Our goal is to develop a method to compute cohomology algebra and rational homotopy group of fiber bundles.

1 CW complex, cellular homology/cohomology

Definition 1. (Attaching space with maps)
Given topological spaces X, Y, closed subset $A \subset X$, and continuous map $f : A \to y$. We define

$$X \cup_f Y \triangleq X \sqcup Y / \sim$$

where $x \sim y$ if $x \in A$ and $f(x) = y$. In the case $X = D^n$, $A = \partial D^n = S^{n-1}$, $D^n \cup_f X$ is said to be obtained by attaching to X the cell (D^n, f).

Proposition 1. If $f, g : S^{n-1} \to X$ are homotopic, then $D^n \cup_f X$ and $D^n \cup_g X$ are homotopic.

Proof. Let $F : S^{n-1} \times I \to X$ be the homotopy between f, g. Then in fact

$$D^n \cup_f X \sim (D^n \times I) \cup_F X \sim D^n \cup_g X$$

□

Definition 2. (Cell space, cell complex, cellular map)

1. A cell space is a topological space obtained from a finite set of points by iterating the procedure of attaching cells of arbitrary dimension, with the condition that only finitely many cells of each dimension are attached.

2. If each cell is attached to cells of lower dimension, then the cell space X is called a cell complex. Define the $n-$skeleton of X to be the subcomplex consisting of cells of dimension less than n, denoted by X_n.

3. A continuous map f between cell complexes X, Y is called cellular if it sends X_k to Y_k for all k.

Proposition 2. 1. Every cell space is homotopic to a cell complex.

2. Every continuous map between cell complexes is homotopic to a cellular map.

Proof. (a)Given $\sigma^k = (D^k, f)$ is attached to X. Since $\partial D^k = S^{k-1}$ is compact, $f(S^{k-1})$ is covered by interior of finitely many cells.

$$f(S^{k-1}) \subset \cup_i (\sigma_i)^\circ$$

Say σ_l has the highest dimension among $\{\sigma_i\}$. If $l > k-1$, then by Sard's theorem, image of f has measure zero in $(\sigma_l)^\circ$. Hence can construct a homotopy $f \sim \tilde{f}$, with $\tilde{f}(S^{k-1}) \cap \sigma_l = \tilde{f}(S^{k-1}) \cap \partial\sigma_l$. Repeat in this way we see f is homotopic to $\tilde{f}$ with $\tilde{f}(S^{k-1}) \subset X_{k-1}$.
(b) Similar. □

Definition 3. (Cellular chain complex)
Define cellular chain as the following

$$C_k^{cell}(X;G) \triangleq \{\sum_l g_l \sigma_l^k \ \mid \ g_l \in G, \sigma_l^k \text{ are cells of dimension } k\}.$$

Also define the boundary operator

$$\begin{aligned} \partial : C_k(X;G) &\to C_{k-1}(X;G) \\ \sigma^k &\mapsto \sum_i [\sigma^k, \sigma_i^{k-1}]\sigma_i^{k-1} \end{aligned}$$

where $[\sigma^k, \sigma_i^{k-1}]$ is called the incidence number, defined to be the degree of the composite map F_i $(\sigma^k = (D^k, f))$

$$\begin{tikzcd} S^{k-1} \arrow[r, "f"] \arrow[rrrd, "F_i"'] & X_{k-1} \arrow[r] & X_{k-1}/X_{k-2} \arrow[r, "\wr"] & \bigvee S_i^{k-1} \arrow[d] \\ & & & S_i^{k-1} \end{tikzcd}$$

where $\bigvee S_i^{k-1}$ is a bouquet of sphere. Each sphere corresponds to a $k-1$ cell. Extend ∂ to be a $G-module$ homomorphism.

Proposition 3.

$$\partial \circ \partial = 0$$

Proof. Assume the following two facts without proof.

