Bott Periodicity Theorem

$$\tilde{\Omega}(SU(2m); I, -I) := \left\{ \begin{array}{l} \text{minimal geodesics (w.r.t } \langle A, B\rangle = Re(Tr(AB^*))) \\ \text{from } I \text{ to } -I \text{ in } SU(2m) \end{array} \right\}$$

$\uparrow$ Killing

⟨Lemma 1.1⟩ $\tilde{\Omega}(SU(2m); I, -I) \simeq Gm(\mathbb{C}^{2m})$.

(p.f.) $\{$geodesics passing $I\} \leftrightarrow \{$one-parameter subgp of $SU(2m)\}$

$(\gamma(t) = \exp(tA)$ for skew-sym A & $Tr A = 0)$

Suppose $A \in su(2m)$ s.t. $\exp A = -I$

$\Rightarrow \exists\, T \in U(2m)$ s.t. $TAT^{-1} = \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{2m} \end{pmatrix}$, where $\sum a_j = 0$.

$\Rightarrow -I = T(\exp A)T^{-1} = \exp(TAT^{-1}) = \begin{pmatrix} e^{ia_1} & & \\ & \ddots & \\ & & e^{ia_{2m}} \end{pmatrix}$

$\Rightarrow a_j = k_j \pi$, where $\sum k_j = 0$, k_j = odd

$\because$ length of $\gamma = \|A\| = \pi\left(\sum k_j^2\right)^{1/2}$

$\therefore k_j \in \{\pm 1\}$ ($\because$ minimal)

$\Rightarrow \dim(\mathrm{Eigen}(i\pi)) = m$ ($\because \sum k_j = 0$).

⟨Prop 1.2⟩ If Ω^d (the space of minimal geodesics from p to q) is a topological manifold, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then the relative homotopy gp $\pi_i(\Omega, \Omega^d) = 0$ for $0 \leq i < \lambda_0$, where Ω is the path space.

⟨Lemma 1.3⟩ Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$

⟨Prop 1.4⟩ $\pi_i(Gm(\mathbb{C}^{2m})) \simeq \pi_{i+1}(SU(2m))$ for $i \leq 2m$.

(p.f.) $\pi_i(Gm(\mathbb{C}^{2m})) \underset{(1.1)}{\simeq} \pi_i(\tilde{\Omega}(SU(2m); I, -I)) \underset{\binom{1.2}{+\,1.3}}{\simeq} \pi_i(\Omega(SU(2m); I, -I)) \underset{(e.g.)}{\simeq} \pi_{i+1}(SU(2m))$.

⟨Def 1.5⟩ $U(m) \longrightarrow U(m+1) \xrightarrow{p} S^{2m+1}$ is a fiber bundle.

$(a_{ij}) \longmapsto (x_{i1}, y_{i1})$ where $a_{i1} = x_{i1} + \sqrt{-1}\, y_{i1}$

$\Rightarrow \pi_i(S^{2m+1}) \longrightarrow \pi_{i-1}(U(m)) \longrightarrow \pi_{i-1}(U(m+1)) \longrightarrow \pi_{i-1}(S^{2m+1})$ is exact

$\because \pi_i(S^{2m+1}) = 0$ for $i \leq 2m+1$ (Sard's Thm)

$\therefore \pi_{i-1}(U(m)) \simeq \pi_{i-1}(U(m+1)) \simeq \pi_{i-1}(U(m+2)) \simeq \cdots$ for $i \leq 2m+1$

We denote these mutually isomorphic gps by $\pi_{i-1}(U)$

⟨Thm 1.6⟩ $\pi_{i-1} U \simeq \pi_{i+1} U$ for $i \geq 1$

(p.f.) · (complex Stiefel manifold)

$V_k(\mathbb{C}^n) := \{k\text{-frames in } \mathbb{C}^n\} = U(n)/U(n-k)$

(i) $U(m) \longrightarrow U(2m) \xrightarrow{p} U(2m)/U(m)$ is a fiber bundle.

$A \longmapsto A\begin{pmatrix} I & 0 \\ 0 & U(m) \end{pmatrix}$

$\Rightarrow \pi_i(U(m)) \longrightarrow \pi_i(U(2m)) \longrightarrow \pi_i(U(2m)/U(m)) \longrightarrow \pi_{i-1}(U(m)) \longrightarrow \pi_{i-1}(U(2m))$ is exact

$\Rightarrow \pi_i(U(2m)/U(m)) = 0$ for $i \leq 2m$.

