Milnor's Exotic 7-Spheres

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0 Introduction

This article is devoted to the explicit construction of Milnor's exotic 7-spheres, together with some prerequisites needed in the construction, such as characteristic classes, cobordisms and signatures.

There will be a nature question rising immediately as soon as we define a smooth structure on a manifold: can a topological manifold admit two different smooth structure? That is, is it true that if two smooth manifolds are homeomorphic, then they are automatically diffeomorphic? The answer is that it depends!!

If the manifold is of dimension 1 or 2, then the answers are sure, proved by Tibor Radó. The case of dimension 3 was also proved to be true by Edwin E. Moise. However, the first counterexample given by John Milnor in 1956 states that there are at least two different smooth structures on 7-sphere. Furthermore, John Milnor and Michel Kervaire proved that there are 28 oriented differentiable structures on 7-sphere (15 if without consideration of orientation) in 1963. As time passed by, Michael Freedman gave a non-standard differentiable structure on $\mathbb{R}^4$, known as exotic $\mathbb{R}^4$, in 1982. By the way, there are no different smooth structures on $\mathbb{R}^n$ for all $n \neq 4$. Even though exotic $\mathbb{R}^4$ has been constructed over 30 years ago, the existence of exotic 4-sphere is still an open problem now.

1 Fiber Bundles and Characteristic Classes

Given a real vector bundle $\xi : E \xrightarrow{\pi} M$ of rank $2n$, one can give each fiber a complex structure in the following sense: $J : E \to E$ is a continuous map which maps each fiber $\mathbb{R}$-linearly to itself and satisfies $J \circ J(v) = -v$ for all $v \in E$. With this complex structure, the real vector bundle ξ can then be viewed as a complex vector bundle. Conversely, given a complex vector bundle ξ, we can also view it as a real vector bundle, denoted by $\xi_{\mathbb{R}}$, by ignoring the complex structure on each fiber and think of each fiber as a real vector space.

Let $\xi : E \xrightarrow{\pi} M$ be a complex vector bundle over a manifold M. We define $\bar{\xi} : \bar{E} \xrightarrow{\bar{\pi}} M$ the complex conjugate vector bundle of ξ by the following way:
(i) $\xi_{\mathbb{R}} = \bar{\xi}_{\mathbb{R}}$.
(ii) The identity map $i : E \to \bar{E}$ is conjugate linear, i.e., $i(cv) = \bar{c}i(v)$ for each $c \in \mathbb{C}$ and $v \in E$.

Now we are ready for the definition of our first characteristic class: Chern classes.

Definition. (Chern Classes)
Let $\xi : E \xrightarrow{\pi} M$ be a complex vector bundle of rank n (complex dimension) over a manifold M. The total Chern class of this bundle is

$$c(\xi) = \sum_{i \geq 0} c_i(\xi) \in H^{2n}(M, \mathbb{Z})$$

satisfying
(i) $c_0(\xi) = 1$, $c_i(\xi) \in H^{2i}(M, \mathbb{Z})$ for all i and $c_i(E) = 0$ for $i > n$.
(ii) Naturality: for any smooth map $f : M \to M'$ from M to another manifold M', $c(f^*\xi) = f^*c(\xi)$.
(iii) Whitney sum formula: $c(\xi \oplus \eta) = c(\xi) \smile c(\eta)$ for any complex vector bundle η over M.
(iv) $c(\gamma) = 1 + g$, where γ is the universal line bundle of $\mathbb{C}\mathbb{P}^\infty$ and $g \in H^2(\mathbb{C}\mathbb{P}^\infty)$ is the generator of the cohomology.

With definition of Chern classes, we define the Euler class $e(\xi_{\mathbb{R}}) = c_n(\xi) \in H^{2n}(M, \mathbb{Z})$, where ξ is a n-dimensional complex vector bundle over M, and the total

Pontryagin class $p(\eta) = \sum_{i\in\mathbb{N}_0} p_i(\eta)$ with $p_i(\eta) = (-1)^i c_{2i}(\eta \otimes \mathbb{C}) \in H^{4i}(M, \mathbb{Z})$, where η is a real vector bundle over M.

There are two facts that indicate us how to compute Pontryagin classes from Chern classes.

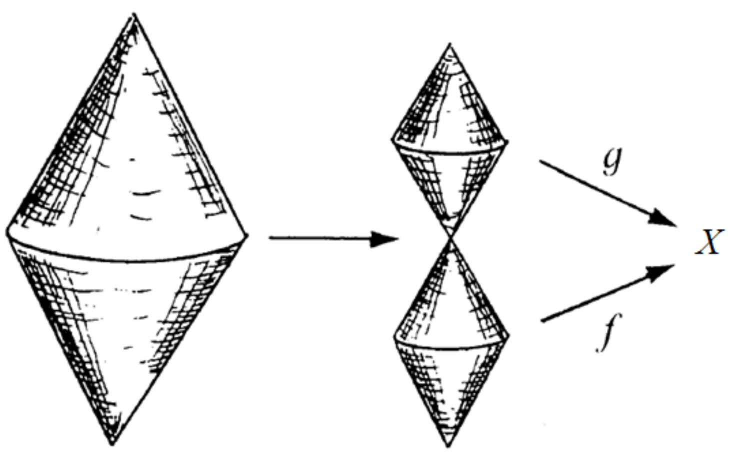
Fact. *If ξ is a real vector bundle, then $\xi \otimes \mathbb{C} \cong_{\mathbb{C}} \overline{\xi \otimes \mathbb{C}}$; if ξ is a complex vector bundle, then $\xi_{\mathbb{R}} \otimes \mathbb{C} \cong_{\mathbb{C}} \xi \oplus \bar{\xi}$.*

Fact. $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$.

Now we recall some basic definitions and properties of homotopy groups, which will frequently appear throughout the whole article.

Definition. (Homotopy Groups)
Define $\pi_n(X, x_0) = \{f : \mathbf{S}^n \to X \mid f \text{ is continuous and } f(t_0) = x_0\}/\sim$, where t_0 is some fixed point of $\mathbf{S}^n$ and the equivalence relation $\sim$ is the homotopic equivalence. The composition law is $[f] + [g] := [f + g]$.

For two continuous maps $f, g : \mathbf{S}^n \to X$ with $f(t_0) = g(t_0) = x_0$, we define $(f + g) : \mathbf{S}^n \to X$ to be the composition of two maps $\Psi \circ \rho$, where $\rho : \mathbf{S}^n \to \mathbf{S}^n \bigvee \mathbf{S}^n \cong \mathbf{S}^n \times \{t_0\} \cup \{t_0\} \times \mathbf{S}^n$ and $\Psi : \mathbf{S}^n \bigvee \mathbf{S}^n \to X$. The space $\mathbf{S}^n \bigvee \mathbf{S}^n$ is obtained by choosing an equator of $\mathbf{S}^n$ passing through t_0 and then identifying the equator as one point, so the resulting space has the isomorphism $\mathbf{S}^n \bigvee \mathbf{S}^n \cong \mathbf{S}^n \times \{t_0\} \cup \{t_0\} \times \mathbf{S}^n$ and $\rho : \mathbf{S}^n \to \mathbf{S}^n \bigvee \mathbf{S}^n$ is defined by sending the upper sphere to $\mathbf{S}^n \times \{t_0\}$, the lower sphere to $\{t_0\} \times \mathbf{S}^n$ and the equator to $\{t_0\} \times \{t_0\}$. The map $\Psi : \mathbf{S}^n \bigvee \mathbf{S}^n \to X$ is given by $\Psi(\{\cdot\} \times \{t_0\}) = f(\cdot)$ and $\Psi(\{t_0\} \times \{\cdot\}) = g(\cdot)$.