1. $f, g : M^n \to S^n$, $f \sim g \Leftrightarrow \deg f = \deg g$

2. If $A \subset X$ is a closed subset, we have a long exact sequence of homotopy group

$$\cdots \to \pi_k(A) \to \pi_k(X) \to \pi_k(X, A) \to \pi_{k-1}(A) \to \cdots$$

First we construct a map $\alpha : \pi_k(X_k, X_{k-1}) \twoheadrightarrow C_k(X; \mathbb{Z})$. Given $f : (D^k, S^{k-1}) \to (X_k, X_{k-1})$, let F_i be the composite map

$$\begin{tikzcd}
D^k \arrow[r, "f"] \arrow[drrr, "F_i"'] & X_l \arrow[r] & X_k/X_{k-1} \arrow[r, "\wr"] & \bigvee S_i^k \arrow[d] \\
 & & & S_i^k
\end{tikzcd}$$

We define $\alpha([f]) = (\deg F_i)\sigma_i^k$. Note that $\sigma^k = \alpha([\sigma^k])$.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(X_{k-1}) \arrow[r] & \pi_k(X_k)\pi_k(X_k, X_{k-1}) \arrow[r, "\hat{\partial}"] & \pi_{k-1}(X_{k-1}) \arrow[r] \arrow[d, "j"] & \cdots \\
 & & & \pi_{k-1}(X_{k-1}, X_{k-2}) \arrow[d, "\hat{\partial}"] & \\
 & & & \pi_{k-1}(X_{k-1}) &
\end{tikzcd}$$

By definition

$$\partial\sigma^k = \alpha(j \circ \hat{\partial}([\sigma^k]))$$
$$\partial\partial\sigma^k = \alpha(j \circ \hat{\partial} \circ j \circ \hat{\partial}([\sigma^k]))$$

where $j \circ \hat{\partial} = 0$ by exactness. $\square$

Definition 4. (Cellular homology, cohomology)

The cellular cycle, boundary, homology is defined as the following

$$Z_n^{cell} \triangleq \ker \partial_n$$
$$B_n^{cell} \triangleq \mathrm{Im}\partial_{n+1}$$
$$H_n^{cell} \triangleq Z_n/B_n$$

For cellular cohomology, we define cellular cochain

$$C^k(X; G) \triangleq \{\text{linear map from } C_k(X; G) \text{ to } G\}$$

$$\delta : C^k(X;G) \to C^{k+1}(X;G)$$
$$\alpha \mapsto \alpha \circ \partial$$

$$Z^n_{cell} \triangleq \ker \delta_n$$
$$B^n_{cell} \triangleq \mathrm{Im}\delta_{n+1}$$
$$H^n_{cell} \triangleq Z^n/B^n$$

In fact cellular homology/cohomology is isomorphic to the singular one, this can be proved by induction of dimension and the long exact sequence of singular homology.

$$\cdots \to H_i(X_k) \to H_i(X_{k+1}) \to H_i(X_{k+1}, X_k) \to H_{i-1}(X_k) \to \cdots$$

I will give another proof using spectral sequence in the following section.

2 Homology/cohomology of product space, cup product

If K_1, K_2 are cell complexes, write $K_1 = \cup\sigma^k,\ K_2 = \cup\sigma^l$. Then $K_1 \times K_2$ has cell decomposition $\cup\sigma^k \times \sigma^l$. Since we can identify $(D^k, f) \sim (I^k, f)$, $(I^k, f) \times (I^l, g) \sim (I^{k+l}, F)$ for some F defines on ∂I^{k+l} appropriately. From the construction of F we see

$$\partial(\sigma_1^i \times \sigma_2^j) = (\partial\sigma_1^i) \times \sigma_2^j + (-1)^i \sigma_1^i \times (\partial\sigma_2^j)$$

Proposition 4. (Kunneth formula)

For G arbitrary ring, F arbitrary field,

$$C_k(K_1 \times K_2; G) \cong \sum_{m+n=k} C_m(K_1;G) \otimes C_n(K_2;G)$$
$$H_k(K_1 \times K_2; F) \cong \sum_{m+n=k} H_m(K_1;F) \otimes H_n(K_2;F)$$
$$\sigma^m \times \sigma^l \hookleftarrow \sigma^m \otimes \sigma^l$$
$$H^k(K_1 \times K_2; F) \cong \sum_{m+n=k} H^m(K_1;F) \otimes H^n(K_2;F)$$
$$\widehat{\sigma^m \times \sigma^l} \hookleftarrow \hat{\sigma^m} \otimes \hat{\sigma^l}$$

When F is not a field, the map is still defined.