(ii) $U(m) \longrightarrow U(2m)/U(m) \xrightarrow{p} U(2m)/U(m)\times U(m)$ is a fiber bundle

$A\begin{pmatrix} I & 0 \\ 0 & U(m) \end{pmatrix} \longmapsto A\begin{pmatrix} U(m) & 0 \\ 0 & U(m) \end{pmatrix}$

$\Rightarrow \pi_i(U(2m)/U(m)) \longrightarrow \pi_i(U(2m)/U(m)\times U(m)) \longrightarrow \pi_{i-1}(U(m)) \longrightarrow \pi_{i-1}(U(2m)/U(m))$ is exact

$\Rightarrow \pi_i(U(2m)/U(m)\times U(m)) \simeq \pi_{i-1}(U(m))$ for $i \leq 2m$

(iii) $SU(m) \longrightarrow U(m) \xrightarrow{\det} S^1$ is a fiber bundle.

$\Rightarrow \pi_i(S^1) \longrightarrow \pi_{i-1}(SU(m)) \longrightarrow \pi_{i-1}(U(m)) \longrightarrow \pi_{i-1}(S^1)$ is exact

$\because \pi_i(S^1) = 0$ for $i > 2$ (a covering space is a fiber bundle).

$\therefore \pi_{i-1}(SU(m)) \simeq \pi_{i-1}(U(m))$ for $i > 2$

· $\underset{\substack{(\text{Def}\\ i\leq 2m+1)}}{\pi_{i-1}(U) \simeq} \pi_{i-1}(U(m)) \underset{\substack{(\text{ii})\\ i\leq 2m}}{\simeq} \pi_i(U(2m)/U(m)\times U(m)) \underset{\substack{(\text{i.4}\\ i\leq 2m+1)}}{\simeq} \pi_{i+1}(SU(2m)) \underset{\substack{(\text{iii})\\ i\geq 1}}{\simeq} \pi_{i+1}(U(2m)) \underset{\substack{(\text{Def}\\ i+2\leq 4m+1)}}{\simeq} \pi_{i+1}(U)$

⟨Thm 1.7⟩ $\pi_{2n}(U) \simeq 0$; $\pi_{2n+1}(U) \simeq \mathbb{Z}$.

(p.f.) $\pi_0(U) \simeq \pi_0(U(1)) = 0$

$\pi_1(U) \simeq \pi_1(U(1)) \simeq \mathbb{Z}$.

Homotopy Theory

⟨Def A⟩ $\pi_i(M, x_0) = \{[\alpha] \mid \alpha : D^i \to M,\ \alpha(\partial D^i) = x_0\}$

$= \{[\alpha] \mid \alpha : S^i \to M,\ \alpha(s_0) = x_0\}$

$= \{[\alpha] \mid \alpha : I^i \to M,\ \alpha(\partial I^{i-1}) = x_0\}$

$\pi_i(M, A, x_0) = \{[\alpha] \mid \alpha : (D^i, S^{i-1}, s_0) \to (M, A, x_0)\}$

$= \{[\alpha] \mid \alpha\ (I^i, I^{i-1}, \partial I^i - I^{i-1}) \to (M, A, x_0)\}$

⟨Prop B⟩ $\cdots \xrightarrow{\partial} \pi_i(A, x_0) \xrightarrow{i_*} \pi_i(M, x_0) \xrightarrow{j} \pi_i(M, A, x_0) \xrightarrow{\partial} \pi_{i-1}(A, x_0) \longrightarrow \cdots$

is exact, where $i : A \to M$ is the inclusion,

j is the inclusion of homotopy classes with $A = \{x_0\}$.

$\partial : \pi_i(M, A, x_0) \longrightarrow \pi_{i-1}(A, x_0)$

$\alpha : D^i \to M$, $\alpha(\partial D^i = S^{i-1}) \subseteq A$, $\alpha(s_0) = x_0$ $\longmapsto$ $\alpha : S^{i-1} \to A$, $\alpha(s_0) = x_0$

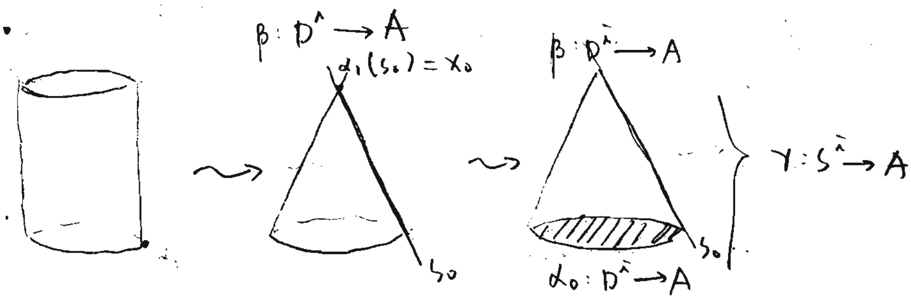
(p.f.). $\ker j = \operatorname{Im} i_*$.

"$\subseteq$": $[\alpha] \in \ker j$ $\quad \alpha : D^i \to M$, $\alpha(\partial D^i) = x_0$.

$\Rightarrow \exists\ \alpha_t : D^i \times I \to M$

$\alpha_t(\partial D^i) \subseteq A$

$\alpha_1(D^i) = x_0$.



"$\supseteq$": trivial.