We will next introduce two important examples of vector bundles, which both play crucial roles in the following story.

Example. (Hopf Quaternion Bundles)

Let $\mathbb{H} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ the quaternions. Let $\{q_0, \cdots, q_n\}$ be the standard basis of $\mathbb{H}^{n+1}$. We now consider $\mathbf{S}^{4n+3}$. Indeed, $\mathbf{S}^{4n+3}$ can be embedded into $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$, i.e., $\mathbf{S}^{4n+3} \hookrightarrow \mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$, so we can represent $\mathbf{S}^{4n+3}$ in quaternion coordinate:

$$\mathbf{S}^{4n+3} := \{(q_0, \cdots, q_n) \in \mathbb{H}^{n+1} \mid |(q_0, \cdots, q_n)|^2 = \sum_{\alpha=0}^{n} |q_\alpha|^2 = 1\}.$$

Then there is a natural action of $\mathrm{SU}(2) \cong \mathbf{S}^3 \hookrightarrow \mathbb{H}$ on $\mathbf{S}^{4n+3}$ given by

$$q \cdot (q_0, \cdots, q_n) := (qq_0, \cdots, qq_n) \in \mathbf{S}^{4n+3} \hookrightarrow \mathbb{H}^{n+1},$$

the left multiplication in quaternions by $q \in \mathbb{H}$ with $|q| = 1$. Hence we obtain a principal $\mathrm{SU}(2)$-bundle with fiber $\mathrm{SU}(2) \cong \mathbf{S}^3$:

$$\begin{array}{ccc} \mathrm{SU}(2) & \xrightarrow{\text{action}} & \mathbf{S}^{4n+3} \\ & & \big\downarrow \pi \\ & & \mathbf{S}^{4n+3} / \mathrm{SU}(2) \end{array}$$

where $\mathbf{S}^{4n+3} / \mathrm{SU}(2) \cong \mathbf{S}^{4n+3} / \mathbf{S}_3 \cong \mathbb{H}\,\mathbb{P}^n$.

Let γ_n be the universal line bundle of $\mathbb{H}\,\mathbb{P}^n$ and define

$$\mathbf{u}_n := c_2(\gamma_{n\,\mathbb{C}}) = e(\gamma_{n\,\mathbb{R}}) \in H^4(\mathbb{H}\,\mathbb{P}^n, \mathbb{Z}).$$

By CW-decomposition of $\mathbb{H}\,\mathbb{P}^n$, one can conclude:

(i) $H^{4i}(\mathbb{H}\,\mathbb{P}^n, \mathbb{Z}) \cong \mathbf{u}_n^i\,\mathbb{Z}$.

(ii) $H^i(\mathbb{H}\,\mathbb{P}^n, \mathbb{Z}) = 0$ if $4 \nmid i$.

So $c(\gamma_{n\,\mathbb{C}}) = 1 + c_1(\gamma_{n\,\mathbb{C}}) + c_2(\gamma_{n\,\mathbb{C}}) = 1 + \mathbf{u}_n$.

For $p(\gamma_{n\,\mathbb{R}})$, notice that $c_0(\gamma_{n\,\mathbb{C}}) = 1$, $c_2(\gamma_{n\,\mathbb{C}}) = \mathbf{u}$ and $c_i(\gamma_{n\,\mathbb{C}}) = 0$ for $i \neq 0, 2$ and compute

$$\begin{aligned} p(\gamma_{n\,\mathbb{R}}) &= \sum_{i \in \mathbb{N}_0} p_i(\gamma_{n\,\mathbb{R}}) = \sum_{i \in \mathbb{N}_0} (-1)^i c_{2i}(\gamma_{n\,\mathbb{R}} \otimes \mathbb{C}) = \sum_{i \in \mathbb{N}_0} (-1)^i c_{2i}(\gamma_{n\,\mathbb{C}} \oplus \bar{\gamma}_{n\,\mathbb{C}}) \\ &= c_0(\gamma_{n\,\mathbb{C}})c_0(\gamma_{n\,\mathbb{C}}) - c_0(\gamma_{n\,\mathbb{C}})c_2(\gamma_{n\,\mathbb{C}}) - c_2(\gamma_{n\,\mathbb{C}})c_0(\gamma_{n\,\mathbb{C}}) + c_2(\gamma_{n\,\mathbb{C}})c_2(\gamma_{n\,\mathbb{C}}) \\ &= 1 - \mathbf{u}_n - \mathbf{u}_n + \mathbf{u}_n^2 = 1 - 2\,\mathbf{u}_n + \mathbf{u}_n^2\,. \end{aligned}$$

Example. (SO(4)-Vector Bundles over $\mathbf{S}^4$ of Rank 4)

We first recall a useful lemma:

<u>Lemma.</u> *The G-bundles over $\mathbf{S}^k$ are determined by $\pi_{k-1}(G)$.*

There is an intuitive explanation of this lemma: View $\mathbf{S}^k$ as the union of the upper sphere and the lower sphere. Each half sphere is contractible and thus their vector bundles are both trivial. Therefore, to obtain a vector bundle over $\mathbf{S}^k$, we should glue the two trivial bundles along the fiber of the equator, and each distinct approach to gluing up two bundles gives distinct vector bundles over $\mathbf{S}^k$.

To study SO(4)-vector bundles over $\mathbf{S}^4$, we should look at $\pi_3(\mathrm{SO}(4))$. Since

$$\pi_3(\mathrm{SO}(4)) \cong \pi_3(\mathrm{SO}(3)) \oplus \pi_3(\mathbf{S}^3) \cong \mathbb{Z} \oplus \mathbb{Z},$$

we know that SO(4)-vector bundles over $\mathbf{S}^4$ are determined by integer pairs $(h, j) \in \mathbb{Z}^2$.

Define $f_{hj} : \mathbf{S}^3 \to \mathrm{SO}(4)$ by $f_{hj}(v)w := v^h w v^j$, where $v, w \in \mathbb{H}$ and $v \in \mathbf{S}^3 \hookrightarrow \mathbb{H}$. Let ξ_{hj} be the vector bundle defined by f_{hj} with fiber $\mathbb{R}^4$ and $\mathbf{S}_{hj}$ the sphere bundle of ξ_{hj}.

Define $\sigma : \mathbf{S}^3 \to \mathrm{SO}(4)$ and $\sigma' : \mathbf{S}^3 \to \mathrm{SO}(4)$ by $\sigma(v)w := vw$ and $\sigma'(vw) := wv$, the left and right multiplication in $\mathbb{H}$. Since $\mathbb{H}\mathbb{P}^1 \cong \mathbf{S}^4$, so this is a special case of Hopf quaternion bundles with $n = 1$. Let $\gamma := \gamma_1$ be the left Hopf bundle corresponding to σ discussing in last example and γ' the vector bundle corresponding to σ'. Then σ and σ' generate $\pi_3(\mathrm{SO}(4))$.