Proof. The first isomorphism is illustrated above.

For the second isomorphism, we choose a 'good' basis for the vector space C_k. We can always find $\{x_{k,i}, y_{k,j}, h_{k,l}\}$ basis for C_k, such that

$$\partial x_{k,i} = y_{k-1,i}, \qquad \partial_{k,j} = 0, \qquad \partial h_{k,l} = 0$$

I.e.

$$\begin{aligned} \{y_{k,j}, h_{k,l}\} &\text{ basis for } Z_k \\ \{y_{k,j}\} &\text{ basis for } B_k \\ \{\overline{h_{k,l}}\} &\text{ basis for } H_k \end{aligned}$$

For $\sum_{k+l=m} H_k(K_1; F) \otimes H_l(K_2)$, we have basis

$$\begin{array}{cccc} x_k \otimes x_l, & y_k \otimes x_l + (-1)^k x_k \otimes y_{l-1}, & x_k \otimes h_l, \quad h_k \otimes x_l \\ y_k \otimes x_l + (-1)^k x_{k+1} \otimes y_{l-1}, & (-1)^{k+1}(y_{k+1} \otimes y_{l-1} + y_{k+1} \otimes y_{l-1}), & y_k \otimes h_l, \quad h_k \otimes y_l, \quad h_k \otimes h_l \end{array}$$

Hence $\{h_k \otimes h_l\}$ form basis for $H(K_1 \times K_2)$. Thus the isomorphism. Cohomology case is similar. It is easy to check the map is still defined when F is not a field.

$\square$

Now we can introduce multiplicative structure on $H^*(X; R)$. Let $\triangle$ be the diagnoal map

$$\begin{aligned} \triangle : X &\to X \times X \\ x &\mapsto (x, x) \end{aligned}$$

Define cup product to be the composite of the following maps

$$\begin{tikzcd} H^k(X;R) \times H^l(X;R) \arrow[r] \arrow[drr, "\smile"] & H^k(X;R) \otimes H^l(X;R) \arrow[r] & H^{k+l}(X \times X) \arrow[d, "\triangle^*"] \\ & & H^{l+k}(X) \end{tikzcd}$$

Proposition 5. 1. Cup product is associative

2. $\alpha^i \smile \beta^j = (-1)^{ij}\beta \smile \alpha$

3. There is a multiplicative identity.

Proof. 1. Define $\triangle_1, \triangle_2 : X \times X \to X \times X \times X$.

$$\triangle_1 : (x, y) \mapsto (x, x, y)$$
$$\triangle_2 : (x, y) \mapsto (x, y, y)$$

$$\begin{tikzcd}
H^k(X)\otimes H^l(X)\otimes H^m(X) \arrow[r] & H^{k+l+m}(X\times X\times X) \arrow[dl, "\triangle_1^*"] \arrow[dr, "\triangle_2^*"] & \\
H^{k+l+m}(X\times X) \arrow[dr, "\triangle^*"] & & H^{k+l+m}(X\times X) \arrow[dl, "\triangle^*"] \\
& H^{k+l+m}(X) &
\end{tikzcd}$$

While $\triangle_1^* \circ \triangle^* = \triangle_2^* \circ \triangle^*$ since $\triangle \circ \triangle_1 = \triangle \circ \triangle_2$

2. It is just orientation.

3.If X is connected, then we can assume there is only one $0-$cell $*$.Let

$$p : X \times X \to X$$
$$(x, y) \mapsto x$$

Then $p \circ \triangle = \mathrm{id}$

$$\widehat{\sigma^i} = (p \circ \triangle)^*\widehat{\sigma^i} = \triangle^*(p^*\widehat{\sigma^i}) = \triangle^*(\widehat{\sigma^i \times *})$$

