g^∞) has bounded curvature.

Step 5. We now extend the limit (M^∞, g^∞) backwards in time. By Step 4, the scalar curvature of (M^∞, g^∞) is bounded from above by a constant $\Lambda > 4$. We claim that, given any $A > 1$, the parabolic neighborhood

$$P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100\eta\Lambda}Q_j^{-1})$$

is surgery-free if j is sufficiently large.

To prove the claim, fix $A > 1$ and suppose that $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100\eta\Lambda}Q_j^{-1})$ contains points modified by surgery. Let $s_j \in [0, \frac{1}{100\eta\Lambda}]$ be the largest number such that $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_jQ_j^{-1})$ is surgery-free. If j is sufficiently large, the pointwise curvature derivative estimate gives

$$\sup_{P(x_j,t_j,AQ_j^{-\frac{1}{2}},-s_jQ_j^{-1})} S \leq 2\Lambda Q_j.$$

Since the scalar curvature is greater than $\frac{1}{2}h_j^{-2}$ at each point modified by surgery, we deduce that $\frac{1}{2}h_j^{-2} \leq 2\Lambda Q_j$ if j is sufficiently large. In particular, $s_jQ_j^{-1} \leq \frac{1}{10\eta}h_j^2$ if j is sufficiently large. Since $\hat{\delta}_j \to 0$, Proposition 6.4 implies that the parabolic neighborhood $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -s_jQ_j^{-1})$ is, after dilating by the factor h_j, arbitrarily close to a piece of the standard solution when j is sufficiently large. Also, Corollary 5.3 shows that (x_j, t_j) lies on a 2ε-neck or a 2ε-cap when j is sufficiently large. If (x_j, t_j) lies on an 2ε-neck, then this neck is actually a strong 2ε-neck, since we are assuming that each $\hat{\delta}_j$-neck on which we perform surgery has a large backward parabolic neighborhood that is surgery-free. Moreover, Corollary 5.3 implies that $|\nabla S| \leq 2\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 2\eta S^2$ at the point (x_j, t_j). Therefore, the point (x_j, t_j) satisfies the conclusion of Theorem 6.7, which contradicts property (iii). Thus, given any $A > 1$, the parabolic neighborhood $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, -\frac{1}{100\eta\Lambda}Q_j^{-1})$ is surgery-free if j is sufficiently large.

Let $\tau_1 := -\frac{1}{200\eta\Lambda}$. In view of the preceding discussion, we may extend (M^∞, g^∞) backwards in time to a complete solution $(M^\infty, g^\infty(t))$ that is defined for $t \in [\tau_1, 0]$ and satisfies $\Lambda_1 := \sup_{t\in[\tau_1,0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda$.

Repeating this process, suppose that we can extend (M^∞, g^∞) backwards in time to a complete solution $(M^\infty, g^\infty(t))$ that is defined for $t \in [\tau_m, 0]$, and satisfies $\Lambda_m := \sup_{t\in[\tau_m,0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\infty$. Let $\tau_{m+1} := \tau_m - \frac{1}{200\eta\Lambda_m}$. The goal is to show that the solution $(M^\infty, g^\infty(t))$ can be extended backward to the interval $[\tau_{m+1}, 0]$, and $\Lambda_{m+1} := \sup_{t\in[\tau_{m+1},0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda_m$.

If it is not possible, then $\exists$ a number $A > 1$ with the property that $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, (\tau_m - \frac{1}{100\eta\Lambda_m})Q_j^{-1})$ contains points modified by surgery for sufficiently large j. Let $s_j \in$

$[0, \frac{1}{100\eta\Lambda_m}]$ be the largest number such that $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, (\tau_m - s_j)Q_j^{-1})$ is surgery-free. If j is sufficiently large. the pointwise curvature derivative estimate gives

$$\sup_{P(x_j,t_j,AQ_j^{-\frac{1}{2}},(\tau_m-s_j)Q_j^{-1})} S \leq 2\Lambda_m Q_j.$$

