Nash isometric imbedding theorem

Def: Given (V^n, g) and (W^q, h) two Riemannian manifolds. A C^1 map $f: V^n \to W^q$ is called <u>isometric</u> if $f^*h = g$, i.e. $d_xf: T_xV \to f_*(T_xV) \subseteq T_{f(x)}W$ is a linear isometry for all $x \in V$.

- An isometric map is already an immersion.
- Locally with frame $\{\partial_i\}_{i=1,\dots,n}$, the isometric condition is $\langle f_*\partial_i, f_*\partial_j\rangle_h = \langle \partial_i, \partial_j\rangle_g$ $(1 \le i \le j \le n)$.

Def: A C^1-map $f: V \to W$ is called <u>short</u> (resp. <u>strictly short</u>)

$\|f_* v\|_h < \|v\|_g$ for all $v \in TV$, denoted by $f^*h < g$ (resp. $\le$).

<u>Theorem</u> (Nash – Kuiper)

If $n < q$, then any strictly short immersion $f: (V^n, g) \to (\mathbb{R}^q, h)$ can be C^0-approximated by isometric C^1-smooth immersions. Moreover, if f is an embedding, then f can be C^0-approximated by isometric C^1-embeddings.

- For a closed C^∞-mfd V^n, by Whitney embedding theorem, V^n can be embedded onto $\mathbb{R}^{2n}$ (or $\mathbb{R}^{2n+1}$). Simply by changing the scale, such an embedding will be short.

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§1. Jet spaces, partially differential relations and h-principle.

- <u>Coordinate definition of Jets</u>

Let $\mathcal{H}_r(n,q)$ denote the real vector space of homogeneous polynomial maps of multidegree r from $\mathbb{R}^n$ to $\mathbb{R}^q$.

Let $f \in C^r(U, W)$; where $U \subseteq_{open} \mathbb{R}^n$, $W \subseteq_{open} \mathbb{R}^q$.

Then $D^r f \in C^0(U, \mathcal{L}^r_{sym}(\mathbb{R}^n, \mathbb{R}^q))$.

Actually, we may consider $D^r f \in C^0(U, \mathcal{H}_r(n,q))$:

For $(x,h) \in U \times \mathbb{R}^n$ and v a basis of $\mathbb{R}^n$.

$$D^r f(x)(h, \dots, h) = (h_1\partial_1 + \dots + h_n\partial_n)^r f(x) = \sum_{|\alpha|=r} \frac{r!}{\alpha!} h^\alpha \partial_v^\alpha f(x) \in \mathcal{H}_r(n,q).$$

(Here, $h = (h_1, \dots, h_n)$ in a basis v, $\partial_v^\alpha = \partial_1^{p_1} \circ \dots \circ \partial_n^{p_n}$ with $p_1 + \dots + p_n = \alpha$)

$$\text{So } \frac{1}{r!} D^r f(x) = (\partial_v^\alpha f(x))_{|\alpha|=r} = \left(\frac{\partial^r f}{\partial u_{i_1} \cdots \partial u_{i_r}}(x) \;\middle|\; 1 \le i_1 \le \dots \le i_r \le n\right)$$

Now we define $J^r(U, W) = U \times W \times \prod_{s=1}^{r} \mathcal{H}_s(n,q)$, which is called the <u>space of r-jets</u> of germs of C^r-maps from U to W.

Note that $\dim \mathcal{H}_s(n,q) = q \cdot \dim \mathcal{H}_s(n,1) = q \cdot \binom{n+r-1}{r} =: q\, d_r$

So $J^r(U,W) \cong U \times W \times \mathbb{R}^{q d_1} \times \cdots \times \mathbb{R}^{q d_r}$.

- There is a continuous map $\pi_r : C^r(U,W) \longrightarrow C^0(U; J^r(U,W))$, here
$$f \longmapsto J^r_f$$
$J^r_f(x) := (x, f(x), \frac{1}{2!}D^2 f(x), \ldots, \frac{1}{r!}D^r f(x))$. Such $J^r_f(x)$ is called the r-jet of f at x.

- <u>Invariant definition of Jets</u>.

In general, we want to define the jet space for arbitrary fibration $p : X \to V$ with fiber F, $\dim F = q$, $\dim V = n$.

Let f, g be C^r sections of the fibration. We define an equivalent relation by $f \sim g \iff \forall v \in V$, $\exists$ an open nbd of v st. [$f(v) = g(v)$ and $J^r_{\varphi_* f}(\varphi(v)) = J^r_{\varphi_* g}(\varphi(v))$ for local trivialization $\varphi : \underset{f(v)=g(v)}{\underset{\Downarrow}{U}} \to \mathbb{R}^n \times \mathbb{R}^q$]

- The jet space $X^{(r)} :=$ the equivalent classes $\langle f \rangle_r(x)$ ($:= J^r_f(x)$) ($x \in V$).
- We have natural maps: $S : X^{(r)} \longrightarrow V$; $\tau : X^{(r)} \longrightarrow X$
$$\langle f \rangle_r(x) \longmapsto x \qquad \langle f \rangle_r(x) \longmapsto f(x)$$
- $X^{(r)}$ is naturally topologized as a manifold with charts $J^r(U,W)$ with $U \underset{open}{\subseteq} \mathbb{R}^n$, $W \underset{open}{\subseteq} \mathbb{R}^q$, where $\langle f \rangle_r(x) \in X^{(r)}$ is represented as $J^r_{f|_U}(x) \in J^r(U,W)$.

Def: • A partially differential relation on sections $f : V \to X$ is a subset $\mathcal{R} \subseteq X^{(r)}$, where r is called the order of r.

- A C^r-section $f : V \to X$ is said to be a (genuine) solution of $\mathcal{R}$ if $J^r_f : V \to X^{(r)}$ maps V into $\mathcal{R}$. Sometimes we may also call sections J^r_f a (generaine) solution of $\mathcal{R}$.
- A section $F : V \to X^{(r)}$ is called a formal solution of $\mathcal{R}$ if $F(V) \subseteq \mathcal{R}$.

- <u>Homotopy principle</u>: A partially differential relation $\mathcal{R}$ satisfies the h-principle if every formal solution of $\mathcal{R}$ is homotopic in $\underset{\{\text{sections from } V \text{ to } \mathcal{R}\}}{\underset{\shortparallel}{\mathrm{Sec}\,\mathcal{R}}}$ to a genuine solution of $\mathcal{R}$.

§2. One dimension convex integration

- Given a differential relation $\mathcal{R} \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$, we can think of it as a differential inclusion $\dot{y} \in \Omega(t,y)$ for $t \in \mathbb{R}$, $y \in \mathbb{R}^q$, where $\Omega(t,y) := \mathcal{R} \cap \mathcal{F}_{t,y}$ with $\underset{\substack{\cong \\ \mathbb{R}^q}}{\mathcal{F}_{t,y}}$ the fiber of the projection $J^1(\mathbb{R},\mathbb{R}^q) \longrightarrow J^0(\mathbb{R},\mathbb{R}^q)$.