⟨Def C⟩ homotopy lifting property (HLP) for $p : E \to B$ w.r.t. X.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "f_t"] & B \end{tikzcd}$$

given f & g_t, where $g_0 = p \circ f$.

f_t : covering homotopy for g_t.

- $p : E \to B$ is a fibration if HLP holds for all X

Serre fibration $\qquad D^i$.

⟨Prop D⟩ If p is a Serre fibration and let $b \in B$, $F = p^{-1}(b)$, $f \in F$,

then $p_*: \pi_i(E, F, f) \longrightarrow \pi_i(B, b)$ is an isomorphism

$$\Rightarrow \cdots \xrightarrow{\partial} \pi_i(F, f) \xrightarrow{i_*} \pi_i(E, f) \xrightarrow{p_* \circ j} \pi_i(B, b) \xrightarrow{\partial} \pi_{i-1}(F, f) \longrightarrow \cdots$$

is exact.

(p.f.) (1-1). $\pi_i(E, F\, f) \xrightarrow{p_*} \pi_i(B, b)$

$[\alpha] \longmapsto [\beta] = 0$ i.e. $\exists\, \beta_t : D^i \to B$

$\alpha: D^i \to E$, $\partial D^i \to F$, $s_0 \mapsto f_0$ $\quad$ $\beta = p \circ \alpha$ $\quad$ $\beta_1(D^i) = b$, $\beta_t(\partial D^i) = b$, $\beta_0 = \beta$

$$\begin{tikzcd} D^i \arrow[r, "\alpha"] \arrow[d] & E \arrow[d, "p"] \\ D^i \times I \arrow[r, "\beta_t"'] \arrow[ur, dashed, "\alpha_t"] & B \end{tikzcd}$$

$\Rightarrow \exists\, \alpha_t : D^i \to E$

$\alpha_1(D^i) \subseteq F$ $\quad$ $\alpha_0 = \alpha$

$\alpha_1(\partial D^i) \subseteq F$ $\quad$ $\alpha_t(s_0) = f$

e.g. (Fiber bundles). $F \to E \xrightarrow{p} B$ $\quad \forall b \in B$, $\exists\, U$ nbhd of b s.t. $\exists\, h: p^{-1}(U) \xrightarrow{\simeq} U \times F$

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\simeq"] \arrow[dr] & U \times F \arrow[d] \\ & U \end{tikzcd}$$

compactness $\Rightarrow$ may assume test space falls in one open set

$$\begin{tikzcd} D^i \arrow[r, "f"] \arrow[d] & U \times F \arrow[d, "p"] \\ D^i \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "f_t"] & U \end{tikzcd}$$

$f_t(x) = (g_t(x), f(x))$

e.g. (path spaces). $E_b := \{\gamma(t) \mid \gamma(0) = b\} \subseteq B^I$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & E_b \arrow[d, "p"] \\ X \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "f_t"] & B \end{tikzcd}$$

$p: B^I \to B$ is cont

$\gamma \longmapsto \gamma(1)$

write $f(x) = \gamma_x$

$\Rightarrow \exists\, f_t : X \longrightarrow E_b \subseteq B^I$

$x \longmapsto (\gamma_x \cdot g_{[0,t]}(x))$ $\quad (g_0(x) = \gamma_x(1))$

$\because \pi_i(E_b) = 0$

$\therefore \pi_i(\Omega(B, b)) \simeq \pi_i(B)$

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§2 Approximation - General Case

⟨Def⟩ $f: M \to \mathbb{R}$

- a critical pt p is called "non-degenerate" if $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$ is non-singular
- the "index" of f at p is the number of negative eigenvalues of $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$
- f is a "Morse function" if it has no degenerate critical pts.
- $M^a := f^{-1}([-\infty, a])$

⟨Thm⟩ $f: M \to \mathbb{R}$ with min 0 s.t. M^c is cpt $\forall c$
If M^0 is a mfd, and if every critical pt in $M - M^0$ has index $\geq \lambda_0$,
then $\pi_r(M, M^0) = 0$ for $0 \leq r < \lambda_0$.

⟨Morse 1⟩ $f: M \to \mathbb{R}$, p: non-degenerate critical pt w/ index λ.
Set $f(p) = c$, suppose $f^{-1}([c-\varepsilon, c+\varepsilon])$ is cpt, and contains no other critical pt of f, for some $\varepsilon > 0$.
Then $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ w/ a λ-cell attached.

⟨Morse 2⟩ $f: \overset{(cpt)}{M} \to \mathbb{R}$ w/ all critical pts on M have index $\geq \lambda_0$,
$\exists g: M \to \mathbb{R}$ s.t. (1) g is a Morse function
(2) $|f(x) - g(x)| < \delta$ $\forall x \in M$, for some $\delta > 0$.
(3) all critical pts of g in M have index $\geq \lambda_0$.