Proposition 1. *$e(\xi_{hj}) = (h + j)\,\mathbf{u}$ and $p_1(\xi_{hj}) = 2(h - j)\,\mathbf{u}$, where $\mathbf{u} := \mathbf{u}_1$.*

Proof. As noted above, this is a special case of Hopf quaternion bundles with $n = 1$. In this case, $e(\gamma_{\mathbb{R}}) = \mathbf{u}$ and

$$p_1(\gamma_{\mathbb{R}}) = -c_2(\gamma_{\mathbb{C}} \otimes \mathbb{C}) = -c_2(\gamma_{\mathbb{C}} \oplus \bar{\gamma}_{\mathbb{C}}) = -[c_0(\gamma_{\mathbb{C}})c_2(\bar{\gamma}_{\mathbb{C}}) + c_2(\gamma_{\mathbb{C}})c_0(\bar{\gamma}_{\mathbb{C}})] = -2\,\mathbf{u}\,.$$

Similarly, $e(\bar{\gamma}_{\mathbb{R}}) = \mathbf{u}$ and $p_1(\bar{\gamma}_{\mathbb{R}}) = 2\,\mathbf{u}$. Then

$$e(\xi_{hj}) = (h + j)\,\mathbf{u} \quad \text{and} \quad p_1(\xi_{hj}) = 2(h - j)\,\mathbf{u}$$

follow from definition. □

Example. (Oriented Canonical Vector Bundles of Grassmannians)

Let $\mathbf{G}_k(\mathbb{R}^{k+p})$ be the Grassmannian consisting of all k-dimensional subspaces of $\mathbb{R}^{k+p}$. Further, define $\tilde{\mathbf{G}}_k(\mathbb{R}^{k+p})$ be the oriented Grassmannian, i.e., its elements are those oriented k-dimensional subspaces of $\mathbb{R}^{k+p}$. Given $\mathbf{G}_k(\mathbb{R}^{k+p})$, there is a canonical vector bundle

$$\gamma_p^k \equiv \gamma^k(\mathbb{R}^{k+p}) := \{(X, v) \mid X \in \mathbf{G}_k(\mathbb{R}^{k+p}) \text{ and } v \in X\}.$$

Similarly, define oriented canonical vector bundle of $\tilde{\mathbf{G}}_k(\mathbb{R}^{k+p})$ by

$$\tilde{\gamma}_p^k \equiv \tilde{\gamma}^k(\mathbb{R}^{k+p}) := \{(\tilde{X}, \tilde{v}) \mid \tilde{X} \in \tilde{\mathbf{G}}_k(\mathbb{R}^{k+p}) \text{ and } \tilde{v} \in \tilde{X}\}.$$

We now introduce two notations about partition numbers:

$$\begin{aligned} \mathfrak{P}(n) &:= \{(i_1, \cdots, i_r) \in \mathbb{N}^r \mid r \in \mathbb{N}, i_1 \leq \cdots \leq i_r, i_1 + \cdots + i_r = n\} \\ \mathfrak{p}(n) &:= |\mathfrak{P}(n)|, \text{ the cardinality of } \mathfrak{P}(n). \end{aligned}$$

Theorem 1. *Let R be a ring with 2 being invertible. $\tilde{\xi}^m$ denotes the universal bundle over $\tilde{\mathbf{G}}_{2n-1}(\mathbb{R}^\infty)$ of rank m. Then*

$$\begin{aligned} H^*(\tilde{\mathbf{G}}_{2n+1}(\mathbb{R}^\infty); R) &= R[p_1(\tilde{\xi}^{2n+1}), \cdots, p_n(\tilde{\xi}^{2n+1})] \\ H^*(\tilde{\mathbf{G}}_{2n}(\mathbb{R}^\infty); R) &= R[p_1(\tilde{\xi}^{2n}), \cdots, p_{n-1}(\tilde{\xi}^{2n}), e(\tilde{\xi}^{2n})]. \end{aligned}$$

2 Thom Spaces and Cobordisms

Let M be an oriented manifold. We denote by $-M$ the same manifold with opposite orientation. For another oriented manifold M', $M + M'$ denotes the disjoint union of M and M'.

Definition. (Cobordism Groups)
We say that two smooth oriented compact manifolds M_1, M_2 both of dimension n are (oriented) codordant if $M_1 + (-M_2) = \partial W$ for some $(n+1)$-dimensional smooth oriented compact manifold-with-boundary W. In this case, we say that M, M' are in the same cobordiam class, denoted by $[M] = [M']$.
The n-th (oriented) cobordism (group) Ω_n is the set of all (oriented) cobordism classes of dimension n.

The above definition indeed make sense because the relation of cobordism classes is clearly an equivalence relation. Another fact is that Ω_n is an abelian group. Our attention will turn to the direct sum of all cobordisms: $\Omega := \bigoplus_{n \in \mathbb{N}_0} \Omega_n$.

Recall that a graded ring is a direct sum of abelian groups $\{G_\alpha\}$ such that $G_\alpha \times G_\beta \subset G_{\alpha+\beta}$. We check that Ω is definitely a graded ring. The only thing we have to verify is that the map $\Omega_i \times \Omega_j \to \Omega_{i+j}$ given by $([M],[N]) \mapsto [M \times N]$ is well-defined. Let $[M] = [M'] \in \Omega_i$ and $[N] = [N'] \in \Omega_j$. Write $M - M' = \partial W$ and $N - N' = \partial V$. Then

$$\begin{aligned} M \times N - M' \times N' &= (M' + \partial W) \times N - M' \times (N - \partial V) \\ &= \partial W \times N + M' \times \partial V \\ &= \partial (W \times N - M' \times V) \end{aligned}$$

Hence $[M] \times [N] = [M'] \times [N']$.

Definition. (Transversality)
Let M, N be two smooth manifolds and N' a submanifold of N. $f : M \to N$ is a smooth map. For a subset A of M we say f is transverse to N' over A if for each $p \in A \cap f^{-1}(N')$ the following condition holds:

$$df(T_p M) + T_{f(p)} N' = T_{f(p)} N.$$

In this case, we denote it by $f \pitchfork_A N'$. In particular, we write $f \pitchfork N'$ if $A = M$. If $f \pitchfork \{y\}$, then we called $y \in N$ is a regular value of f.

In other words, $f \pitchfork_A N'$ if and only if the composition of maps

$$T_x M \xrightarrow{df_x} T_{f(x)}N \to T_{f(x)}N/T_{f(x)}N'$$

is surjective for all $x \in A \cap f^{-1}(N')$. Specifically, if $N' = \{y\}$ is a set containing only one point, then df_x is surjective for all $x \in A \cap f^{-1}(y)$.