Select $\alpha_m \in [0, \frac{n-1}{2})$ so that $\frac{\alpha_m}{K_{\text{stnd}}(1-\frac{2\alpha_m}{n-1})} \geq 8\Lambda_m(\frac{1}{100\eta\Lambda_m} - \tau_m)$. If $(s_j - \tau_m)Q_j^{-1} \geq \alpha_m h_j^2$ for sufficiently large j , then Proposition 6.4 together with the lower bound for the scalar curvature on the standard solution (cf. Theorem 5.1) implies

$$\begin{aligned}
\sup_{P(x_j,t_j,AQ_j^{-\frac{1}{2}},(\tau_m-s_j)Q_j^{-1})} S &\geq \frac{\alpha_m}{K_{\text{stnd}}(1-\frac{2\alpha_m}{n-1})} h_j^{-2} \\
&\geq \frac{\alpha_m}{2K_{\text{stnd}}(1-\frac{2\alpha_m}{n-1})}(s_j - \tau_m)^{-1} Q_j \\
&\geq 4\Lambda_m Q_j
\end{aligned}$$

for sufficiently large j, which is impossible. Consequently, $(s_j - \tau_m)Q_j^{-1} \leq \alpha_m h_j^2$ for sufficiently large j. Since $\hat{\delta}_j \to 0$, Proposition 6.4 implies that the parabolic neighborhood $P(x_j, t_j, AQ_j^{-\frac{1}{2}}, (\tau_m - s_j)Q_j^{-1})$ is, after dilating by the factor h_j, arbitrarily close to a piece of the standard solution when j is sufficiently large. Also, Corollary 5.3 shows that (x_j, t_j) lies on a 2ε-neck or a 2ε-cap when j is sufficiently large. If (x_j, t_j) lies on an 2ε-neck, then this neck is actually a strong 2ε-neck, since we are assuming that each $\hat{\delta}_j$-neck on which we perform surgery has a large backward parabolic neighborhood that is surgery-free. Moreover, Corollary 5.3 implies that $|\nabla S| \leq 2\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 2\eta S^2$ at the point (x_j, t_j). Therefore, the point (x_j, t_j) satisfies the conclusion of Theorem 6.7, which contradicts property (iii). Thus, the flow $(M^\infty, g^\infty(t))$ can be extended backward to the interval $[\tau_{m+1}, 0]$, where $\tau_{m+1} := \tau_m - \frac{1}{200\eta\Lambda_m}$, and we have $\Lambda_{m+1} := \sup_{t\in[\tau_{m+1},0]} \sup_{M^\infty} S_{g^\infty(t)} \leq 2\Lambda_m$.

Step 6. Now, let $\tau^* = \lim_{m\to\infty} \tau_m \leq -\frac{1}{100n\Lambda}$. Standard diagonal sequence argument leads to a complete, smooth limit flow $(M^\infty, g^\infty(t))$ that is defined on the interval $(\tau^*, 0]$ and that has bounded curvature for each $t \in (\tau^*, 0]$. The goal is show that $\tau^* = -\infty$. Suppose not, then $\lim_{m\to\infty}(\tau_m - \tau_{m+1}) = 0$, hence $\lim_{m\to\infty} \Lambda_m = \infty$. Arguing as in Step 6 in the proof of Theorem 7.1, we can show that the limit flow $(M^\infty, g^\infty(t)), t \in (\tau^*, 0]$ has bounded curvature, which contradicts the fact that $\lim_{m\to\infty} \Lambda_m = \infty$. Hence, $\tau^* = -\infty$.

Step 7. As a consequence of Step 6, if we dilate the flow $(M, g(t))$ around the point (x_j, t_j) by the factor Q_j, then , after passing to a subsequence, the rescaled flows converge to an ancient solution that is complete, has bounded curvature, is weakly PIC2, and satisfies $R - \theta S \,\mathrm{id} \owedge \mathrm{id} \in \mathrm{PIC}$. By Proposition 6.6, the limiting ancient solution is κ-noncollapsed for some $\kappa > 0$ that depends only on the initial data.

By Corollary 1.14, the point (x_j, t_j) has a Canonical Neighborhood that is either a strong 2ε-neck with center at x_j; or a 2ε-cap; or a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$; or a quotient neck. Recall that we have discarded all connected components that are diffeomorphic to $\mathbb{S}^n/\Gamma$. Hence, the Canonical Neighborhood of (x_j, t_j) cannot be a closed manifold diffeomorphic to $\mathbb{S}^n/\Gamma$. If the Canonical Neighborhood of (x_j, t_j) is a quotient neck, then Theorem A.1 in [5] implies that the underlying manifold contains a nontrivial incompressible $(n-1)$-dimensional space form, contrary to our assumption. Consequently, the point (x_j, t_j) has a Canonical Neighborhood that is either a strong 2ε-neck with center at x_j or a 2ε-cap.

Finally, Corollary 2.4 implies that $|\nabla S| \leq 2\eta S^{\frac{3}{2}}$ and $|\partial_t S| \leq 2\eta S^2$ at (x_j, t_j). In summary, we have shown that the point (x_j, t_j) satisfies the conclusion of Theorem 10.7, which contradicts property (iii).