(pf. of Thm). $h: (I^r, \dot{I}^r) \to (M^c, M^0) \subseteq (M, M^0)$, where $c = \max\{f(h(I^r))\}$
$\wr$ (we seek)
$h': (I^r, \dot{I}^r) \to (M^0, M^0)$.
Choose g as in ⟨Morse 2⟩, then $M^c \subseteq g^{-1}((-\infty, c+\delta])$
If $M^0 \supseteq g^{-1}((-\infty, \gamma])$ for some γ, then we're done!
Instead, we consider U: some nbhd of M^0, of which M^0 is a retract.
Now, $U \supseteq g^{-1}((-\infty, 2\delta])$, where $3\delta = \min\{f(M-U)\}$.
$g^{-1}[2\delta, c+\delta]$ being cpt together with ⟨Morse 1⟩ $\Rightarrow$ Done! □

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(p.f. of Morse 1)

(Claim) $\exists (u^1, \cdots, u^n)$ in a nbhd U of p s.t. $f = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^n)^2$ on U

Choose $\varepsilon > 0$ s.t. $\{(u^1, \cdots, u^n) \mid \sum (u^i)^2 \leq 2\varepsilon\} \subseteq U$

$e^\lambda := \{(u^1, \cdots, u^n) \mid \sum_{i=1}^{\lambda} (u^i)^2 \leq \varepsilon,\ u^{\lambda+1} = \cdots = u^n = 0\}$.

$\mu : \mathbb{R} \to \mathbb{R}$. s.t. (1) $\mu(0) > \varepsilon$

(2) $\mu(r) = 0$ for $r \geq 2\varepsilon$

(3) $-1 < \mu'(r) \leq 0 \ \forall r$

$F : M \to \mathbb{R}$. $\begin{cases} f & \text{outside } U \\ f - \mu(\xi + \eta) & \text{inside } U \end{cases}$ where. $\begin{cases} \xi = \sum_{i=1}^{\lambda} (u^i)^2 \\ \eta = \sum_{i=\lambda+1}^{n} (u^i)^2 \end{cases}$

Check (1) $F^{-1}(-\infty, c+\varepsilon] = M^{c+\varepsilon}$

(2) F and f have the same critical pts.

(3) $F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract of $M^{c+\varepsilon}$

(4) $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$

⟨Lemma 1⟩ $f : M \to \mathbb{R}$.

Suppose $f^{-1}([a, b])$ is cpt. and contains no critical pts of f.
Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.

(p.f.)

$X := \operatorname{grad} f / \|\operatorname{grad} f\|^2$

$\varphi_t : M \to M$. : diffeomorphisms.

$df(\varphi_t(q))/dt = \langle d\varphi_t(q)/dt,\ \operatorname{grad} f \rangle = \langle X, \operatorname{grad} f \rangle = 1$

$r_t : M^b \to M^b$

$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_t(a - f(q))(q). & \text{if } a \leq f(q) \leq b. \end{cases}$

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⟨Morse 2.1⟩ Any bounded $f: M \to \mathbb{R}$ can be uniformly approximated by g: Morse up to the ith derivative on a compact set K for some fixed i.

⟨Morse 2.2⟩ Let $g: U \to \mathbb{R}$ s.t. $\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \varepsilon$, $\left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| < \varepsilon$ uniformly throughout K: cpt in U for some sufficiently small $\varepsilon > 0$.

If all critical pts of f in K have index $\geq \lambda_0$, then so does that of g.

(p.f.) $k_g(x) = \sum_i \left|\frac{\partial g}{\partial x_i}\right| \geq 0$

$e_g^1(x) \leq e_g^2(x) \leq \cdots \leq e_g^n(x)$: eigenvalues of $\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)$

$m_g(x) := \mathrm{Max}\{k_g(x), -e_g^{\lambda_0}(x)\}$

Likewise, we have $k_f(x)$, $e_f^{\lambda_0}(x)$, $m_f(x)$

Let $\delta > 0$ be the minimum of m_f on K

If $|k_g(x) - k_f(x)| < \delta$, $|e_g^{\lambda_0}(x) - e_f^{\lambda_0}(x)| < \delta$, then we're done!

§3. Approximation – How path space fits in

⟨Thm⟩ Let M be a complete Riemannian manifold.
If $\tilde{\Omega}(M;p,q)$ is a topological manifold, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$.
Then $\pi_i(\Omega, \tilde{\Omega}) = 0$ for $0 \leq i < \lambda_0$.