Fact. *Let $f : M \to N$ be a smooth map and $y \in N$ a point. Assume* $\dim M = m$ *and* $\dim N = n$. *If $f \pitchfork \{y\}$, then $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.*

Proof. Let $x \in f^{-1}(y)$. Since the map $df_x : T_xM \to TyN$ is surjective, we then have $\mathfrak{N} := \ker df_x$ is a smooth manifold of dimension of $m-n$. Embed M into $\mathbb{R}^k$ for some k. Choose a linear map $L : \mathbb{R}^k \to \mathbb{R}^{m-n}$ to be non-singular on $\mathfrak{N} \subset T_xM \hookrightarrow \mathbb{R}^k$. Define

$$\begin{aligned} F : M &\to N \times \mathbb{R}^{m-n} \\ x &\mapsto (f(x), L(x)). \end{aligned}$$

Its differential is given by $dF_x(v) = (df_x(v), L(v))$, so dF_x is surjective. Hence F maps some neighborhood U of x diffeomorphically to some neighborhood V of $(y, L(x))$. Therefore, F maps $f^{-1}(y) \cap U$ diffeomorphically to $(\{y\} \times \mathbb{R}^{m-n}) \cap V$. That is, $f^{-1}(y)$ is a smooth manifold of dimension $m - n$. $\square$

Fact. (*Brown's Theorem*)
Let $f : M \to N$ be a smooth map. Then the set of regular values of f is dense in N, i.e., the set $\{y \in N \mid f \pitchfork \{y\}\}$ is dense in N.

Proof. This is a corollary of Sard's theorem. We first recall the statement of ssard's theorem.

<u>Sard's Theorem.</u> Let $f : M \to N$ be a smooth map between manifolds M, N. Then the set

$$C := \{x \in M \mid df_x \text{ is not surjective.}\}$$

has (Lebesgue) measure 0 in N.

Then Brown's theorem follows from $N - f(C) = \{y \in N \mid f \pitchfork \{y\}\}$ and $N - f(C)$ is dense in N. $\square$

The task for us is to approximate a map transverse to $\{0\}$ over a closed subset by a map transverse to $\{0\}$ over a larger closed subset.

Lemma 1. *Suppose M is an open subset of $\mathbb{R}^m$. Let $X \subset M$ be a closed subset in M and K a compact subset of M. Suppose $f : M \to \mathbb{R}^n$ is a smooth map and $f \pitchfork_X \{0\}$. Fix a compact $K' \subset M$ with $K \subset (K' - \partial K')$. Given $\varepsilon > 0$ then there exists a smooth map $g : M \to \mathbb{R}^n$ satisfying:*
(i) $g \pitchfork_{X \cup K} \{0\}$.
(ii) $f \mid_{^cK'} = g \mid_{^cK'}$, *where* $^cK' := M - K'$.
(iii) $|f(x) - g(x)| < \varepsilon$ *for all* $x \in M$.

Proof. Let $\lambda : M \to [0, 1]$ be a smooth cut-off function such that $\lambda \mid_K \equiv 1$ and $\lambda \mid_{^cK'} \equiv 0$. According to the fact mentioned above, we can take $y \in \mathbb{R}^n$ with $|y| < \varepsilon$ such that $f \pitchfork y$. Define $g(x) = f(x) - \lambda(x)y$ and check that
(a) $f \mid_{^cK'} = g \mid_{^cK'}$.
(b) $|f(x) - g(x)| < \varepsilon$ for all $x \in M$.
(c) $g \pitchfork_K \{0\}$.
(a) and (b) are obvious. For (c), if $x_0 \in g^{-1}(0) \cap K$, then $0 = g(x_0) = f(x_0) - \lambda(x_0)y = f(x_0) - y \implies x_0 \in f^{-1}(y)$. By our assumption, $f \pitchfork y$, i.e.,

$$df(T_{x_0}M) = T_y \mathbb{R}^n = T_0 \mathbb{R}^n .$$

In addition,

$$dg(T_{x_0}M) = d(f + \lambda y)(T_{x_0}M) = df(T_{x_0}M) = T_0 \mathbb{R}^n .$$

Hence we verify that (c) holds.

Claim: If y is chosen to be sufficiently close to 0, then $g \pitchfork_{K' \cap X} \{0\}$.

With this claim, together with (b) and (c), we then prove $g \pitchfork_{X \cup K} \{0\}$.

Now we prove the claim. $f \pitchfork_{X \cap K'} \{0\}$ implies $Df(x)$ is of full rank for all $x \in X \cap K' \cap f^{-1}(0)$, where $Df = (\partial f_i / \partial x_j)_{ij}$. Since $X \cap K'$ is compact, one can find K'' compact such that $(X \cap K') \subset (K'' - \partial K'')$ and Df is of full rank on K''. Let $U := X \cap K' \cap g^{-1}(0)$. By taking $|y|$ sufficiently small, one has $U \subset K''$. Then for each $x \in U$,

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{ij} = \left(\frac{\partial f_i}{\partial x_j} - y_i \cdot \frac{\partial \lambda_i}{\partial x_j}\right)_{ij}$$

is of full rank by choosing $|y|$ small enough again (view $\det(Df)$ as a continuous function). Hence the claim. $\square$

Remark. In the above proof, the verification of $g \pitchfork_K \{0\}$ can also be carried out as we do in proving $g \pitchfork_{K' \cap X} \{0\}$. In this case, it is much easier because of $K \cap g^{-1}(0) = f^{-1}(y)$. By our assumption, $\det(Df) \neq 0$ on $f^{-1}(y)$. In fact, these two proofs are the same but write in different ways.

Definition. (Thom Spaces)

Let ξ be a vector bundle over a smooth manifold M. Define the Thom space of ξ to be $\mathbf{T}(\xi) := \mathbf{D}(\xi)/\,\mathbf{S}(\xi)$, where $\mathbf{D}(\xi)$ contains all elements in ξ with length ≤ 1, and $\mathbf{S}(\xi)$ contains all elements in ξ with length $= 1$. Let t_0 be the point of $\mathbf{T}(\xi)$ identified by $\mathbf{S}(\xi)$.

Fact. *Suppose a smooth manifold M is also a CW-complex. If ξ is a k-dimensional vector bundle over M, then $\mathbf{T}(\xi)$ is a $(k-1)$-connected CW-complex.*

Theorem 2. *Let ξ be a vector bundle of a closed manifold M of rank k. Then there is a group homomorphism $\tau : \pi_{n+k}(\mathbf{T}(\xi)) \to \Omega_n$.*

Proof. There are several steps.

For convenience, denote $\mathbf{T}(\xi)$ by $\mathbf{T}$. Given a map $f : \mathbf{S}^{n+k} \to \mathbf{T}$, one can approximate f by a map $f_0 : \mathbf{S}^{n+k} \to \mathbf{T}$ on $f_0^{-1}(\mathbf{T} - t_0) = f^{-1}(\mathbf{T} - t_0)$. Choose an open covering $\{W_1, \cdots, W_r\}$ of compact set $f_0^{-1}(M)$ such that $f_0(W_i)$ is contained in some $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^k$, where U_i are some open subsets of M. Let $\rho_i : \pi^{-1}(U_i) \to \mathbb{R}^k$ be the projection. Pick compact $K_i \subset W_i$ such that $f_0^{-1}(M) \subset (K_1 \cup \cdots \cup K_r)$.

Our strategy is to modify f_0 within one W_i after another, and then define the last one to be our desired function. We want to construct maps $f_1, \cdots, f_r$ satisfying:

(a) Every f_i is smooth in $f_i^{-1}(\mathbf{T} - t_0)$ and $f_i \,|_{W_i - K_i} = f_{i-1} \,|_{W_i - K_i}$.

(b) $f_i \pitchfork_{K_1 \cup \cdots \cup K_r} M$ for $i = 1, \cdots, r$.