□

7. Global existence of surgically modified flows

As previous sections, fix a compact initial manifold (M, g_0) of dimension $n \geq 12$ that has positive isotropic curvature and does not contain any nontrivial incompressible $(n-1)$-dimensional space forms. In this section, the goal is to show that there exists a Ricci flow with surgery starting from (M, g_0), which exists globally and becomes extinct in finite time. We begin by finalizing our choice of the surgery parameters. As usual, we fix a small number $\varepsilon > 0$. Having chosen ε, we choose numbers $\hat{r}, \hat{\delta}$ such that the conclusion of Theorem 6.7 holds. Having chosen $\varepsilon, \hat{r}, \hat{\delta}$, we choose h so that the following holds:

Proposition 7.1 (cf. G. Perelman [25, Lemma 4.3])**.** *Given $\varepsilon, \hat{r}, \hat{\delta}$, we can find a small number $h \in (0, \hat{\delta}\hat{r})$ with the following property: Suppose that we have a Ricci flow with surgery with parameters $\varepsilon, \hat{r}, \hat{\delta}, h$ that is defined on the time interval $[0, T)$ and goes singular at time T. Let x be a point that lies in an 4ε-horn in $(M, g(T))$ and has curvature $S(x, T) = h^{-2}$. Then the parabolic neighborhood $P(x, T, \hat{\delta}^{-1}h, -\hat{\delta}^{-1}h^2)$ is surgery-free. Moreover, $P(x, T, \hat{\delta}^{-1}h, -\hat{\delta}^{-1}h^2)$ is a strong $\hat{\delta}$-neck.*

Proof. Suppose not. Then $\exists$ a sequence of positive numbers $h_j \to 0$, a sequence of Ricci flows with surgery $\mathcal{M}^{(j)}$ and a sequence of points x_j with the following properties:

(i) The flow $\mathcal{M}^{(j)}$ has surgery parameters $\varepsilon, \hat{r}, h_j, \hat{\delta}$. It is defined on the time interval $[0, T_j)$ and goes singular as $t \to T_j$.
(ii) The point (x_j, T_j') lies on an 4ε-horn and $S(x_j, T_j') = h_j^{-2}$.
(iii) The parabolic neighborhood $P(x_j, T_j, \hat{\delta}^{-1}h_j, -\hat{\delta}^{-1}h_j^2)$ contains points modified by surgery, or it is not a strong $\hat{\delta}$-neck.

Note that from the definition of Ricci flows with surgery, we have estimate $|\nabla S| \leq 4\eta S^{\frac{3}{2}}$ whenever $S \geq \hat{r}^{-2}$. Since $h_j \to 0$, it follows that $\inf_{x\in B_{g(T_j)}(x_j,Ah_j)} S(x,T_j) \geq (1+2\eta A)^{-2}h_j^{-2}$. In particular, if j is sufficiently large (depending on A), then

$$\inf_{x\in B_{g(T_j)}(x_j,Ah_j)} S(x,T_j) \geq 10(\hat{\delta}\hat{r})^{-2}.$$

We claim that for each $A > 1$, $\exists$ a constant $Q(A)$, which depends on A, but not on j, such that $\sup_{x\in B_{g(T_j)}(x_j,Ah_j)} S(x,T_j) \leq Q(A)h_j^{-2}$ if j is sufficiently large. Suppose that such constant $Q(A)$ does not exist, then $\exists$ a sequence of points (y_j,T_j), lying on the same horn as (x_j,T_j), such that the blow-up limit around (y_j,T_j) is a piece of nonflat metric cone. Since the flow $\mathcal{M}^{(j)}$ satisfies the Canonical Neighborhood Assumption with accuracy 4ε, the point (y_j,T_j) either lies on a strong 4ε-neck or on a 4ε-cap. The second case can easily be ruled out (cf. Claim 2 in Theorem 12.1 in [23]), so (y_j,T_j) must lie on a strong 4ε-neck. In particular, there exists a small parabolic neighborhood of (y_j,T_j) that is surgery-free. Due to Proposition 6.2, the blow-up limit around (y_j,T_j) is weakly PIC2. Hence, Proposition 1.5 implies that the limit cannot be a piece of a nonflat metric cone, which proves the claim.