□

3 Spectral sequence

In the section above, we know how to get cell decomposition and homology/cohomology from those of base space and fiber space if the fiber bundle is trivial. For general fiber bundles, we can still derive their cell decomposition from the bases and fibers. Since each cell is contractible, every fiber bundle over a cell is trivial. If the base space B and the fiber space F have cell decomposition $B = \cup\sigma_B^k, F = \cup\sigma_F^l$, then the total space E has cell decomposition $E = \cup \sigma_B^k \times \sigma_F^l$. While in this case we do not have the formula

$$\partial(\sigma_E^{q+j}) = \partial(\sigma_B^i \times \sigma_F^j) = (\partial\sigma_B^i) \times \sigma_F^j + (-1)^i \sigma_B^i \times (\partial\sigma_F^j)$$

In fact if the base space is simply-connected we will have

$$\partial(\sigma_E^{q+j}) = (\partial\sigma_B^i) \times \sigma_F^j + (-1)^j \sigma_B^i \times (\partial\sigma_F^j) + \partial_2\sigma_E^{q+j} + \partial_3\sigma_E^{q+j} + \cdots$$

where $\partial_k\sigma_E^{q+j}$ is a linear combination of the product of the $(q-k)-$dimensional cell of the space and $(j+k-1)-$dimensional cell of the fiber. In this chapter we introduce spectral sequence to deal with this case.

Definition 5. (filtered complex)

Given a chain complex C_*, a filtration is an increasing sequence of subset of C_*

$$\cdots \subset F_{p-1}C_i \subset F_pC_i \subset F_{p+1}C_i \subset \cdots \subset C_i$$

With the boundary operator ∂ sends F_pC_i to F_pC_{i-1}. (F_pC_*, ∂) is called a filtered complex.

In our case, we take C_* to be the sellular chain complex of fiber bundle E.

$$F_pC_i \triangleq \{\sigma \in C_i(E) \mid \pi(\sigma) \subset B_p\}$$

where π is the projection map, B_p is the $p-$skeleton of the base space B.

Let $G_pC_i \triangleq F_pC_i/F_{p-1}C_i$, $\partial : G_pC_i \to G_pC_{i-1}$ is well-defined. Define

$$\begin{aligned} E_{p,q}^0 &\triangleq G_pC_{p+q} = F_pC_{p+q}/F_{p+q-1} \\ \partial_0 &\triangleq \partial : E_{p,q}^0 \to E_{p,q-1}^0 \\ E_{p,q}^1 &\triangleq \frac{\ker\partial_0}{\operatorname{Im}\partial_0} \end{aligned}$$

In fact,

$$E_{p,q}^1 = \frac{\{x \in F_pC_{p+q} \mid \partial x \in F_{p-1}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial(F_pC_{p+q+1})}$$

Define ∂_1 on $E_{p,q}^1$ as following. For representative $\beta \in F_pC_{p+q}$, $\partial\beta \in F_{p-1}C_{p+q-1}$,

$$\begin{aligned} \partial_1 : E_{p,q}^1 &\to E_{p-1,q}^1 \\ [\beta] &\mapsto [\partial\beta] \end{aligned}$$

It is well-defined and $\partial_1 \partial_1 = 0$. Define $E^2_{p,q} = \frac{\ker \partial_1}{\operatorname{Im} \partial_1}$. Can define ∂_2 and so on.

Theorem 1. Let $(F_p C_*, \partial)$ be a filtered complex, define

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p-r} C_{p+q-1})}$$

here the fraction $\frac{A}{B}$ means $\frac{A}{A \cap B}$.