- For a section $\mathcal{F} = (f, \varphi) : \mathbb{R} \longrightarrow \mathcal{R} \subseteq J^1(\mathbb{R},\mathbb{R}^q)$, we define

  $\mathrm{Conv}_{\mathcal{F}(t)} \mathcal{R}$: the convex hull of the path connected component of $\Omega(t, f(t))$ that contains $\mathcal{F}(t)$ (in the fiber $\mathcal{F}_{t,f(t)}$).

  $$\mathrm{Conv}_{\mathcal{F}} \mathcal{R} := \bigcup_{t \in \mathbb{R}} \mathrm{Conv}_{\mathcal{F}(t)} \mathcal{R}$$

<u>Def</u>: A formal solution $\mathcal{F} = (f, \varphi)$ of $\mathcal{R} \subseteq J^1(\mathbb{R},\mathbb{R}^q)$ is called <u>short</u> if $df(\mathbb{R}) \subseteq \mathrm{Conv}_{\mathcal{F}} \mathcal{R}$.

- <u>Key lemma</u>: (One dimensional convex integration)

Let $\underset{\text{open}}{\mathcal{R} \subseteq} J^1(\mathbb{R},\mathbb{R}^q)$ be an open differential relation and $\mathcal{F} = (f,\varphi) : \overset{[0,1]}{\overset{\shortparallel}{I}} \longrightarrow \mathcal{R}$ be a short formal solution of $\mathcal{R}$. Then there exists a continuous family of short formal solutions $\mathcal{F}_\tau = (f_\tau, \varphi_\tau) : I \longrightarrow \mathcal{R}$, $\tau \in [0,1]$ s.t.

(0) $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1$ is a genuine solution of $\mathcal{R}$

(1) f_τ is (arbitrarily) C^0-close to f for all $\tau \in [0,1]$.

(2) $\mathcal{F}_\tau(0) = \mathcal{F}(0)$ and $\mathcal{F}_\tau(1) = \mathcal{F}(1)$ for all $\tau \in [0,1]$.

pf.) <u>Step 1</u>: (Reduction)

First we know $\mathcal{R} \sim \{\dot{y} \in \Omega(t,y)\} \xleftrightarrow{z = y - f(t)} \tilde{\mathcal{R}} \sim \{\dot{z} \in \tilde{\Omega}(t,z) = \Omega(t, z + f(t)) - \dot{f}(t)\}$ with initial $\underset{\text{short}}{\wedge}$ formal solution $(0, \varphi - \dot{f})$.

To reduce further, we define an abstract flower $S := \dfrac{\coprod_{i=0}^{n} [0,1]_i}{\{0\}}$, which means we paste n intervals at one point 0, denoted by 0_S.

Such $I_0 = [0,1]_0$ is called the stem of the flower and $I_i = [0,1]_i$ $(i \geq 1)$ are called the petals of the flower. We say the image of $\psi : S \longrightarrow \mathbb{R}^q$ a flower and write $\psi_i : I \longrightarrow \mathbb{R}^q$, $\psi_i(0) = \psi(0_S)$ to be the restriction of ψ on $[0,1]_i$.

With this definition, we have the following reduction:

It suffices to prove when $\mathcal{R} = \mathbb{R} \times \underset{\substack{\uparrow \\ \text{closed } \varepsilon\text{-ball}}}{B^q_\varepsilon} \times \underset{\substack{\uparrow \\ \text{a flower}}}{\Psi}$ with $0 \in \mathrm{Int}\,\mathrm{Conv}(\{\psi_i(1)\}_{1 \leq i \leq n})$.

and $\mathcal{F} = (0, \varphi) : I \longrightarrow \mathcal{R}$ with $\varphi := \psi_0$.

(pf. of reduction) We claim that for $\mathcal{R} \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ and $\mathcal{F} = (0, \varphi) : I \to \mathcal{R}$ short formal solution, there exists a number $\delta > 0$ s.t. for any $t_0 \in [0, 1-\delta]$ we can choose a flower $\Psi = \Psi(t_0) \subseteq \mathcal{F}_{t_0, 0}$ s.t.
(0) $0 \in \mathrm{Int\,conv}(\{\psi_i(1)\}_{i=1 \cdots n})$
(1) $\psi_0(t) = \varphi(t_0 + \delta t)$
(2) $[t_0, t_0+\delta] \times B^q_\varepsilon \times \Psi \subseteq \mathcal{R}$ for small ε.

With this claim, by choosing an appropriate subdivision of the interval I our reduction holds. To prove the claim, let $t_0 \in I$. We can pick a finite set of points in the connected component of $\Omega(t_0, 0)$ containing $\varphi(t_0)$ such that 0 belongs to the interior of the convex hull of these points. Now connect $\varphi(t_0)$ with the chosen points by some paths. These paths are petals the flower $\Psi = \Psi(t_0)$ while the stem is the paths $\psi_0(t) = \varphi(t_0 + \delta t)$. This has (0), (1). And (2) just follows from the openess of $\mathcal{R}$ and the compactness of I.

Step 2: (Construction of smooth genuine solutions).

We define the weighted product of $p_i : I \to \mathbb{R}^q$ in the weight of $(\alpha_1, \dots, \alpha_k)$ $(\alpha_1 + \cdots + \alpha_k = 1, \alpha_i > 0)$ by $p = p_1 \cdot \cdots \cdot p_k : I \to \mathbb{R}^q$ sending $t \in I$ to $p(t) = p_i\left(\frac{t - t_{i-1}}{\alpha_i}\right)$, $t \in (t_{i-1}, t_i]$, $p(0) = p_1(0)$, $t_i = \sum_{j=1}^{i} \alpha_j$ and $t_0 = 0$.

If $\alpha_i = \frac{1}{k}$ for all i, we call such product a uniform product.

For any $p : I \to \mathbb{R}^q$, we may also define $\int p(\sigma)\, d\sigma : I \to \mathbb{R}^q$
$$t \longmapsto \int_0^t p(\sigma)\, d\sigma.$$

Then we have the following property:

If $\int_0^1 p_i(\sigma)\, d\sigma = 0$ for $1 \le i \le k$, then $\int p_1 \cdot \cdots \cdot p_k(\sigma)\, d\sigma = \frac{1}{k} \int p_1(\sigma) d\sigma \cdot \cdots \cdot \int p_k(\sigma)\, d\sigma$

and then $\| \int p_1 \cdot \cdots \cdot p_k(\sigma)\, d\sigma \|_{C^0} = \frac{1}{k} \max\{ \| \int p_i(\sigma)\, d\sigma \|_{C^0} \mid 1 \le i \le k \}$

Now if $\psi = \{\psi_0, \psi_1, \dots, \psi_k\}$ is the parametrizing map for the flower Ψ.
(To ensure the smoothness of the construction of maps, we assume $\psi_i(t) = \varphi(0)$ near $t=0$ and $\psi_i(t) = \psi_i(1)$ near $t=1$.) Write $\psi_i(1) = a_i$ for $1 \le i \le k$.

Consider the product $\psi := \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \cdots \cdot \psi_k \cdot a_k \cdot \psi_k^{-1}$, where the weights of constant paths a_i are $(1-\rho)\alpha_i$ and others are $\frac{\rho}{2k}$, where α_i is s.t. $\alpha_1 a_1 + \cdots + \alpha_n a_n = 0$ ($\because$ $0 \in \mathrm{Int\,conv}(\{\psi_i(1)\})$).