In particular, if j is sufficiently large, which depends on A, then the distance of the point x_j from either end of the horn is at least Ah_j. Now, fix a number $A > 1$. Since the point (x_j,T_j) lies on a 4ε-horn, no point in $B_{g(T_j)}(x_j,Ah_j)$ can lie on a 4ε-cap. Hence, the Canonical Neighborhood Assumption implies that every point in $B_{g(T_j)}(x_j,Ah_j)$ lies on a strong 4ε-neck. Like before, Shi's estimate (cf. [26]) leads to bounds for all the covariant derivatives of the curvature tensor in $B_{g(T_j)}(x_j,\frac{1}{2}Ah_j)$. Note that these bounds may depend on A, but are independent of j. Passing to the limit, we sending $j \to \infty$ first and $A \to \infty$ second. In the limit, we obtain a complete manifold with two ends that, by Proposition 6.2, is uniformly PIC and weakly PIC2. By the Cheeger-Gromoll splitting theorem (cf. [10]), the limit is isometric to a product $X \times \mathbb{R}$; moreover, the cross-section X is compact and is nearly isometric to $\mathbb{S}^{n-1}$. Since every point in $B_{g(T_j)}(x_j,Ah_j)$ lies on a strong 4ε-neck, we conclude that, for each $A > 1$, the parabolic neighborhood $P(x_j,T_j,Ah_j,-\frac{3h_j^2}{4})$ is surgery-free if j is sufficiently large (depending on A). After rescaling and passing to the limit, we obtain a solution to the Ricci flow that is defined on the time interval $[-\frac{1}{2},0]$ and that splits off a line. Now, if j is sufficiently large (depending on A), then no point in $P(x_j,T_j,Ah_j,-\frac{h_j^2}{2})$ can lie on a 4ε-cap. Hence, if j is sufficiently large, then every point in the parabolic neighborhood $P(x_j,T_j,Ah_j,-\frac{h_j^2}{2})$ lies on a strong 4ε-neck. This allows us to extend the limit solution backward in time to the interval $[-1,0]$. Repeating this argument, we can extend the limit solution backwards in time, so that it is defined on $[-1,0],[-\frac{3}{2},0],[-2,0]$, etc. To summarize, we produce a limit solution that is ancient, uniformly PIC, weakly PIC2, and splits as a product of a line with a manifold diffeomorphic to $\mathbb{S}^{n-1}$. By the work of Brendle, Huisken and Sinestrari (cf. [7]), the limiting solution is a family of standard cylinders.

Therefore, if j is sufficiently large, then the parabolic neighborhood $P(x_j, T_j, \hat{\delta}^{-1}h_j, -\hat{\delta}^{-1}h_j^2)$ is surgery-free, and $P(x_j, T_j, \hat{\delta}^{-1}h_j, -\hat{\delta}^{-1}h_j^2)$ is a $\hat{\delta}$-neck, which contradicts (iii). □

We are now able to prove the main result of this section:

Theorem 7.2. *Fix a small number $\varepsilon > 0$. Let $\hat{r}, \hat{\delta}$ be chosen as described at the beginning of this section, and let h be chosen as in Proposition 7.1. Then there exists a Ricci flow with surgery with parameters $\varepsilon, \hat{r}, \hat{\delta}, h$, which is defined on some finite time interval $[0, T)$ and becomes extinct as $t \to T$.*

Proof. Evolve the initial metric g_0 by smooth Ricci flow until the flow becomes singular for the first time. It follows from Theorem 6.7 and a standard continuity argument that the flow satisfies the Canonical Neighborhood Property with accuracy 2ε on all scales less than $2\hat{r}$, up until the first singular time. At the first singular time, we perform finitely many surgeries on $\hat{\delta}$-necks that have curvature level h^{-2}. The existence of such necks is ensured by Proposition 7.1. After performing surgery, we restart the flow and continue until the second singular time. Again, Theorem 6.7 and a standard continuity argument, we conclude that the flow with surgery satisfies the Canonical Neighborhood Property with accuracy 2ε on all scales less than $2\hat{r}$, up until the second singular time. Consequently, Proposition 7.1 ensures that, at the second singular time, we can again find $\hat{\delta}$-necks on which to perform surgery. After performing surgery, we continue the flow until the third singular time. Theorem 6.7 also guarantees that the flow with surgery satisfies the Canonical Neighborhood Property with accuracy 2ε on all scales less than $2\hat{r}$, up until the third singular time. We can now perform surgery again and repeat the process.

Since each surgery reduces the volume by at least $c(n)h^n$, we have an upper bound for the number of surgeries. By Proposition 6.1, the flow with surgery must become extinct by time $\frac{n}{2\inf_{x\in M} S(x,0)}$ at last. The conclusion of Theorem 7.2 follows. □

Corollary 7.3. *The manifold M is diffeomorphic to a connected sum of finitely many spaces, each of which is a quotient of $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$ by standard isometries.*

Proof. Combining Theorem 7.2 with Proposition 4.1 leads to the desired result. □

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