- $\Omega(M;p,q) = \Omega(M) = \Omega := \{$piecewise smooth paths from p to q in $M\}$
- $T\Omega_\omega := \{$piecewise smooth v.f. W along ω for which $W(0) = W(1) = 0\}$
- $E_a^b(\omega) = \int_a^b \| d\omega/dt \|^2 dt$ (we write E for E_0^1)
- $d(\omega, \omega') := \max_{t\in[0,1]} \rho(\omega(t), \omega'(t)) + \left(\int_0^1 \left(\frac{ds}{dt} - \frac{ds'}{dt}\right)^2 dt\right)^{1/2}$, where ρ is the topological metric on M coming from its Riemannian metric, and s, s' are the arclengths of ω, ω' respectively.
- d is a metric on Ω, under which E is continuous
- $\Omega^c := E^{-1}([0,c]) \subseteq \Omega$
- $\mathrm{Int}\,\Omega^c := E^{-1}([0,c)) \subseteq \Omega$
- $\Omega(t_0, \cdots, t_k) := \{\omega : [0,1] \to M \mid (1)\ \omega(0) = p,\ \omega(1) = q$
  (Space of broken geodesics) $(2)\ \omega|_{[t_{i-1}, t_i]}$ is a geodesic $\forall i\}$,
  where $0 = t_0 < t_1 < \cdots < t_k = 1$ is a subdivision of $[0,1]$
- $\mathrm{Int}\,\Omega(t_0, \cdots, t_k)^c := (\mathrm{Int}\,\Omega^c) \cap (\Omega(t_0, \cdots, t_k))$

⟨Prop 1⟩ Let c be a fixed positive number s.t. $\Omega^c \neq \phi$.
Then for all sufficiently fine subdivisions $(t_0, \cdots, t_k)$ of $[0,1]$, $\mathrm{Int}\,\Omega(t_0, \cdots, t_k)^c$ can be given the structure of a smooth finite dimensional manifold in a natural way.

(p.f.) 1° $S := \{x \in M \mid \rho(x,p) \leq \sqrt{c}\}$ is cpt, and contains Ω^c:
$\because$ M: complete $\therefore$ by Hopf-Rinow, S is cpt
2° Choose $(t_0, \cdots, t_k)$ of $[0,1]$ s.t. $\omega|_{[t_{i-1}, t_i]}$ is uniquely and smoothly determined by the two end points:
- $(L_{t_{i-1}}^{t_i} \omega)^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i} \omega) \leq (t_i - t_{i-1})(E\omega) \leq (t_i - t_{i-1}) c$.

3° $\mathrm{Int}\,\Omega(t_0,\dots,t_k)^c \approx M\times\cdots\times M$ $(k-1)$-fold

$\omega \longmapsto (\omega(t_1),\dots,\omega(t_{k-1}))$ □

- $B := \mathrm{Int}\,\Omega(t_0,\dots,t_k)^c$
- E' be the restriction of E to B.

⟨Thm of §2⟩ Let $f: M \to \mathbb{R}$ be smooth with minimum 0 s.t. each M^c is compact. If M^0 is a manifold, and if every critical point in $M-M^0$ has index $\geq \lambda_0$. Then $\pi_r(M,M^0)=0$ for $0\leq r<\lambda_0$

⟨Prop 2⟩ (1) $E': B \to \mathbb{R}$ is smooth

(2) $\forall a<c$. $B^a = (E')^{-1}([0,a])$ is compact

(3) B is a deformation retract of $\mathrm{Int}\,\Omega^c$

(4) critical points of E' are precisely that of E in $\mathrm{Int}\,\Omega^c$, i.e. unbroken geodesics from p to q of length $<\sqrt{c}$.

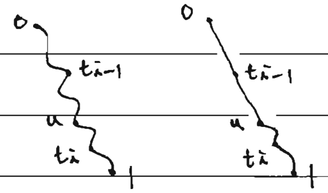
(5) the index (nullity) of $(E')_{**}$ at each critical point γ is equal to that of E_{**} at γ

(p.f.) (1) $E'(\omega) = \sum_{i=1}^{k} \rho(\omega(t_{i-1}),\omega(t_i))^2 / (t_i - t_{i-1})$

(2) $B^a \approx \left\{ (p_1,\dots,p_{k-1}) \in S^{k-1} \;\middle|\; \sum_{i=1}^{k} \rho(p_{i-1},p_i)^2/(t_i-t_{i-1}) \leq a \right\}$.

(3) $r: \mathrm{Int}\,\Omega^c \to B$ defined as in Prop 1

$r_u: \mathrm{Int}\,\Omega^c \to \mathrm{Int}\,\Omega^c$



for $t_{i-1} \leq u \leq t_i$

(4) (1st variation formula): $\frac{1}{2} E_*(W) = -\sum_i \langle W(t_i), \Delta \dot\gamma(t_i)\rangle - \int_0^1 \langle W(t), \nabla_t \dot\gamma(t)\rangle dt$

(5) $TB_\gamma \longleftrightarrow T\Omega_\gamma(t_0,\dots,t_k) := \{ W$ along $\gamma \mid (1)\ W|_{[t_{i-1},t_i]}$ is a Jacobi field along $\gamma|_{[t_{i-1},t_i]}$ $\forall i$ (space of broken Jacobi fields).