Now we want to adjust the weights so that $\int_0^1 \psi(t)\, dt = 0$.

To do this, note that if we weight $a_1 \cdot a_2 \cdot \cdots \cdot a_k$ by $\alpha_1 \cdots \alpha_k$, then $\int_0^1 a_1 \cdot \cdots \cdot a_k(t)\, dt = 0$.

Hence, $d = \int_0^1 \psi(t)\,dt = \int_0^1 \psi(t)\,dt - \int_0^1 a_1 \cdot \cdots \cdot a_k(t)\,dt \in \mathbb{R}^q$.

Then $\|d\| \le \left(\max_{t \in I} \|\psi(t)\|\right) \rho$.

For sufficiently small ρ, we have $d \in \operatorname{Int}\{(1-\rho)\Delta\}$ with $\Delta = \operatorname{conv}\{a_1, \dots, a_k\}$.

Hence $\exists$ $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k$ s.t. $-d = \tilde{\alpha}_1(1-\rho)a_1 + \cdots + \tilde{\alpha}_k(1-\rho)a_k$.

Therefore assign the new weights $\tilde{\alpha}_1(1-\rho), \dots, \tilde{\alpha}_k(1-\rho)$ to the constant paths a_i, then $\int_0^1 \psi(t)\,dt = 0$.

Similarly, we can adjust the weights of $\tilde{\psi} = \psi_1 \cdot a_1 \cdot \psi_1^{-1} \cdot \cdots \cdot \psi_k \cdot a_k \cdot \psi_k^{-1} \cdot \varphi$ s.t. $\int_0^1 \tilde{\psi}(t)\,dt = 0$.

Now let $\varphi_1 = \underbrace{\psi \cdot \psi \cdot \ldots \cdot \psi}_{N-1} \cdot \tilde{\psi}$ to be the uniform product of N factors.

And let $f_1(t) = \int_0^t \varphi_1(\sigma)\,d\sigma$. Then we have $\|f_1\|_{C^0} = \frac{1}{N} \max\left\{ \left\|\int \psi(\sigma)\,d\sigma\right\|_{C^0}, \left\|\int \tilde{\psi}(\sigma)\,d\sigma\right\|_{C^0} \right\}$

So for N large, $F_1 = (f_1, \varphi_1)$ is a genuine solution of $\mathcal{R}$ and F_1 satisfies the boundary conditions.

Now construction of (f_2, φ_2) is easy: $f_2(t) = \tau f_1(t)$ and φ_2 can be easily constructed from the fact that $\psi_i \cdot \psi_i^{-1} \sim \varphi(0)$. This proves the key lemma.

Reference : 1. Introduction to the h-principle, Y. Eliashberg and N. Mishachev

2. Convex Integration theory, D. Spring

3. Partial Differential Relation, M. Gromov.

§3 One-dimensional convex integration

Motivation of convex integration:

Eg. Consider a path $\gamma : I = [0,1] \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$. It can be shown that any path of length l can be uniformly approximated by a path of length L if $l < L$ (fixing $\gamma(0), \gamma(1)$). Hence, the solutions to $\dot{x}^2 + \dot{y}^2 < 1$ can be C^0-approximated by those to $\dot{x}^2 + \dot{y}^2 = 1$. (i.e. the world line of a particle can be C^0-approximated by that of photon) This says that the space of solutions $I \to \mathbb{R}^2$ of $\dot{x}^2 + \dot{y}^2 = 1$ is C^0-dense to the space of solutions to $\dot{x}^2 + \dot{y}^2 < 1$.

This motivates us to consider the "fiberwise convex hull" of the original relation.

<u>Recall</u>: A formal solution $F = (f, \varphi)$ of $\mathcal{R} \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ is short if $df(\mathbb{R}) \subseteq \operatorname{Conv}_F(\mathcal{R})$

We have proved last time the following lemma:

<u>Lemma</u>: (One dimensional convex integration):

Let $\mathcal{R} \subseteq J^1(\mathbb{R}, \mathbb{R}^q)$ be an open differential relation and $F = (f, \varphi) : I \to \mathcal{R}$ be a short formal solution of $\mathcal{R}$. Then there exists a conti. family of short formal solutions $F_\tau = (f_\tau, \varphi_\tau) : I \to \mathcal{R}$, $\tau \in [0,1]$ st.

(1) $F_0 = F$; F_1 = genuine solution; $F_\tau(0) = F(0)$, $F_\tau(1) = F(1)$ for all $\tau \in [0,1]$.

(2) f_τ is (arbitrarily) C^0-close to f

We will start to consider a parametric family of such one dimensional open relation: for any fiber bundle $p : X \to V$, we may consider parametric sections:

$$f : P \times V \longrightarrow P \times X, \quad (p, v) \longmapsto (p, f_p(v))$$

, where P is the space of parameters and provided that f is conti. wrt. $p \in P$. Then we can define "fibered differential relation" $\mathcal{R} \subseteq P \times X^{(1)}$ and genuine/formal solution to $\mathcal{R}$ similarly.

Now let $P = I^l$ be a cube.

<u>Theorem</u> (Parametric one-dimensional convex integration)

Let $\mathcal{R} \subseteq I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ be an open fibered differential relation and $F = F(p,t) = (f(p,t), \varphi(p,t)) : I^l \times I \to \mathcal{R}$ be st. for all $p \in I^l$, the section

$F(p,t) : \{p\} \times I \longrightarrow \mathcal{R}_p := \mathcal{R} \cap (\{p\} \times J^1(\mathbb{R}, \mathbb{R}^q))$ is a short formal solution of $\mathcal{R}_p$,

(briefly, we will call it "fiberwise short formal solution".)

Suppose $f(p,t)$ smoothly depends in p and when $p \in Op(\partial I^l)$. $f(p,t)$ is a genuine solution of R_p.

Then there exists a homotopy of fiberwise short formal solutions

$F_\tau = F_\tau(p,t) = (f_\tau(p,t), \varphi_\tau(p,t)) = I^l \times I \longrightarrow R$, $\tau \in [0,1]$ s.t.

1) $F_0 = F$; F_1 = genuine solution of R; $F_\tau(p,0) = F(p,0)$, $F_\tau(p,1) = F(p,1)$ for all $p \in I^l$

2) f_τ is arbitrarily C^0-close to f

3) F_τ is a constant for $p \in Op(\partial I^l)$.

(4) $\partial_p f_1(p,t)$ is arbitrarily C^0-close to $\partial_p f(p,t)$

pf.) It is really similar to the non-parametric case.