1. ∂ induces $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, $\partial_r \partial_r = 0$

2. E^{r+1} is the homology chain complex (E^r, ∂_r). I.e.

$$E^{r+1}_{p,q} = \frac{\ker(\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{Im}(\partial_r : E^r_{p-r,q-r+1} \to E^r_{p,q})}$$

3. $E^1_{p,q} = H_{p+q}(G_p C_*)$

4. If the filtration of C_i is bounded for each i. I.e. $F_{-s} = 0, F_s = C_i$ for $s >> 0$. Then for r sufficiently large

$$E^r_{p,q} = G_p H_{p+q}(C_*)$$

   And

$$\sum_{p+q=k} E^r_{p,q} = H_k(C_*)$$

Proof. The proof is straight forward check and is notationally tedious. If you have problem, feel free to ask me for help. □

Similarly for cohomology case. Run through the proof of the theorem, we'll see in our case we have the following theorem.

Theorem 2. Given fiber bundle
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$
with B, F are cell complexes and B is simply-connected. We have the following tool to compute homology/cohomology with coefficient is a field.

Homology:

1. (Define) $E^2_{p,q} = H_p(B) \otimes H_q(F)$

2. There is $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ induced by boundary operator on E.

$$E^{r+1}_{p,q} = \frac{\ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d_r : E^r_{p-r,q-r+1} \to E^r_{p,q})}$$

3. For $r >> 0$,

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots \triangleq E^\infty_{p,q}, \quad \sum_{p+q=k} E^\infty_{p,q} = H_k(E)$$

Cohomology:

1. (Define) $E^{p,q}_2 = H^p(B) \otimes H^q(F)$

2. There is $d^*_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$, $(d^*_r)^2 = 0$, $E^*_{r+1} = H^*(E^*_r, d^*_r)$

3. For $r >> 0$,

$$E^{p,q}_r = E^{p,q}_{r+1} = \cdots \triangleq E^{p,q}_\infty, \quad \sum_{p+q=k} E^{p,q}_\infty = H^k(E)$$

4. Given cup product on $C^*(E)$, it induces sup product on $E^{p,q}_r$.

$$d^*_r(\alpha \smile \beta) = (d^*_r) \smile \beta \pm \alpha \smile (d^*_r\beta)$$

1 Example for computing cohomology ring by using spectral sequence

Recall the last theorem in the previous note.

Theorem 1. Given fiber bundle
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$
with B, F are cell complexes and B is simply-connected. We have the following tool to compute homology/cohomology with coefficient is a field.

Homology:

1. $E^2_{p,q} = H_p(B) \otimes H_q(F)$
2. There is $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ induced by boundary operator on E.
$$E^{r+1}_{p,q} = \frac{\ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d_r : E^r_{p-r,q-r+1} \to E^r_{p,q})}$$
3. For $r >> 0$,
$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots \triangleq E^\infty_{p,q}, \quad \sum_{p+q=k} E^\infty_{p,q} = H_k(E)$$

Cohomology:

1. $E_2^{p,q} = H^p(B) \otimes H^q(F)$
2. There is $d^*_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$, $(d^*_r)^2 = 0$, $E^*_{r+1} = H^*(E^*_r, d^*_r)$
3. For $r >> 0$,
$$E^{p,q}_r = E^{p,q}_{r+1} = \cdots \triangleq E^{p,q}_\infty, \quad \sum_{p+q=k} E^{p,q}_\infty = H^k(E)$$
4. Given cup product on $C^*(E)$, it induces sup product on $E^{p,q}_r$.
$$d^*_r(\alpha \smile \beta) = (d^*_r) \smile \beta \pm \alpha \smile (d^*_r\beta)$$

Furthermore, we can derive the following fact for homology from the proof.

1. $i_* : H_n(F) \to H_n(E)$ may be computed as follows:

$$H_n(F) = E^2_{0,n} \to E^r_{0,n} = E^\infty_{0,n} \subset H_n(E)$$

2. $p_* : H_n(E) \to H_n(B)$ may be computed as follows:

$$H_n(E) = E^\infty_{n,0} = E^r_{n,0} \subset E^2_{n,0} \subset H_n(B)$$

We give some examples.