(c) $\pi(f_i(x)) \in M$ equals $\pi(f_0(x))$ for all $x \in f_0^{-1}(\mathbf{T} - t_0)$.

We start from f_0 and construct f_i inductively. Assume f_{i-1} has already been constructed. Condition (c) implies that $f_{i-1}(W_i) \subset \pi^{-1}(U_i) \cong U_i \times \mathbb{R}^k$. Also, it follows from condition (b) that the map $\rho_i \circ f_{i-1} : W_i \to \mathbb{R}^k$ has

$$\rho_i \circ f_{i-1} \pitchfork_{(K_1 \cup \cdots \cup K_i) \cap W_i} \{0\}.$$

By lemma 1, we can approximate $\rho_i \circ f_{i-1} : W_i \to \mathbb{R}^k$ by a map $\rho_i \circ f_i$: such that

(a′) $\rho_i \circ f_i \mid_{W_i - K_i} = \rho_i \circ f_{i-1} \mid_{W_i - K_i}$.

(b′) $f_i \pitchfork_{K_1 \cup \cdots \cup K_r} \{0\}$ $N := g^{-1}(\mathbf{T}(\xi) - t_0)$.

So we can define $f_i : W_i \to \pi^{-1}(U_i) \cong U_i \times \mathbb{R}^k$ whose first coordinate $\pi(f_i(x))$ is determined by condition (c) and the second coordinate $\rho_i \circ f_i(x)$ is determined by condition (a), (a′) and (b′). It is then clear that condition (b) holds. Hence we define $f_1, f_2, \cdots, f_r$ inductively.

Now let $g := f_r$. We must prove the following claim.

Claim: $g^{-1}(M) \subset (K1 \cup \cdots \cup K_r)$.

Indeed, $g \pitchfork_{K1 \cup \cdots \cup K_r} M$. If we can prove the claim, then we will conclude $g \pitchfork M$.

Now we prove the claim. Since $K_1 \cup \cdots K_r$ is a compact neighborhood of $f_0^{-1}(M)$ in the compact manifold $\mathbf{S}^{n+k}$, one can find $c \in (0,1)$ such that $|f_0(x)| < c$ for all $x \notin (K_1 \cup \cdots \cup K_r)$. Let f_i is chosen to satisfy

$$|f_i(x) - f_{i-1}(x)| < \frac{c}{r} \qquad \text{for all } x \in \mathbf{S}^{n+k}.$$

Consequently, $|g(x) - f_0(x)| < c$ for all $x \in \mathbf{S}^{n+k}$ and thus $|g(x)| \neq 0$ for any $x \notin (K_1 \cup \cdots \cup K_r)$. That is, $g^{-1}(M) \subset (K1 \cup \cdots \cup K_r)$. Hence $g \pitchfork M$. So we naturally define $\tau([f]) = [g^{-1}(M)]$, where $g^{-1}(M)$ is a compact manifold of dimension n by our construction.

Next we have to check that this map is well defined, i.e., $[f_0] = [f_1] \in \pi_{n+k}(\mathbf{T}(\xi))$ such that $f_0 \pitchfork M$ and $f_1 \pitchfork M$ implies $f_0^{-1}(M) = f_1^{-1}(M)$. We take a smooth map $F : \mathbf{S}^{n+k} \times [0,1] \to \mathbf{T}(\xi)$ such that

$$F(x, [0, \frac{1}{3}]) = f_0(x) \qquad F(x, [\frac{2}{3}, 1]) = f_1(x).$$

Since $f_0 \pitchfork M$ and $f_1 \pitchfork M$, we have

$$F \pitchfork_{\mathbf{S}^{n+k} \times (0, \frac{1}{3}] \cup N \times [\frac{2}{3}, 1)} M.$$

By lemma 1 and similar process of above construction, we can approximate F by $F' : \mathbf{S}^{n+k} \times [0,1] \to \mathbf{T}(\xi)$ such that

$$F' \pitchfork_{\mathbf{S}^{n+k} \times (0,1)} \quad \text{and} \quad F'(x,[0,\delta)) = f_0(x), \ F'(x,(1-\delta,1]) = f_1(x),$$

where δ is some positive number less than 1/3. Then $\partial F'(M) = f_1^{-1}(M) - f_0^{-1}(M)$. Hence $[f_0^{-1}(M)] = [f_1^{-1}(M)] \in \Omega_n$.

The final thing is to verify τ is definitely a group homomorphism. It is obvious that the addition in homotopy group corresponds to the disjoint union in cobordism group. Hence we construct the homomorphism. □

Recall in section 1 we have defined the oriented canonical vector bundle $\tilde{\gamma}_p^k := \tilde{\gamma}^k(\mathbb{R}^{k+p})$ over $\tilde{\mathbf{G}}_k(\mathbb{R}^{k+p})$.

Lemma 2. *If $k \geq n$ and $p \geq n$, then the homomorphism $\tau : \pi(\mathbf{T}(\tilde{\gamma}_p^k)) \to \Omega_n$ is surjective.*

Proof. Let M^n be a compact smooth manifold of dimension n. By Whitney embedding theorem, one can embed M^n into $\mathbb{R}^{n+k}$ for some k. Let TN^k be the normal vector bundle of M^n in $\mathbb{R}^{n+k}$ (the superscript indicates that TN is a k-dimensional vector space). By the existence of tubular neighborhood of M^n, there exists a neighborhood U of M^n in $\mathbb{R}^{n+k}$ diffeomorphic to TN^k. Thus

$$U \cong TN^k \xrightarrow{\text{Gauss map}} \tilde{\gamma}_n^k \hookrightarrow \tilde{\gamma}_p^k \xrightarrow{\text{canonical map}} \mathbf{T}(\tilde{\gamma}_p^k).$$

Let $g : U \to \mathbf{T}(\tilde{\gamma}_p^k)$ be the resulting map. There is no doubt that $g \pitchfork M$ and $g^{-1}(\tilde{\mathbf{G}}_k(\mathbb{R}^{k+p})) = M$. Extend g to a map $\hat{g} : \mathbf{S}^{n+k} \to \mathbf{T}(\tilde{\gamma}_p^k)$ by viewing $\mathbf{S}^{n+k} \cong \mathbb{R}^{n+k} \cup \{\infty\}$ and sending $\mathbf{S}^{n+k} - U$ to t_0. Hence

$$[\hat{g}] \in \pi(\mathbf{T}(\tilde{\gamma}_p^k)) \quad \text{and} \quad [M^n] = [\hat{g}^{-1}(\tilde{\mathbf{G}}_k(\mathbb{R}^{k+p}))]$$

□

Corollary 1. *The manifolds*

$$\mathbb{C}\mathbb{P}^{2i_1} \times \cdots \times \mathbb{C}\mathbb{P}^{2i_r},$$

where $(i_1, \cdots, i_r) \in \mathfrak{P}(m)$, are independent, i.e., free of relations, in Ω_{4m}. Hence Ω_{4m} has rank $\geq \mathfrak{p}(m)$.

Theorem 3. *Let X be a finite CW complex which is r-connected with $r \geq 1$. Then we have the isomorphism $\pi_n(X) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q})$ for $n \leq 2r$.*

Theorem 4. (*Thom*)
$\Omega \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4, \mathbb{C}\mathbb{P}^6, \cdots]$.