A) (Reduction). As before, we may assume $f \equiv 0$. To reduce further, we define a "fibered flower": $\Psi: I^l \times S \longrightarrow I^l \times \mathbb{R}^q$, $\Psi = \Psi(I^l \times S)$

$\Psi_p := \Psi(\{p\} \times S) \subseteq \{p\} \times \mathbb{R}^q$. ($S$: abstract flower)

Claim: We only need to prove the case when

- $R_p = \{p\} \times \mathbb{R} \times \overline{B^q_\varepsilon(0)} \times \Psi_p \subseteq \{p\} \times J^1(\mathbb{R}, \mathbb{R}^q)$ for all $p \in I^l$ for a fibered flower Ψ s.t. $0 \in \mathrm{Int\,Conv}(\partial \Psi_p)$ for each $p \in I^l$.
- $F = (0, \varphi) = I^l \times I \longrightarrow R$ with $\varphi \equiv \psi_0$

Actually, the claim follows from the sublemma below:

Sublemma: Let $R \underset{\text{open}}{\subseteq} I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ and $F = (0,\varphi) : I^l \times I \to R$ be fiberwise short formal solution of R. Then there exists $\delta > 0$ s.t. $\forall\, t_0 \in [0, 1-\delta]$ we can chose a fibered flower $\Psi = \Psi(t_0) \subset I^l \times F_{t_0,0}$ s.t. for each $p \in I^l$

(1) $0 \in \mathrm{Int\,Conv}(\partial \Psi_p)$

(2) $\psi_0(p,t) = \varphi(p, t_0 + \delta t)$, $t \in I$

(3) $\{p\} \times [t_0, t_0+\delta] \times \overline{B^q_\varepsilon(0)} \times \Psi_p \subseteq R_p$ for sufficiently small $\varepsilon > 0$.

pf.) Let $t_0 \in I$. Let $\psi_0 = (p,t) \longmapsto \varphi(p, t_0 + \delta t)$ with δ chosen later.

For all $p_0 \in I^l$, as before, we can choose a flower Ψ_{p_0} as in the proof of Lemma and using the openess of R extend Ψ_{p_0} over a neighborhood U of $p_0 \in I^l$ s.t. for all $p \in U$, $0 \in \mathrm{Int\,Conv}(\partial \Psi_p)$ and $\psi_i(p,t)$ are paths in $\mathrm{Conn}_{F(p,t_0)} R$.

Now chose a finite covering of I^l by U_j, $j = 1, \dots, L$, and denote the corresponding flower by Ψ^{U_j}, which consists of petals $\psi_i^{U_j} = U_j \times I \longrightarrow R$, $i = 1, \dots, N_j$, $j = 1, \dots, L$.

May chosen $U_j' \subseteq U_j$ (slightly smaller open sets) s.t. $\overline{U_j'} \subseteq U_j$ and $\bigcup_j U_j' \supseteq I^l$.

Chose cut off function $\beta^j : I^l \longrightarrow [0,1]$ s.t. $\beta^j|_{U_j'} = 1$, $\beta^j|_{I^l \setminus U_j} = 0$.

For $i = 1, \dots, N_j$, set $\psi_i^j(p,t) = \psi_i^{U_j}(p, \beta^j(p) t)$ for $p \in U_j$ and $\psi_i^j(p,t) = \varphi(p, t_0)$ for $p \in I^l \setminus U_j$.

Then the fibred flower with stem $\varphi_0(p,t) = \ell(p, t_0 + t\delta)$ and petals φ_i^j $\begin{matrix} j=1,\dots,L \\ i=1,\dots,N \end{matrix}$ satisfies 1) and 2). Using openness of $\mathcal{R}$ and compactness of I, δ can be chosen. $\#$

(B) (Convex Decomposition of a Section)

Let Ψ be a fibred flower. Set $a_i(p) = \psi_i(p,1)$, $i=1,\dots,N$, $p \in I^l$.

For any smooth section $d : p \in I^l \longmapsto d(p) \in \operatorname{int} \operatorname{Conv} \partial \Psi_p$. There exists smooth functions $\alpha_i : I^l \longrightarrow [0,1]$, $i=1,\dots,N$ st. $\alpha_1(p) + \dots + \alpha_N(p) = 1$. $\alpha_1 a_1 + \dots + \alpha_N a_N = d$.

Pf.) We can do it locally at any open neighborhood of p. Then it is just a partition of unity argument.

(C) (Construction of homotopy F_τ)

Apply the proof of the Main lemma parametrically. Consider "0 and $d(p)$" instead of "0 and d" in the previous proof with weights $\alpha_i(p)$, $\lambda_i(p)$ that depends on p.

Then the functions $\varphi_\tau(p,t)$ can be constructed to satisfy 1) and 2).

As for 3), it suffices to set $F_\tau = F_{\beta(p)\tau}$, where $\beta : I^l \to [0,1]$ is 0-valued near ∂I^l and equal to 1 on a slightly smaller cube $I_1^l \subseteq I^l$, so that $F_0(p,t) = F(p,t)$ is a genuine solution of R_p for all $p \in I^l \setminus I_1^l$.

Now for (4), For $F = (0, \varphi)$, we want to show that f_1 can be chosen st. $\partial_p f_1(p,t)$ is arbitrary close to 0.

Take the uniform product $\varphi_1(p,\cdot) = \psi(p) \cdot \dots \cdot \psi(p) \cdot \tilde{\Psi}(p)$ of N factors and set $f_1(p,t) = \int_0^t \varphi_1(p,\sigma)\, d\sigma$. Then $\partial_p f_1(p,t) = \int_0^t \partial_p \varphi_1(p,\sigma)\, d\sigma$, where

$\partial_p \varphi_1(p,\cdot) = \partial_p \psi(p) \cdot \dots \cdot \partial_p \psi(p) \cdot \partial_p \tilde{\Psi}(p)$.

Since $\int_0^1 \partial_p \psi(p,\sigma)\, d\sigma = \partial_p \int_0^1 \psi(p,\sigma)\, d\sigma \equiv 0$. Hence by the previous property, we have $\| \partial_p f_1 \|_{C^0} = \frac{1}{N} \max\{ \| \partial_p g \|_{C^0}, \| \partial_p h \|_{C^0} \}$ with $\begin{cases} g(p,t) = \int_0^t \psi(p,\sigma)\, d\sigma \\ h(p,t) = \int_0^t \tilde{\Psi}(p,\sigma)\, d\sigma \end{cases}$

Then for N large, $\| \partial_p f_1 \|_{C^0}$ is arbitrarily small. $\#$

Now we will use these two 1-D convex integration lemma to complete an 'approximation step' in the proof of C^1-isometric immersion.

Let (V^n, g), $(\mathbb{R}^q, h)$ be Riemannian.
standard metric

We will consider the bundle $p : V^n \times \mathbb{R}^q \longrightarrow V^n$ and the isometric relation $\mathcal{R}_{iso}$ defined by $f^* h = g$ for sections $f : V^n \longrightarrow (V^n \times) \mathbb{R}^q$.

• Given a pair of metrics g and $\tilde g$ on V ($\tilde g$ can be semi-Riemannian), we define

$r(\tilde g, g) : TV \setminus V \longrightarrow \mathbb{R}$

$v \longmapsto r(\tilde g, g)(v) = \dfrac{\|v\|_{\tilde g}}{\|v\|_g}$ and

given a pair of maps $f, \tilde f : (V, g) \longrightarrow \mathbb{R}^q$ we define

$d_g(f, \tilde f) : TV \setminus V \longrightarrow \mathbb{R}$

$v \longmapsto d_g(f, \tilde f)(v) = \dfrac{\|f_* v - \tilde f_* v\|_h}{\|v\|_g}$.

Then we have (r1) If $g_1 \geq g_2$, then $r(\tilde g, g_1) \leq r(\tilde g, g_2)$

(r2) $r(\tilde g, g) \leq r(\tilde g + g_1, g + g_1)$ if $r(\tilde g, g) < 1$

(d1) If $g_1 \geq g_2$, $d_{g_1}(f, \tilde f) \leq d_{g_2}(f, \tilde f)$

Note $r(g, \tilde g)$ and $d_g(\tilde f, f)$ will not depend on the length of v.