1.1 $H^*(\mathbb{CP}^n)$

consider the fiber bundle

$$\begin{tikzcd} S^1 \arrow[r] & S^{2n+1} \arrow[d] \\ & \mathbb{CP}^n \end{tikzcd}$$

Since $E_2^{p,q} \cong H^p(\mathbb{CP}^n) \otimes H^q(S^1)$, $E_2^{p,q} = 0$ for $q > 1$. Hence $E_3^{p,q} = 0$ for $q > 2$. Thus $d_r^* = 0$ for $r \geq 3$. Thus

$$E_\infty^{p,q} = E_3^{p,q}, \quad H^k(S^{2n+1}) = E_3^{k,0} + E_3^{k-1,1}$$

This implies $E_3^{k,0} = E_3^{k-1,1} = 0$ for $k \neq 0, 2n+1$.

Note $E_2^{1,0}$ is in kernel of d_2^* and not in image of it. Hence $H^1(\mathbb{CP}^1) \cong E_2^{1,0} = E_3^{1,0} = 0$. This gives $E_2^{1,1} \cong H^1(\mathbb{CP}^1) \otimes H^1(S^1) = 0$. Therefore $E_2^{3,0}$ is not in image and hence equal to zero. Repeat in this way we see

$$H^{2k+1}(\mathbb{CP}^n) = 0.$$

Let u denotes a generator of $E_2^{0,1}$. $d_2^*(u)$ must be nonzero since $E_3^{1,0} = 0$. Say $d_2^*(u) = v$. By $E_3^{2,0} = 0$ we see $E_2^{2,0} = \langle v \rangle$.

$$d_2^*(u \otimes v) = d_2^*(u)v \pm u d_2^*(v) = v^2$$

Similarly $v^2 = d_2^*(u \otimes v) \neq 0$ if $2n \neq 2$, and $E_2^{4,0} = \langle v^2 \rangle$. Repeat in this way,

$$H^{2k}(\mathbb{CP}^n) = \langle v^k \rangle, \quad \text{for } k \leq n$$

When $k = n+1$ we have

$$H^{2n+1}(S^{2n+1}) = E_3^{2n+1,0} + E_3^{2n,1} = E_3^{2n,1}$$

So $0 = d_2^*(u \otimes v^n) = v^{n+1}$.

Conclusion:

$$H^k(\mathbb{CP}^n) = \begin{cases} \langle v^j \rangle & \text{, for } k = 2j,\ 0 \le j \le n \\ 0 & \text{, otherwise.} \end{cases}$$

1.2 $H^*(SU(n); \mathbb{R})$

First we have $SU(2) \cong S^3$, thus $H_0(SU(2)) = H_3(SU(2)) = \mathbb{R}$, $H^0(SU(2)) = H^3(SU(2)) = \mathbb{R}$. For $SU(n)$, $n \ge 3$, we need the following fact.

Fact: The cohomology algebra of a (finite-dimensional) connected Lie group is a finitely generated free exterior algebra $\bigwedge[y_1, \cdots y_n]$.

This fact comes from the group structure of Lie groups (or generally an H-space).

$$\begin{aligned} \psi : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

It induces

$$\psi^* : H^*(G) \to H^*(G) \otimes H^*(G)$$

After some purely algebraic argument we can obtain the fact. Now we use this fact to compute $H^*(SU(3))$. We can let $SU(3)$ acts on a five-dimensional sphere in $\mathbb{C}^3$, with isotropy group $SU(2)$. This give us a fiber bundle

$$\begin{array}{ccc} SU(2) & \longrightarrow & SU(3) \\ & & \downarrow \\ & & S^5 \end{array}$$

We have cell decomposition $SU(2) = \sigma_F^0 \cup \sigma_F^3$, $S^5 = \sigma_B^0 \cup \sigma_B^5$. Thus $SU(3)$ has cell decomposition of $SU(3)$

$$\sigma^0 = \sigma_B^0 \times \sigma_F^0,\quad \sigma^3 = \sigma_B^0 \times \sigma_F^3,\quad \sigma^5 = \sigma_B^5 \times \sigma_F^0,\quad \sigma^8 = \sigma_B^5 \times \sigma_F^3$$

Thus the cohomology group is $\mathbb{R}$ in degree $0, 3, 5, 8$. from the fact we know $H^*(SU(3)) = \bigwedge[y_3, y_5]$ where y_3, y_5 are generators of $H^3(SU(3)), H^5(SU(3))$.