$(2)\ W(0) = W(1) = 0 \}$

The rest follows from the Lemma below. □

⟨Lemma⟩ γ as in Prop 1

(1) $T\Omega_\gamma = T\Omega_\gamma(t_0, \cdots, t_k) \oplus T'$ (orthogonal sum)

(2) E_{**} restricted to T' is positive-definite,

where $T' := \{ W \in T\Omega_\gamma \mid W(t_0) = \cdots = W(t_k) = 0 \}$

(p.f.) (1) Given $W \in T\Omega_\gamma$, choose $W_1 \in T\Omega_\gamma(t_0, \cdots, t_k)$ s.t. $W_1(t_i) = W(t_i)$

(This is possible, since $J(\gamma) \longrightarrow TM_{\gamma(t_{i-1})} \times TM_{\gamma(t_i)}$ is an isomorphism by being onto).

$W = W_1 + (W - W_1)$ is the unique decomposition

(2^{nd} variation formula): $\frac{1}{2} E_{**}(W_1, W_2) = -\sum_i \langle W_2(t_i), \Delta(\nabla_{\dot\gamma} W_1(t_i)) \rangle$
$- \int_0^1 \langle W_2(t), \nabla_{\dot\gamma}\nabla_{\dot\gamma} W_1(t) + R(\dot\gamma, W_1)\dot\gamma \rangle dt$

(2) $\geq$: γ is a broken geodesic

$>$: Suppose $W, W' \in T'$ and $E_{**}(W, W) = 0$.

Then $0 \leq E_{**}(W + cW', W + cW') = 2c E_{**}(W, W') + c^2 E_{**}(W', W')$

$\Rightarrow E_{**}(W, W') = 0 \Rightarrow W = 0$ □

⟨Def⟩ X : topological space with $X_0 \subseteq X_1 \subseteq \cdots$

$X_\Sigma := X_0 \times [0,1] \cup X_1 \times [1,2] \cup \cdots \subseteq X \times \mathbb{R}$

• X is the "homotopy direct limit" (HDL) of $\{X_i\}$ if $p : X_\Sigma \longrightarrow X$, $(x, t) \longmapsto x$ is a homotopy equivalence (HE).

⟨Prop 3⟩ Let X, Y be the HDL of $\{X_i\}, \{Y_i\}$ resp.

If $f : X \rightarrow Y$ restricted as $f_i : X_i \rightarrow Y_i$ is a HE $\forall i$, then f is a HE.

(p.f.) By some homotopy arguement! □

(p.f. of Thm) Prop 1 + Prop 2 $\Rightarrow \pi_i(\mathrm{Int}\,\Omega^c, \tilde{\Omega}) = 0 \quad \forall c$

Ω is paracompact $\Rightarrow$ admits partition of unity

Construct $f: \Omega \to \mathbb{R}$ s.t. $f(\gamma) \geq c+1$ for $\gamma \notin \mathrm{Int}\,\Omega^c$ & $f(\gamma) \geq 0 \ \forall \gamma$

$\Omega \approx \Omega_{\xi} \dots$: deformation retract

$\gamma \mapsto (\gamma, f(\gamma))$

$\Rightarrow \Omega$ is a HDL of $\{\mathrm{Int}\,\Omega^c\}$

Prop 3 $\Rightarrow \pi_i(\Omega, \tilde{\Omega}) = 0$. □

<Recall> $\pi_i(\tilde{\Omega}(SU(2m)); I, -I) \approx \pi_i(\Omega(SU(2m))) \approx \pi_{i+1}(SU(2m))$.

· $\Omega^* := \{w : [0,1] \to M \mid$ all continuous paths w from p to $q\}$

· $d^*(w, w') = \max_{t \in [0,1]} \rho(w(t), w'(t))$ induces the compact-open topology $(d^* \leq d)$

<Prop 4> The natural map $i : \Omega \to \Omega^*$ is a HE

§4. The Index of Geodesics

⟨Errata⟩ In proving ⟨Thm of §3⟩, "$\pi_i(\Omega, \tilde{\Omega})=0$" was merely an consequence of "$\pi_i(\mathrm{Int}\,\Omega^c, \tilde{\Omega})=0$ for c arbitrarily large." I was too caught up with all that "Homotopy Direct Limit" stuff, and gave a misleading proof.

⟨Thm of §3⟩ M: complete Riem. mfd. $p, q \in M$.
If $\tilde{\Omega}$ is a top. mfd, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$.
Then $\pi_i(\Omega) \approx \pi_i(\tilde{\Omega})$ for $0 \leq i < \lambda_0 - 1$

⟨Thm⟩ Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.

⟨Lemma of §3⟩ (1) $T\Omega_\gamma = T\Omega_\gamma(t_0, \cdots, t_k) \oplus T'$, where $T' := \{W \in T\Omega_\gamma \mid W(t_i)=0\}$
(2) E_{**} restricted to T' is positive definite.