Proof. By lemma 2, we know that Ω_n is a homomorphic image of $\pi_{n+k}(\mathbf{T}(\tilde{\gamma}_p^k))$. By theorem 3, we have

$$\pi_{n+k}(\mathbf{T}(\tilde{\gamma}_p^k)) \otimes \mathbb{Q} \cong H_{n+k}(\mathbf{T}(\tilde{\gamma}_p^k); \mathbb{Q}).$$

Note that we have the natural isomorphism $H_{n+k}(\mathbf{T}(\tilde{\gamma}_p^k); \mathbb{Q}) \cong H^{n+k}(\mathbf{T}(\tilde{\gamma}_p^k); \mathbb{Q})$. By theorem 1, we have

$$\begin{aligned} \operatorname{rank} \Omega_n &\leq \mathfrak{p}(m) && \text{if } \ n = 4m \\ \operatorname{rank} \Omega_n &= 0 && \text{if } \ 4 \nmid n. \end{aligned}$$

However, corollary 1 tells us that $\operatorname{rank} \Omega_n \geq \mathfrak{p}(m)$ when $n = 4m$. As a result, $\operatorname{rank} \Omega_n = \mathfrak{p}(m)$ when $n = 4m$. In fact, corollary 1 implies more:

$$\mathbb{C}\mathbb{P}^{2i_1} \times \cdots \times \mathbb{C}\mathbb{P}^{2i_r} \qquad (i_1, \cdots, i_r) \in \mathfrak{P}(m)$$

is a set of basis of $\Omega_{4m} \otimes \mathbb{Q}$. Hence the Thom's theorem. □

3 Hirzebruch's Signature Formula

We first recall some properties about homology and cohomology of manifolds.

Fact. *Let F be a field. If M is a n-dimensional compact oriented manifold, then*

$$H^n(M;F) \cong F$$
$$H^m(M;F) = 0 \quad \textit{for all } m > n.$$

Theorem 5. (*Poincaré Duality*)
If M is a n-dimensional oriented compact manifold, then

$$H^k(M;R) = H_{n-k}(M;R)$$

for $k = 0, 1, \cdots, n$, where R can be any coefficient ring.

We consider the pairing

$$H_i(M;F) \times H_{n-i}(M;F) \to F.$$

Since $\dim H^i(M;F) = \dim H_{n-i}(M;F) = \dim H^{n-i}(M;F)$ by Poincaré duality and natural isomorphism of vector spaces, we can view $H^{n-i}(M;F)$ as the dual space of $H^i(M;F)$. In particular, we have

$$H_i(M;\mathbb{Z}) \times H_{n-i}(M;\mathbb{Z}) \to \mathbb{Z}\,.$$

For n being even, we can define a non-degenerate bilinear form on $H_{n/2}(M;\mathbb{Z})$ by

$$\langle x, y\rangle := \tilde{x}(y),$$

where $x, y \in H_{n/2}(M;\mathbb{Z})$ and $\tilde{x} \in H^{n/2}(M;\mathbb{Z})$ is the isomorphic image of x. Note that $\langle\,\cdot,\cdot\,\rangle$ is symmetric if $n/2$ is even an is alternating if $n/2$ is odd.

Definition. (Fundamental Classes)
Let M be a n-dimensional compact oriented manifold. The fundamental (homology) class of M, denoted by μ_M, is the generator of $H(M;\mathbb{Z}) \cong \mathbb{Z}$, which is compatible with the orientation of M.

Now given a compact oriented manifold M of dimension $4n$. For any two $\alpha, \beta \in H^{2n}(M, \mathbb{R})$, we define the pairing

$$\langle \alpha, \beta \rangle := (\alpha \smile \beta)(\mu_M).$$

This pairing is non-degenerate. Recall that deRham theorem tells us that the deRham cohomology is isomorphic singular cohomology, i.e.,

$$H^p_{dR}(M; \mathbb{R}) \cong H^p(M; \mathbb{R}).$$

In some situations, the viewpoint of deRham cohomology is easier to compute than singular cohomology.

Definition. (Signatures of Compact Oriented Smooth Manifolds)
Let M be a compact oriented smooth manifold of dimension n. If $4 \nmid n$, then its signature, denoted by $\sigma(M)$, is defined to be zero. If $n = 4k$, then $\sigma(M)$ is defined to be the signature of the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $H^{2k}(M; \mathbb{R})$.

Remark.
(a) Recall that the signature of a symmetric real bilinear form is the difference of the number of positive eigenvalues and the number of negative eigenvalues.
(b) We now have two bilinear forms: one on homology, the other on cohomology. However, we use the same notation $\langle \cdot, \cdot \rangle$ because the signature of a compact oriented smooth manifold can be defined to be the signature of $\langle \cdot, \cdot \rangle$ on homology or $\langle \cdot, \cdot \rangle$ on cohomology.

For short, $\sigma : \Omega \to \mathbb{Z}$ is a map between rings. As we expected, σ is actually a ring homomorphism. We state and prove this result in the following theorem, which was first presented in Thom's paper.

Theorem 6. (*Thom*)
$\sigma : \Omega \to \mathbb{Z}$ is a ring homomorphism, i.e., σ satisfies
(a) $\sigma(M + N) = \sigma(M) + \sigma(N)$.
(b) $\sigma(M \times N) = \sigma(M) \times \sigma(N)$.
(c) $\sigma(M) = 0$ if $M = \partial W$.

Proof. Let $\dim M = m, \dim N = n$ and $\dim W = m + 1$.

(a) This is obvious.

(b) Let $V := M \times N$. If $4 \nmid \dim V$, then $4 \nmid m$ or $4 \nmid n$. Thus $\sigma(M \times N) = 0$ and $\sigma(M) \times \sigma(N) = 0$.

Now suppose $\dim V = 4k$. By Künneth theorem,

$$H^{2k}(V; \mathbb{R}) \cong \bigoplus_{s+t=2k} H^s(M; \mathbb{R}) \otimes_{\mathbb{R}} H^t(N; \mathbb{R}).$$

Two elements $x, y \in H^{2k}(V; \mathbb{R})$ are said to be orthogonal if $xy(\mu_V) := x \smile y(\mu_V) = 0$. Let $\{v_i^s\}, \{w_j^t\}$ be basis of $H^s(M; \mathbb{R}), H^t(N; \mathbb{R})$ such that $v_i^s v_j^{m-s} = \delta_{ij}, w_i^t w_j^{n-t} = \delta_{ij}$ for $s \neq m/2, t \neq n/2$. Let $A = H^{m/2}(M; \mathbb{R}) \oplus H^{n/2}(N; \mathbb{R})$ if m, n are both even and $A = 0$ for other cases. Define $B := A^{\perp}$ in $H^{2k}(V; \mathbb{R})$. So

$$\{v_i^s \otimes w_j^t \mid s + t = 2k, s \neq \frac{m}{2}, t \neq \frac{n}{2}\}$$

is an orthogonal basis of B.