• Def: An embedding $f : (V, g) \longrightarrow (\mathbb{R}^q, h)$ is ε-isometric if $(1-\varepsilon) g < f^* h < (1+\varepsilon) g$

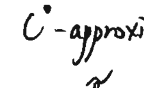
Lemma: $\forall \varepsilon > 0$ any strictly short embedding $f : (I, g) \longrightarrow (\mathbb{R}^q, h)$ can be C^0-approximated by ε-isometric embeddings. Moreover, for all $\rho > 0$, the approximating map $\tilde f$ can be chosen to be s.t. $d_g(f, \tilde f) < r(g - f^* h, g) + \rho$.

Pf. Let $f : (I, g) \longrightarrow (\mathbb{R}^q, h)$ be a strictly short embedding.

Let τ be an orienting g-unit vector field on I, i.e. $\partial_\tau t > 0$, $\|\tau\|_g = 1$.

Then $R_{iso} \subseteq J^1(I, \mathbb{R}^q) = I \times \mathbb{R}^q \times \mathbb{R}^q$ over a point $(t, y) \in I \times \mathbb{R}^q$ is the unit sphere $\Omega(t, y) = \{ w \in \mathbb{R}^q \mid \|w\|_h = 1 \}$.

Choose a normal vector field n to $f(I) \subseteq \mathbb{R}^q$. Instead of the relation R_{iso} over $I \times \mathbb{R}^q$, we can consider $R_f \subseteq R_{iso}$ over $f(I) \subseteq I \times \mathbb{R}^q$, which consists of vectors $w \in \Omega(t, y)$ s.t. $w \in \mathrm{Span}\{ f_* \tau, n \}$ and $\langle w, f_* \tau \rangle_h \geq \|f_* \tau\|_h^2$.



Now $\left(f, \dfrac{f_* \tau}{\|f_* \tau\|_h}\right)$ is a short formal solution of R_f.

Let $\tilde R_f \subseteq J^1(I, \mathbb{R}^q)$ be a small open neighborhood of $R_f \subseteq J^1(I, \mathbb{R}^q)$

By one-dimensional convex integration lemma, $\exists\, \tilde f$ solution of $\tilde R_f$ which is C^0-close to f.

If $\tilde f$ is sufficiently C^0-close to f, then $\tilde f$ is also an embedding.

(the angle between $f_* \tau$ and $\tilde f_* \tau$ $< \frac{\pi}{2}$ - a constant). $r(\Delta, g)(\tau) + \rho$, where $\rho \to 0$ as $\tilde R_f \to R_f$

$\forall \rho > 0$, $d_g(f, \tilde f)(\tau) = \|\tilde f_* \tau - f_* \tau\|_h < \sqrt{1 - \|f_* \tau\|_h} + \rho = \sqrt{(g - f^* h)(\tau)} + \rho = \|\tau\|_{g - f^* h} + \rho$ #

§3. Proof of Nash–Kuiper Theorem

Theorem (Nash–Kuiper) If $n<q$ then any strictly short embedding $f:(V^n,g)\longrightarrow(\mathbb{R}^q,\underset{\text{standard metric}}{h})$ can be C^0-approximated by isometric C^1-embeddings.

Pf.) (A) Decomposition of a metric :

- A positive quadratic form $\underset{\text{on } \mathbb{R}^n}{Q}$ can be written as $\sum_{i=1}^{n} l_i^2$, when l_i are linear.

  (Write $Q(x)=x^TAx$. A - positive definite $\Rightarrow$ $A=B^TB$ for some invertible B
  $\Rightarrow Q(x)=(Bx)^T(Bx)$).

- A Riemannian metric g on V^n can be decomposed as $\sum_{\text{finite}} \alpha(x)\,dl^2$, where $\alpha\geq 0$ and $l=l(x)$ is linear.

  pf.) Choose a set of local parametrizations $\{u_i:\mathbb{R}^n\xrightarrow{\sim}U_i\subseteq V^n\}$ and a partition of unity $\{\alpha_i\}$ subordinate to $\{U_i\}$. (i.e. $\sum\alpha_i=1$, $\operatorname{supp}\alpha_i\subseteq U_i$).

  - Let $A_i=\operatorname{supp}(\alpha_i\circ u_i)$ and $g^i=(u_i^*g)|_{A_i}$.

    For each i, take positive quadratic forms Q_{ij}, $j=1,\dots,N(i)$ s.t.
    $g^i_x:=g^i|_{T_x\mathbb{R}^n}\in\operatorname{int\,conv}(\{Q_{ij}\mid 1\leq j\leq N(i)\})$ for all $x\in A_i$.

    Then we can write $g^i=\sum_j \beta_j(x)\,Q_{ij}$ and decompose Q_{ij} as $\sum_{k=1}^{n}(l_{ijk})^2$.

    Then $g=\sum_{i,j,k}\alpha_i\left((u_i^{-1})^*(d\,\beta_j(l_{ijk})^2)\right)$

(B) ε-isometric approximation.

Recall: For metrics $g,\tilde g$ on V, $r(g,\tilde g):TV\backslash V\longrightarrow\mathbb{R}$ $\left(v\longmapsto\dfrac{\|v\|_{\tilde g}}{\|v\|_g}\right)$

For $f,\tilde f:(V,g)\to\mathbb{R}^q$, $d_g(f,\tilde f):TV\backslash V\longrightarrow\mathbb{R}$ $\left(v\longmapsto\dfrac{\|f_*v-\tilde f_*v\|_h}{\|v\|_g}\right)$.

- ① (Convergence lemma)

  Given $f_i:V\longrightarrow\mathbb{R}^q$: C^∞. If $f_i\xrightarrow{C^0}\tilde f$ and $d_g(f_i,f_{i+1})<C_i$ with $\sum_i C_i<\infty$, then $\tilde f$ is C^1 and $f_i\xrightarrow{C^1}\tilde f$. (Only a Cauchy convergence argument.)

② (Approximation theorem)

(1) (1 D case) : $\forall\varepsilon>0$ $\forall\rho>0$ Any short embedding $f:(I,g)\longrightarrow(\mathbb{R}^q,h)$ can be C^0-approximated by ε-isometric embeddings $\tilde f$ s.t. $d_g(f,\tilde f)<r(g-f^*h,g)+\rho$.

(Rk: This proof follows from 1-dimensional convex integration lemma.)

(2). ($g-f^*h=\alpha(x)\,dl^2$): Let $f:(V^n,g)\longrightarrow(\mathbb{R}^q,h)$ $(n<q)$ be a short embedding s.t. $g-f^*h=\alpha(x)\,dl^2$ (in local frame). Then $\forall\varepsilon>0,\rho>0$ f can be C^0-approximated by ε-isometric embeddings $\tilde f$ s.t. $d_g(f,\tilde f)<r(g-f^*h,g)+\rho$.

(pf). We will reduce it to the parametric 1-D convex integration lemma.

It suffices to consider $(V, g) = (\mathbb{R}^n, g)$.