⟨Prop 1⟩ λ (index of E_{**}) $= \#\{\gamma(t) \mid \gamma(t)$ is conjugate to $\gamma(0)$ along γ, $0<t<1\}$ counted with multiplicity

(p.f.) $\gamma_\tau := \gamma|_{[0,\tau]}$ is a geodesic

$\lambda(\tau) :=$ the index of $(E_0^\tau)_{**}$ associated with γ_τ

(2^{nd} variation formula) $\frac{1}{2}E_{**}(W_1, W_2) = -\sum_i \langle W_2(t_i), \Delta(\nabla_{\dot\gamma} W_1(t_i))\rangle - \int_0^1 \langle W_2(t), \nabla_{\dot\gamma}\nabla_{\dot\gamma} W_1(t) + R(\dot\gamma, W_1)\dot\gamma\rangle dt$

Assertions: (1) "λ is an increasing function of τ": extend by 0

(2) "$\lambda(\tau)=0$ for small τ": minimal geodesics

(3) "$\lambda(\tau)$ is left cont": $(E_0^\tau)_{**}$ is cont on τ within a small nbhd

$\Rightarrow |\lambda(\tau')| \geq |\lambda(\tau)|$ if $|\tau'-\tau|<\varepsilon$. this together w/ (2) implies (3)

(4) "$\lambda(\tau+\varepsilon) = \lambda(\tau)+\nu$, for small $\varepsilon>0$, where $\nu=$ nullity of $(E_0^\tau)_{**}$":

$W_1, \cdots, W_{\lambda(\tau)} \in T\Omega_{\gamma_\tau}$ s.t. $((E_0^\tau)_{**}(W_i, W_j))$ is negative definite

$J_1, \cdots, J_\nu$: L.I. Jacobi fields vanishing at endpoints.

$\because \nabla_{\dot\gamma_\tau} J_k(\tau)$ are L.I.

$\therefore \exists X_1, \cdots, X_\nu$ along $\gamma_{\tau+\varepsilon}$ vanishing at endpts of $\gamma_{\tau+\varepsilon}$

s.t. $\langle \nabla_{\dot\gamma_\tau} J_k(\tau), X_k(\tau) \rangle = I_\nu$

Extend W_i, J_k over $\gamma_{\tau+\varepsilon}$ by 0

Then w.r.t. $W_1, \cdots, W_{\lambda(\tau)}, c^{-1}J_1 - cX_1, \cdots, c^{-1}J_\nu - cX_\nu$

$$\left(E_0^{\tau+\varepsilon}\right)_{**} = \begin{pmatrix} ((E_0^\tau)_{**}(W_i, W_j)) & cA \\ cA^t & -4I + c^2 B \end{pmatrix} \quad \text{for some fixed } A, B$$

□

⟨Prop 2⟩ M: symmetric space, γ: geodesic with $\dot\gamma(0) = V$

Define $K_V: TM_p \to TM_p$ by $K_V(W) = R(V, W)V$, and let $e_1, \cdots, e_n$ be the eigenvalues

Then $\{\text{conj pts of } p \text{ along } \gamma\} = \{\gamma(\pi k/\sqrt{e_i}) \mid k \in \mathbb{Z}, e_i > 0\}$

(p.f.) $\because \langle R(V, W)X, Y\rangle = \langle R(X, Y)V, W\rangle$

$\therefore K_V$ is self-adjoint: $\langle K_V(W), X\rangle = \langle W, K_V(X)\rangle$

i.e. $\exists \{U_1, \cdots, U_n\}$: ONB of $TM_{\gamma(0)}$ s.t. $K_V(U_i) = e_i U_i$

Extend U_i along γ by parallel translation so that $R(V, U_i)V = e_i U_i$

along γ since M is symmetric

Therefore, "$\nabla_{\dot\gamma}\nabla_{\dot\gamma} W + K_V(W) = 0$" becomes "$\sum \frac{d^2 W_i}{dt^2} U_i + \sum e_i W_i U_i = 0$",

and by $\{U_i\}$ being ONB along γ, "$\frac{d^2 W_i}{dt} + e_i W_i = 0$"

Note that we want solutions that vanish at $t = 0$.

(i) $e_i > 0$: $W_i(t) = c_i \sin(\sqrt{e_i}\, t) \Rightarrow t = \pi/\sqrt{e_i}$ yields conjugate pts.

(ii) $e_i = 0$: $W_i(t) = c_i t \Rightarrow$ no non-zero solution.

(iii) $e_i < 0$: $W_i(t) = c_i \sinh(\sqrt{|e_i|}\, t) \Rightarrow$ no non-zero solution. □

(p.f. of Thm) Recall that if $\gamma(t) = \exp(tX)$, $0 \le t \le 1$, then $X = i\pi \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_{2m} \end{pmatrix}$

w/ $\sum k_j = 0$, k_j: odd

$K_X: su(2m) \to su(2m)$ above in Prop 2 is in fact given by:

$K_X(Y) = \frac{1}{4}[[X, Y], X] = \frac{1}{4}[i\pi(k_i - k_j) y_{ij}, X]$, $Y = (y_{ij})$

$= \frac{\pi^2}{4}(k_i - k_j)^2 y_{ij}$.