(c) The coefficient ring of the following diagram is $\mathbb{R}$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{2k+1}(W^{4k+1}, M^{4k}) & \xrightarrow{\partial_*} & H_{2k}(M^{4k}) & \xrightarrow{i_*} & H_{2k}(W^{4k+1}) & \longrightarrow \cdots \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
\cdots \longrightarrow & H_{2k}(W^{4k+1}) & \xrightarrow{i^*} & H^{2k}(M^{4k}) & \xrightarrow{\delta^*} & H^{2k+1}(W^{4k+1}, M^{4k}) & \longrightarrow \cdots
\end{array}$$

Note that $\mathrm{im} i^*$ is of half dimension of $H^{2k}(M)$. For any two cocycles x, y in M^{4k} are obtained by restricting cocycles x', y' in W^{4k+1}, i.e., $i^*(x') = x, i^*(y') = y$. Then

$$\langle\, x, y \,\rangle = (x \smile y)(\mu_M) = i^*(x' \smile y')(\mu_M) = (x' \smile y')i_*(\mu_M).$$

Thus the number of positive and negative eigenvalues are the same. $\square$

Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$ be a graded ring. Define $A^{\Pi} := \{a_0 + a_1 + a_2 + \cdots \mid a_i \in A^i\}$. Particularly, we are interested in $A_1^{\Pi} := \{1 + a_1 + a_2 + \cdots \mid a_i \in A^i\} \subset A^{\Pi}$.

Definition. (Multiplicative Sequences)

Let $x \in A^{\Pi}$. We say that $\{K_n\}_{n \in \mathbb{N}_0}$ is a multiplicative sequence if

(i) each $K_n(x_1, \cdots, x_n)$ is a homogeneous polynomial of degree n.

(ii) $K(ab) = K(a)K(b)$ for any $a, b \in A_1^{\Pi}$, where

$$K(x) := 1 + K_1(x_1) + K_2(x_1, x_2) + \cdots.$$

Proposition 2. *Let A be the graded polynomial ring $R[t]$ where t is a variable of degree 1. Given $f(t) \in 1 + \lambda_1 t + \lambda_2 t^2 + \cdots \in R[[t]]$, there is an unique multiplicative sequence $\{K_n\}$ such that $K(1+t) = f(t)$.*

Proof. By definition, we want to find $\{K_n\}$ satisfying

$$K(1+t) = 1 + K_1(t) + K_2(t,0) + K_3(t,0,0) + \cdots = 1 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 \cdots .$$

That is, we want to find $K_n(x_1, \cdots, x_n)$ whose coefficient of x_1^n-term is λ_n for each n.

We prove the existence of $\{K_n\}$ at first. Fix $n \in \mathbb{N}$. Let $\{t_1, \cdots, t_n\}$ be algebraically independent and all of degree 1. For $I = (i_1, \cdots, i_r) \in \mathfrak{P}(n)$, define $\lambda_I := \lambda_{i_1} \cdots \lambda_{i_r}$. Let $\mathbf{s}_1, \cdots, \mathbf{s}_n$ be the elementary symmetric polynomials of $\{t_1, \cdots, t_n\}$. Note that $\{\mathbf{s}_1, \cdots, \mathbf{s}_n\}$ is also algebraically independent. We claim that

$$K_n(\mathbf{s}_1, \cdots, \mathbf{s}_n) := \sum_{I \in \mathfrak{P}(n)} \lambda_I g_I(\mathbf{s}_1, \cdots, \mathbf{s}_n)$$

is the desired multiplicative sequence. The definition of g_I is as following:

$$g_I(\mathbf{s}_1, \cdots, \mathbf{s}_n) = \sum t_{j_1}^{i_1} \cdots t_{j_r}^{i_r}$$

with $1 \leq j_1, \cdots, j_r \leq n$ all distinct and no "repeated terms". [1] It is clear that we have the formula

$$g_I(ab) = \sum_{HJ=I} g_H(a) g_J(b).$$

Hence $K(ab) = K(a)K(b)$. □

Remark. In the case of proposition 2, we call the multiplicative sequence $\{K_n\}$ belongs to the formal power series $f(t)$.

Now we define the action of multiplicative sequence $\{K_n\}$ on m-dimensional compact oriented smooth manifold M^m. If $4 \nmid m$, define $K(M^m) = 0$. If $m = 4k$, define

$$K(M^4 k) := K_k(p_1, \cdots, p_k)(\mu_M).$$

[1]See chapter 16 of [MS] for the complete definition of g_I. Notice that the notations of [MS] are different from this article.

Theorem 7. (*Hirzebruch's Signature Formula*)
Let $\{L_n\}$ be the multiplicative sequence belonging to

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + \frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}t^n + \cdots ,$$

where B_k denotes the k-th Bernoulli number. Then $\sigma(M^{4k}) = L(M^{4k})$.

Proof. By Thom's cobordism theorem, we only need to check that $\sigma(\mathbb{C}\mathbb{P}^{2k}) = L(\mathbb{C}\mathbb{P}^{2k})$ for each $k \in \mathbb{N}$. We have already computed that $\sigma(\mathbb{C}\mathbb{P}^{2k}) = 1$. To compute $L(\mathbb{C}\mathbb{P}^{2k})$, we recall that $p(\mathbb{C}\mathbb{P}^{2k}) = (1+a^2)^{2k+1}$, where $a := -c_1(\gamma^1)$ with γ_1 the canonical line bundle of $\mathbb{C}\mathbb{P}^{2k}$. By definition,

$$L(1 + a^2 + 0 + \cdots) = \frac{\sqrt{a^2}}{\tanh\sqrt{a^2}} \qquad \Longrightarrow \qquad L(p(\mathbb{C}\mathbb{P}^{2k})) = \left(\frac{a}{\tanh a}\right)^{2k+1}.$$

Now we replace a by a complex variable z. We want to compute the coefficient of z^2k in the Taylor expansion of $\left(\frac{z}{\tanh z}\right)^{2k+1}$. The substitution $u = \tanh z$ with

$$dz = \frac{du}{1-u^2}$$

gives

$$\frac{1}{2\pi i}\int \frac{dz}{(\tanh z)^{2k+1}} = \frac{1}{2\pi i}\int \frac{1+u^2+u^4+\cdots}{u^{2k+1}}du = 1.$$

Hence $L(\mathbb{C}\mathbb{P}^2 k) = 1$. □

4 Construction of Exotic 7-Spheres

We recall some definitions and properties in Morse theory.

Definition. (Morse Function)
A Morse function f on a manifold M is a real-valued function whose critical points (the points where the first derivative of f vanishes) are all non-degenerate, i.e., its Hessian matrix is non-singular.

Note that the index of a non-degenerate critical point y of f is the dimension of the largest subspace of the tangent space of M at y on which the Hessian is negative definite.