Note that f is isometric on the leaves of $\mathcal{P} = \{l(x) = \text{const.}\}$ — the $(n-1)$-dimension affine foliation.

Let v be the vector field on $\mathbb{R}^n$ normal to the leaves of $\mathcal{P}$ (wrt g).

$\rightsquigarrow$ The integral curves of v forms a foliation $\mathcal{L}$ normal to $\mathcal{P}$.

Choose a global frame $\{\partial_i\}_{1\le i\le n}$ on $\mathbb{R}^n$ s.t. ∂_1 tangent to $\mathcal{L}$ $(v = \partial_1)$
∂_i $(2\le i\le n)$ tangent to $\mathcal{P}$.

Then
$$\begin{cases} \langle f_*\partial_i, f_*\partial_j\rangle_h = \langle \partial_i, \partial_j\rangle_g, & 2\le i\le j\le n. \\ \langle f_*\partial_1, f_*\partial_j\rangle_h = \langle \partial_1, \partial_j\rangle_g, & 2\le j\le n. \\ \langle f_*\partial_1, f_*\partial_1\rangle_h = \langle \partial_1,\partial_1\rangle_g - \langle \partial_1,\partial_1\rangle_{\alpha(x)dl^2} \end{cases}$$

Choose a normal vector field n to $f(\mathbb{R}^n) \subseteq \mathbb{R}^q$. Then f can be considered as a family of maps $f_p : \mathcal{L}_p \to \mathbb{R}^q$, where $\mathcal{L}_p$ are the leaves of $\mathcal{L}$, and hence by parametric one-dimensional convex integration lemma, for R_{f_p}: the isometric relation on $\mathcal{L}_p$ and $\tilde{R}_{f_p}$ the ε-isometric relation, we can choose $\tilde{f}$ s.t. $\partial_i\tilde{f} = \tilde{f}_*\partial_i$ is arbitrary C^0-close to $\partial_i f = f_*\partial_i$ $(i\ge 2, \dots, n)$. In particular, $\tilde{f}$ will be an embedding. Moreover, $\langle \tilde{f}_*\partial_1, \tilde{f}_*\partial_j\rangle_h$, $2\le j\le n$ can be made arbitrarily small by choosing $\tilde{R}_{f_p}$ sufficiently close to R_{f_p}. Then $\tilde{f}$ is s.t. $\tilde{f}^*h$ is arbitrarily close to g.

As in 1-D case, $\forall \rho > 0$, by choosing $\tilde{R}_{f_p}$ sufficiently close to R_{f_p}, $\tilde{f}$ is s.t. $d_g(f, \tilde{f}) \le r(g - f^*h, g) + \rho$.

(3). (General case). Let $n < q$. $\forall \varepsilon > 0\ \forall \rho > 0$ any short embedding $f : (V^n, g) \to (\mathbb{R}^q, h)$ can be C^0-approximated by ε-isometric embeddings $\tilde{f}$ s.t. $d_g(f, \tilde{f}) < N r(g - f^*h, g) + \rho$, where N is the number s.t. $g - f^*h = \sum_{i=1}^N \alpha_i(x)(dl_i)^2$ (local frame).

(pf). Let $g - f^*h = \sum_{i=1}^N \rho_i$ be the decomposition.

Let $g_k = f^*h + \sum_{i=1}^k \rho_i$, $k = 1, \dots, N$ $(g_N = g)$.

Then by (2) we can construct embeddings $f_1, \dots, f_N = \tilde{f}$ s.t. f_i^*h is arbitrarily close to g_i, $i = 1, \dots, N$. In particular, $\tilde{f} = f_N$ is s.t. $\tilde{f}^*h$ is arbitrarily close to $g = g_N$.

Moreover, $f_1, \dots, f_N$ can be chosen s.t. $d_{g_i}(f_{i-1}, f_i) < r(\rho_i, g_i) + \rho'$, where $f_0 = f$, $\rho' = \frac{\rho}{N}$.

Since $g \ge g_i$ and $r(\rho_i, g_i) < 1$, $i = 1, \dots, N$, we have

$$d_g(f_0, f_1) \le d_{g_1}(f_0, f_1) < r(\rho_1, g_1) + \rho' \le r(\rho_1 + \rho_2 + \dots + \rho_N, g_1 + \rho_2 + \dots + \rho_N) + \rho' = r(g - f^*h, g) + \rho'$$

$$d_g(f_1, f_2) \le d_{g_2}(f_1, f_2) < r(\rho_2, g_2) + \rho' \le r(\rho_2 + \rho_3 + \dots + \rho_N, g_2 + \rho_3 + \dots + \rho_N) + \rho' \le r(g - f^*h, g) + \rho'$$

$d_g(f_{N-1}, f_N) \le d_{g_N}(f_{N-1}, f_N) < r(\rho_N, g_N) + \rho' \le r(g - f^*h, g) + \rho'$

Then $d_g(\tilde{f}, \tilde{f}) - d_g(f_0, f_N) \le d_g(f_0, f_1) + \cdots + d_g(f_{N-1}, f_N) < N\, r(g - f^*h, g) + \rho'$

(C) <u>Proof of Nash - Kuiper Theorem</u>

Choose constants $\rho_i > 0$ s.t. $\sum_{i=1}^{\infty} \rho_i < \infty$ and a constant $k > 0$ s.t. $k^2 f^*h > g$.

Fix a decomposition of $g - f^*h$ $(= \sum_{i=1}^{N} \rho_i)$ and fix a positive sequence $\delta_i \uparrow 1$ s.t.

$\sqrt{\delta_1} + \sqrt{\delta_2 - \delta_1} + \cdots + \sqrt{\delta_m - \delta_{m-1}} + \cdots < \infty$.

Note that if we set $g_i = f^*h + \delta_i(g - f^*h)$, then $g_i \to g$.

On the other hand, $f^*h < g_1$ and hence $f_0 = f : (V, g_1) \to (\mathbb{R}^q, h)$ is strictly short.

By approximation theorem, we can C^0-approximate the embedding f_0 by ε_1-isometric embedding $f_1 : (V, g_1) \to (\mathbb{R}^q, h)$ s.t. $d(f_0, f_1) < N\, r(\delta_1(g - f^*h), g_1) + \rho_1$

$= N\sqrt{\delta_1}\, r(g - f^*h, g_1) + \rho_1 \le N\sqrt{\delta_1}\, r(g - f^*h, f^*h) + \rho_1 = Nk\sqrt{\delta_1}\, r(g - f^*h, k^2 f^*h) + \rho_1$

$< Nk\sqrt{\delta_1}\, r(g - f^*h, g) + \rho_1$.

For ε_1 sufficiently small, $f_1^*h \approx f_0^*h + \delta_1(g - f^*h)$, and hence $f_1^*h < \underbrace{f_0^*h + \delta_2(g - f^*h)}_{= g_2}$, which means $f_1 : (V, g_2) \to (\mathbb{R}^q, h)$ is strictly short.