We find that the following eigenvectors of K_x form a basis of $su(2m)$:

(1) $E_{ij} = \begin{array}{cc} & \\ \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & i \\ j & \end{array}$ $i<j$ corr. to $\frac{\pi^2}{4}(k_i-k_j)^2$

(2) $E'_{ij} = \begin{array}{cc} & \\ \begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix} & i \\ j & \end{array}$ $i<j$ corr. to $\frac{\pi^2}{4}(k_i-k_j)^2$

(3). diagonal matrices corr to 0

$\Rightarrow t = \frac{2}{k_i-k_j}, \frac{4}{k_i-k_j}, \cdots$ gives rise to conjugate pts.

There are $\left(\frac{k_i-k_j}{2}\right)-1$ such values within $(0,1)$, each w/ multiplicity 2.

Prop 1 + Prop 2 $\Rightarrow \lambda = \sum_{k_i>k_j}(k_i-k_j-2)$, which may be divided into the following cases:

(1) at least $(m+1)$ of the k_i's are <0:

then at least one of the positive k_i must be ≥ 3.

$\Rightarrow \lambda \geq \sum_1^{m+1}(3-(-1)-2) = 2m+2$

(2) exactly m of the k_i's are <0:

then by being a non-minimal geodesic, at least one of the k_i's is ≥ 3 with another being ≤ -3

$\Rightarrow \lambda \geq \sum_1^{m-1}((3-(-1)-2)) + \sum_1^{m-1}(1-(-3)-2) + (3-(-3)-2) \geq 2m+2$ $\square$

Bott Periodicity Theorem – Orthogonal Groups

⟨Thm⟩ $\pi_i O \cong \pi_{i+8} O$ for $i \geq 0$.

$i \pmod 8$	$\pi_i O$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	0
3	$\mathbb{Z}$
4	0
5	0
6	0
7	$\mathbb{Z}$

⟨Rmk⟩ $\pi_i(O(n))$ stabilizes since we have $O(n) \to O(n+1) \to S^n$

⟨Def⟩ n: even

· A "cpx structure" J on $\mathbb{R}^n$ is a linear transf. $J: \mathbb{R}^n \to \mathbb{R}^n$ belonging to $O(n)$.
satisfying $J^2 = -I$

· $\Omega_1(n) := \{$all cpx structures on $\mathbb{R}^n\}$

⟨Lemma 1⟩ $\tilde{\Omega}(O(n); I, -I) \cong \Omega_1(n)$

⟨Lemma 2⟩ Any non-minimal geodesic from I to $-I$ in $O(2m)$ has index $\geq 2m-2$.

⟨Prop 1⟩ $\pi_i \Omega_1(n) \cong \pi_{i+1} O(n)$ for $i \leq n-4$.

⟨Def⟩ n: divisible by a high power of 2.

$J_1, \cdots, J_{k-1}$: fixed cpx structures on $\mathbb{R}^n$ s.t. $J_r J_s + J_s J_r = 0$ for $r \neq s$ (anti-commute)

· $\Omega_k(n) := \{$all cpx structures J on $\mathbb{R}^n \mid J$ anti-commutes w/ $J_1, \cdots, J_{k-1}\}$

· $\Omega_0(n) := O(n)$.

⟨Lemma 3⟩ Each $\Omega_k(n)$ is a smooth, totally geodesic submfd of $O(n)$.

(p.f.) Assertions : (1) $J \exp A$ is a cpx structure iff A anti-commutes w/ J

(2) $J \exp A$ anti-comm. w/ $J_1, \cdots, J_{k-1}$ iff A comm. w/ $J_1 \cdots J_{k-1}$

⟨Lemma 4⟩ $\tilde{\Omega}(\Omega_\ell(n); J_\ell, -J_\ell) \cong \Omega_{\ell+1}(n)$ for $0 \leq \ell < k$.

⟨Prop 2⟩ For each $k \geq 0$, $\exists\ g_k$: real-valued function s.t.

(1) any non-minimal geodesic from J to $-J$ in $\Omega_k(n)$ has index $\geq g_k(n)$

(2) $g_k(n) \to \infty$ as $n \to \infty$

⟨Cor⟩ $\pi_i \Omega_k(n) \cong \pi_{i-1} \Omega_{k+1}(n)$ for sufficiently large n

⟨Def⟩ $\Omega_k(n) \hookrightarrow \Omega_k(n+n')$, choose fixed cpx structures $J_1', \cdots, J_k'$ on $\mathbb{R}^{n'}$

$J \mapsto J \oplus J_k'$

⟨Prop 3⟩ $\pi_h \Omega_0 \cong \pi_{h-1} \Omega_1 \cong \cdots \cong \pi_1 \Omega_{h-1}$