Lemma 3. (*Morse Lemma*)
Let y be a non-degenerate critical point of $f : M^n \to \mathbb{R}$ with index α. Then there exists a chart $(x_1, x_2, \cdots, x_n)$ in a neighborhood U of y such that

$$x_i(y) = 0 \quad \forall 1 \leq i \leq n$$

and

$$f(x) = f(b) - x_1^2 - \cdots - x_\alpha^2 + x_{\alpha+1}^2 + \cdots + x_n^2 \quad \forall x \in U \setminus \{y\}.$$

Recall that we have introduced the SO(4)-vector bundles over $\mathbf{S}^4$ of rank 4. Let ξ_{hj} be the vector bundle of rank 4 defined by $f_{hj} : \mathbf{S}^3 \to \mathrm{SO}(4)$ where $f_{hj}(v)w := v^h w v^j$ (By viewing $v \in \mathbf{S}^3 \hookrightarrow \mathbb{H}$, the multiplication $v^h w v^j$ is doing in $\mathbb{H}$). Let $\mathbf{S}_{hj}$ be the sphere bundle of ξ_{hj}. Now we are ready to construct the exotic seven spheres.
Idea: Suppose $h + j = 1$.We will show that $M_k^7 := \mathbf{S}_{hj}$ is a topological 7-sphere by constructing a Morse function on M_k^7, where k is an odd number and assume $h - j = k$. Finally, if we can show that M_k^7 is not diffeomorphic to standard $\mathbf{S}^7$, then we complete the construction of exotic 7-spheres by explicitly constructing a exotic 7-sphere, M_k^7. The final step will be carried out by computing characteristic classes.

Step 1. If $f : M_k^7 \to \mathbb{R}$ is a Morse function with two critical points. Let y_0, y_1 be the two critical points. Since M_k^7 is compact, the two critical points are actually the maximun and minimun of f. By rescaling the function f, we may assume $f(y_0) = 0$ and $f(y_1) = 1$. Now consider the gradient flow:

$$\frac{d\mathbf{x}}{dt} = \nabla f(\mathbf{x}).$$

Note that this flow is orthogonal to each level set $f^{-1}(a)$ for $a \in (0,1)$. Hence we have $f^{-1}([0,a]) \cong f^{-1}([0,b])$ for each $a, b \in (0,1)$. For a sufficiently small, it follows from Morse lemma that there exists a chart $(x_1, \cdots, x_7)$ of neighborhood $f^{-1}([0,a])$ of y_0 such that

$$f(\mathbf{x}) = x_1^2 + \cdots + x_7^2.$$

That is, $f^{-1}([0,a]) \cong \mathbf{D}^7$. Therefore, $f^{-1}([0,1)) = M_k^7 - \{y_1\} \cong \mathbf{D}^7$. Hence $7_k \cong \mathbf{S}^7$ topologically.

Step 2. Now our mission is to construct a Morse function and apply step 1 to conclude that $7_k \cong \mathbf{S}^7$ topologically. To construct a Morse function on M_k^7, we have to realize M_k^7 as a more understandable structure. We will check in this step that M_k^7 can be realized as gluing two copies of $\mathbb{R}^4 \times \mathbf{S}^3$ along $(\mathbb{R}^4 - \{0\}) \times \mathbf{S}^3$ via a diffeomorphism g of $(\mathbb{R}^4 - \{0\}) \times \mathbf{S}^3$ given by

$$g : (u, v) \to (u', v') \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|} \right).$$

(Notice that the operation are done in $\mathbb{H}$ for $\mathbb{R}^4 - \{0\} \hookrightarrow \mathbb{H}$ and $\mathbf{S}^3 \hookrightarrow \mathbb{H}$.) This map is well-defined because of $h + j = 1$. Now take the case $u = u'$ into consideration. In this case, we get a restricted map of g on $\mathbf{S}^3$:

$$\tilde{g} := \mathbf{S}^3 \to \mathrm{SO}(4)$$

$$\tilde{g}(u)v := u^h v u^j.$$

This is exactly the map that we define M_k^7. Hence the result.

Step 3. From step 2, we have two coordinate charts (u, v) and (u'', v'). In this step, we will verify that

$$f(u, v) = \frac{Re(v)}{\sqrt{1 + |u|^2}} = \frac{Re(u'')}{\sqrt{1 + |u''|^2}} \quad \text{where } u'' := u'(v')^{-1} = \frac{u}{|u| \cdot u^h v u^j}$$

is our desired Morse function on M_k^7, i.e., it has two non-degenerate critical points. First of all, direct computation shows that the second equality holds. It is clear that our definition of f forces f to be an increasing function in the first variable under the chart (u'', v'). As a result, the critical points lie in the chart (u, v), and thus the critical points look like $(0, v)$. However, f reduce to be height function of $\mathbf{S}^3$ in this case. Hence the critical points are $(0, 1)$ and $(0, -1)$.

Step 4. We want to find some condition of k that will make M_k^7 not diffeomorphic to standard $\mathbf{S}^7$. Now suppose that M_k^7 is diffeomorphic to standard $\mathbf{S}^7$. We can attach a standard 8-dimensional disk to $\mathbf{D}(\xi_{hj})$ along M_k^7 because we assume that $M_k^7 \cong \mathbf{S}^7$. Denote the resulting 8-dimensional space by W_k^8. By our construction, $W_k^8 \cong \mathbf{T}(\xi_{hj})$. As a consequence,

$$H^i(\mathbf{S}^4) \cong H^{4+i}(\mathbf{D}(\xi_{hj}), \mathbf{S}^{hj}) \cong H^{4+i}(\mathbf{T}(\xi_{hj}), t_0).$$

This implies

$$\begin{aligned} H^i(W_k^8) &\cong \mathbb{Z} && \text{if } i = 0, 4, 8 \\ H^i(W_k^8) &= 0 && \text{if } i \neq 0, 4, 8. \end{aligned}$$

So the signature of W_k^8 equals 1 or -1 up to our choice of the orientation of W_k^8. Assume $\sigma(W_k^8) = 1$. Now Hirzebruch's signature formula gives $1 = \dfrac{7p_2 - p_1^2}{45}$. Our task now turns to compute the Pontryajin classes of W_k^8. Recall that we have computed in section 1 that $e(\xi_h j) = \mathbf{u}$ and $p_1(\xi_{hj}) = 2k\,\mathbf{u}$, where $\mathbf{u} := e(\gamma_{1\,\mathbb{R}}) \in H^4(\mathbb{H}\,\mathbb{P}^1, \mathbb{Z})$ ($\gamma_{1\,\mathbb{R}}$ is the canonical line bundle of $\mathbb{H}\,\mathbb{P}^1$). Let $\pi : \xi_{hj} \to \mathbf{S}^4$ be the canonical projection. We have $T\xi_{hj} \cong \pi^*(T\,\mathbf{S}^4) \oplus \pi^*(\xi_{hj})$, and then apply Whitney sum formula and naturality of characteristic classes to obtain (note that $p(T\,\mathbf{S}^4) = 1$)

$$\begin{aligned} p(T\xi_{hj}) &= \pi^* p(\xi_{hj}) \\ p_1(T\xi_{hj}) &= \pi^* p_1(\xi_{hj}) = \pi^*(2k\,\mathbf{u}) = 2k\,\mathbf{u} = 2ke(\xi_{hj}). \end{aligned}$$

Hence $p_1^2(TW_k^8) = p_1^2(T\xi_{hj}) = 4k^2$. Finally, write $4k^2 + 45 = 7p_2 \equiv 0 \pmod 7$ and use this to conclude $k \equiv 0 \pmod 7$. However, our assumption is that k is any odd number. That is, the equality $k \equiv 0 \pmod 7$ fails to hold for any odd k. Hence we conclude that our original hypothesis is wrong: M_k^7 is not diffeomorphic to standard $\mathbf{S}^7$.

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Note.

The outline of this article is mainly based on [Wa], where some contents are referred to other books.