Hence, $\forall \varepsilon_2 > 0$ we can C^0-approximate the embedding f_1 by an ε_2-isometric embedding $f_2 : (V, g_2) \to (\mathbb{R}^q, h)$. Choose ε_1 sufficiently small, we can make $f_2 - f_1^*h$ close to $(\delta_2 - \delta_1)(g - f^*h)$

$d(f_1, f_2) < N\, r((\delta_2 - \delta_1)(g - f^*h), g_2) + \rho_2 = N\sqrt{\delta_2 - \delta_1}\, r(g - f^*h, g_2) + \rho_2$

$\le N\sqrt{\delta_2 - \delta_1}\, r(g - f^*h, f^*h) + \rho_2 \le Nk\sqrt{\delta_2 - \delta_1}\, r(g - f^*h, k^2 f^*h) + \rho_2$

$\le Nk\sqrt{\delta_2 - \delta_1}\, r(g - f^*h, g) + \rho_2$.

The sequence $\{f_i\}$ can be chosen C^0-converging to some conti. map $\bar{f}$.

On the other hand, $\sum_i d(f_i, f_{i+1}) < Nk\, r(g - f^*h, g)(\sqrt{\delta_1} + \sqrt{\delta_2 - \delta_1} + \cdots) + \sum_i \rho_i < \infty$.

By convergence lemma, $\bar{f}$ is C^1-isometric embedding and $f_i \xrightarrow{C^1} \bar{f}$.

<u>Remark</u>: Such C^1-isometric embedding may not be C^2 due to curvature reason.

<u>Eg</u>: A flat torus is $\mathbb{R}^2/\Lambda \cong T^2$, where $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$.

- The Gaussian curvature of $\mathbb{R}^2/\Lambda$ is 0.
- A compact C^2 surface in $\mathbb{R}^3$ has a point with positive Gaussian curvature.
- By Theorem Egregium, no C^2-isometric embedding of a flat torus in $\mathbb{R}^3$ exists.

§4. C^1-isometric embedding of the flat torus

- (Another proof of one-dimensional convex integration)

Let $\mathcal{R} \underset{open}{\subseteq} J^1(\mathbb{R}, \mathbb{R}^q)$ and $\mathcal{F} = (f, \varphi) : I \to \mathcal{R}$ be a short formal solution, i.e. $df(\mathbb{R}) \subseteq \mathrm{conv}_f \mathcal{R}$. Then there exist genuine solutions $\tilde{f}$ arbitrarily C^0-close to f.

(Pf.) •• (Integral representation lemma)

For such $\mathcal{F}$, $\exists\ h : [0,1]\times[0,1] \longrightarrow \bigcup_{x,y} \mathcal{R}_{x,y}$,

$(x,t) \longmapsto h(x,t) \in \mathcal{R}_{x,y} := \{ z \mid (x,y;z) \in \mathcal{R},\ x \in \mathbb{R},\ y \in \mathbb{R}^q \}$

such that $f'(x) = \int_0^1 h(x,t)\,dt$, $h(x,0) = h(x,1)$. (h is called the loop)

(Pf). For any $f'(x)$, take $a_1, \dots, a_k$ in a path connected component of $\mathcal{R}_{x,f(x)}$ whose convex hull contains $f'(x)$. By the path-connectedness, there exists a smooth map $\psi : [0,1] \to \mathcal{R}_{x,f(x)}$ s.t. $\psi(0) = \psi(1)$, $\psi(s_i) = a_i$ for some $s_i \in [0,1]$, $i = 1, \dots, k$.

Let $d\mu_i$ be a continuous positive measure on $[0,1]$ s.t. $\int_0^1 d\mu_i = 1$ and $d\mu_i \approx \delta(s - s_i)$ [Eg. $d\mu_i = f_i(s)\,ds$ with f_i a smooth function concentrating near s_i]. (the Dirac delta)

Then the integrals $b_i = \int_0^1 \psi(s)\,d\mu_i(s)$ is close to $a_i = \psi(s_i) = \int_0^1 \psi(s)\,\delta(s - s_i)\,ds$ for $i = 1, \dots, k$.

Hence $f'(x)$ will lie in the convex hull of b_i $(1 \le i \le k)$ if b_i is close to a_i.

Since "being in the convex hull" is an open condition, actually on a small neighborhood of x, we can pick $\psi : \mathcal{O}p(x) \times [0,1] \to \bigcup_{x \in \mathcal{O}p(x)} \mathcal{R}_{x,f(x)}$ s.t. $\psi(x,0) = \psi(x,1)$, $\psi(x, s_i) = a_i(x)$ and $f'(x)$ lies in the convex hull of $\{a_i(x)\}_{1 \le i \le k}$. Then we may have $b_i(x) = \int_0^1 \psi(x,s)\,d\mu_i(s)$ close to $a_i(x)$. Then since $f'(x)$ lies in the convex hull of $b_i(x)$ for $x \in U$.

We can pick $p_i(x) \ge 0$, $\sum_{i=1}^k p_i(x) = 1$ s.t. $\sum_{i=1}^k p_i(x)\, b_i(x) = f'(x)$. Letting $d\mu$ be a $\mathcal{O}p(x)$-parametrized family of measures s.t. $d\mu_x = \sum_{i=1}^k p_i(x)\,d\mu_i$, we get $\int_0^1 \psi(x,s)\,d\mu_x(s) = \sum_{i=1}^k p_i(x) \int_0^1 \psi(x,s)\,d\mu_i(s)$ $= \sum_{i=1}^k p_i(x)\, b_i(x) = f'(x)$. A partition of unity argument can make sure that we can define $d\mu_x$ to be a $[0,1]$-parametrized family. Now define $\lambda : [0,1]\times[0,1] \to [0,1]$ by $\lambda(x,t) = \int_0^t d\mu_x$, then $\frac{\partial \lambda}{\partial t} > 0 \rightsquigarrow \lambda^{-1}$ exists and is smooth.

Let $h(x,s) = \psi(x, \lambda^{-1}(x,s))$, and we have $f'(x) = \int_0^1 h(x,s)\,ds$ for all $x \in [0,1]$.

•• Now setting $F : I \to \mathbb{R}^q$ by $F(t) = f(0) + \int_{x=0}^t h(x, \{Nx\})\,dx$, where $N \in \mathbb{N}$ and $\{Nx\}$ is the fraction part of Nx. Then $\| F - f \|_{C^0} \le \frac{M}{N}$, where M is a constant depending on h, and hence F will be a genuine solution and C^0 close to f if N large. ([I], Page 17, 18). #

[I]: Isometric embeddings of square flat tori in ambient space. V. Borrelli, S. Jabrane, F. Lazarus, B. Thibert.

- In our case, the isometric embedding problem in one dimensional is st. $R_{a,y}$ is a sphere of radius $r(s)$ in $\mathbb{R}^n$. In the case, f is short iff $\|f'(s)\| \le r(s)$. Assume f' is never 0 and let $n: I \to \mathbb{R}^n$ be the vector field normal to f. Let $t(s) = \frac{f'(s)}{\|f'(s)\|}$. We choose the loop $h(x,\cdot)$ by

$$h(s,u) = r(s)\left(\cos(d_s \cos(2\pi u))\, t(s) + \sin(d_s \cos(2\pi u))\, n(s)\right) \quad \text{with } d_s := J_0^{-1}\left(\frac{\|f'(s)\|}{r(s)}\right),$$

where $J_0(x) = \int_0^1 \cos(x\cos(2\pi u))\,du$ the Bessel function restricted to $[0, \underset{2.4}{z}]$. $\sim (*)$

- For $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ case, we want to find $f_{iso} : (T^2, \mu) \to (\mathbb{R}^3, \langle\cdot,\cdot\rangle_{\mathbb{R}^3})$ satisfying $f_{iso}^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3} = \mu$. We start with the primitive case $\mu - f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = \underset{\substack{\uparrow \\ \text{function}_{\ge 0}}}{\rho}\, \ell \otimes \underset{\substack{\uparrow \\ \text{linear}}}{\ell}$.

We choose $V \in \ker \ell$, with relatively prime integer coordinate.

Then $\gamma : [0,1] \to T^2$, $t \mapsto O + tV$ is simple closed in T^2.

Consider U st. U, V is an orthogonal basis of $\mathbb{R}^2$ and $\|U\|\,\|V\| = 1$

We first consider the "Cylinder" case: $O + tV + sU$, $t \in \mathbb{R}/\mathbb{Z}$, $s \in [0,1]$.

(WLOG, we may assume $\ell(U) = \|U\|$.)

- "Cyl": (the convex integration on cylinder)

We want to get ε-isometric embedding for (Cyl, μ) into $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_{\mathbb{R}^3})$

Define $\phi_t : I \to Cyl$, $s \mapsto O + tV + sU$, $t \in \mathbb{R}/\mathbb{Z}$.

We would apply to each $f \circ \phi_t$ the $\underset{\text{parametric}}{\text{one}}$ dimensional convex integration: $n = \frac{U\cdot f \wedge V\cdot f}{\|U\cdot f \wedge V \cdot f\|}$

$h(t,s,u) = \bar{h}(\phi_t(s), \cos(2\pi u))$

$\bar{h}(p,c) = r(p)\left(\cos(d(p)c)\, t(p) + \sin(d(p)c)\, n(p)\right)$, $r := \sqrt{\mu(U,U)}$, $t := \frac{U\cdot f}{\|U\cdot f\|}$, $d = J_0^{-1}\left(\frac{\|U\cdot f\|}{r}\right)$

And the "ε-isometric solution" is $F \circ \phi_t(s) := f(O+tV) + \int_{u=0}^{s} h(t,u,\{Nu\})\,du$.

But in this case, we will find $F^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}(U,V) \ne \frac{\cos(d\cos(2\pi Ns))}{\|U f\|}\mu(U,V)$.

Hence, unless $\mu(U,V)$ is null, F will not be a ε-isometric solution.

Now we adjust U by $W = U + \xi V$ with $\xi = -\frac{\mu(U,V)}{\mu(V,V)}$ so that $\mu(W,V) = 0$.

After lots of computation, we can see such h is an ε-isometric solution.

- "T^2": (the convex integration on the torus T^2)

Though we have constructed an ε-isometric embedding of cylinders, but this will not be an ε-isometric embedding of the torus since the boundary may not coincide. Thus we will make some adjustment.

We define $\bar{F} \circ \varphi(t,s) = F\circ\varphi(t,s) - w(s)\left(F\circ\varphi(t,1) - f\circ\varphi(t,1)\right)$, where φ is the integral curve of W and $w : I \to I$ is S-shaped with $w(0) = 0$, $w(1) = 1$, $w^{(k)}(0) = w^{(k)}(1) = 0$

After computation, we can see $\bar{F}$ is an ε-isometric embedding st.

1. $\|\bar{F} - f\|_\infty \le \frac{k_1(h)}{N}$ ← a constant depending on h.

2. $\|d\bar{F} - df\|_\infty \le \frac{k_2(h, \xi, \psi, w')}{N} + \sqrt{7}\, \|\rho\|_\infty^{\frac{1}{2}}$

3. $\|V\cdot\bar{F} - V\cdot f\| \le \frac{1}{N} k_3(h, \psi)$

4. $\|W\cdot\bar{F} - W\cdot f\| \le \sqrt{7}\, \|V\| \left(1 + \|w'\|_\infty\right) \|\rho\|_\infty^{\frac{1}{2}}$

5. $\|\mu - \bar{F}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}\|_\infty \le \frac{1}{N} K_4(f\circ\psi, r, h, w', \psi^{-1})$. | We will denote $\bar{F}$ by $IC(f, \mu, N)$.

- Now we start to decompose the metric.

We assume f is s.t. $D = g - f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ belongs to the open cone $\mathcal{C} = \{\rho_1 l_1\otimes l_1 + \rho_2 l_2\otimes l_2 + \rho_3 l_3\otimes l_3\}$, where $l_1 = dx$, $l_2 = \frac{1}{\sqrt{5}}(dx + 2dy)$, $l_3 = \frac{1}{\sqrt{5}}(dx - 2dy)$.

Let $\mu_1 = f^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \rho_1(D_1)\, l_1\otimes l_1$, $D_1 := D$. $\rightsquigarrow$ We have $f_1 = IC(f, \mu_1, N_1)$.

Then we can prove that for N large, $D_2 := g - f_1^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} \in \mathcal{C}$, $\rho_1(D_2) \approx 0$, $\rho_3(D_2) \approx \rho_3(D_1)$. Then $\rho_2(D_2) > 0$, and we can set $\mu_2 := f_1^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \rho_2(D_2)\, l_2\otimes l_2$, and build the quasi isometry $f_2 := IC(f_1, \mu_2, N_2)$.

Similarly, for N_2 large enough, we can prove $D_3 := g - f_2^*\langle\cdot,\cdot\rangle \in \mathcal{C}$, write

$D_3 = \rho_1(D_3)\, l_1\otimes l_1 + \rho_2(D_3)\, l_2\otimes l_2 + \rho_3(D_3)\, l_3\otimes l_3$, and then $\rho_1(D_3) \approx 0$, $\rho_2(D_3) \approx 0$, $\rho_3(D_3) \approx \rho_3(D_1)$.

Since $\rho_3(D_3) > 0$, we can set $\mu_3 := f_2^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \rho_3(D_3)\, l_3\otimes l_3$. with $f_3 = IC(f_2, \mu_3, N_3)$.

For N_3 large enough, f_3 is almost an isometry of the metric g.

We denote f_3 by $IC(f, g, N_1, N_2, N_3)$.

- Now for the last step, we set $F_0 : T^2 \longrightarrow \mathbb{R}^3$ be an embedding s.t. $\Delta := \langle\cdot,\cdot\rangle_{\mathbb{R}^2} - F_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ lies in the cone $\mathcal{C}$.

  Set $\delta_k \nearrow 1$ and $\sum \sqrt{\delta_k - \delta_{k-1}} < \infty$, $g_k = F_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \delta_k \Delta$

  $F_k := IC(F_{k-1}, g_k, N_{k,1}, N_{k,2}, N_{k,3})$.

  Then F_∞ is a C^1-isometry.

- The choice of F_0 :

$$\begin{cases} X(x,y) = \frac{1}{2\pi}\left(r_2 + r_1\cos(2\pi x)\right)\cos 2\pi y \\ Y(x,y) = \frac{1}{2\pi}\left(r_2 + r_1\cos(2\pi x)\right)\sin 2\pi y \\ Z(x,y) = \frac{r_1}{2\pi}\sin(2\pi x) \end{cases}$$

, choose $r_1 = \frac{1}{5}$, $r_2 = \frac{1}{2}$.

↑ the standard torus parametrization