Claim 1. $H^*(SU(n)) = \bigwedge[y_3, y_5, \cdots, y_{2n-1}]$.

Proof. Use induction on n. Let $SU(n)$ acts on a $(2n-1)-$dimensional sphere in $\mathbb{C}^n$, we get fiber bundle

$$\begin{tikzcd} SU(n-1) \arrow[r] & SU(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

Let's see $E_2^{p,q} \cong H^p(B) \otimes H^q(F)$. Since $H^k(S^{2n-1}) = 0$ for $1 < k < 2n-1$, $E_2^{k,q} = 0$ for $1 < k < 2n-1$. So $d_r^* : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are all zero for $1 < r < 2n-2$, so every nonzero element is in kernel and not in image of d_r^*, thus $E_r^{p,q} = E_2^{p,q}$ for $2 \le r \le 2n-1$.

Next see $d_{2n-1}^* : E_{2n-1}^{0,2n-2} \to E_r^{2n-1,0}$ or equivalently $E_2^{0,2n-2} \to E_r^2$. By induction hypothesis we know every element in $E_2^{0,2n-2} = H^{2n-2}(SU(n-1))$ is linear combination of $\alpha \smile \beta$. By the Leibniz rule of d_{2n-1}^* and triviality of d_{2n-1}^* on $E_{2n-1}^{0,k}$ for $k < 2n-2$, we have $d_{2n-1}^* : E_{2n-1}^{0,2n-2} \to E_r^{2n-1,0}$ is trivial, and similarly $d_{2n-1}^* : E_{2n-1}^{p,q} \to E_{2n-1}^{p+r,q-r+1}$ is trivial for all p, q. Thus

$$E_\infty^{p,q} = E_2^{p,q}$$

And let y_{2n-1} denote a generator of $E_2^{2n-1,0}$.

$$E_2^{p,q} = \begin{cases} H^q(SU(2n-1)) & \text{, for } p = 0 \\ \langle y_{2n-1} \rangle_{\mathbb{R}} \otimes H^q(SU(2n-1)) & \text{, for } p = 2n-1 \\ 0 & \text{, otherwise} \end{cases}$$

Hence

$$H^*(SU(n)) = \sum E_\infty^{p,q} = \sum E_2^{p,q} = \bigwedge[y_3, y_5, \cdots y_{2n-1}]$$

$\square$

2 Eilenberg-Maclane Space

Definition 1. (Eilenberg-Maclane space)

We say a topological space X is a $K(D,n)$ if

$$\pi_i(X) = \begin{cases} D & \text{, if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

If D is a finitely generated abelian group, then such space always exists. In fact we can construct a cell complexes which is $K(D,n)$ as follows.

Step1.

Assume D is generated by $\{a_1, \cdots a_k\}$. Let K^0 is a single point. Attach k $n-$cell to K^0 to get $K^n \cong \bigvee_{i=1}^k S_i^n$. Next for each relation $\sum b_i a_i = 0$, attach a $(n+1)-$cell to K^n with the attaching map compose with projection to S_i^n has degree b_i. We get a $(n+1)-$dimensional complex K^{n+1}, with $\pi_n(K^{n+1}) = D$, $\pi_i(K^{n+1}) = 0$ for $i < n$.

Step2.

Just as we attach cells to kill the relation in $\pi_n(K^n)$, now we attach cells to kill π_j for $j > n$. What needs to be checked is after we attach $(n+2)-$cell to K^{n+1}, the $n-$th homotopy group is invariant. But this holds by Sard's theorem.

Proposition 1. We have several facts:

1. If cell complexes X, Y are $K(D,n)$, then $X \sim Y$.
2. $K(D_1, n) \times K(D_2, n) = K(D_1 \times D_2, n)$.
3. $\Omega(K(D,n)) = K(D, n-1)$ for $n > 1$. Where $\Omega(X)$ is the loop space of X. (We can omit the base point since the fundamental group is trivial).

Proof. We skip the proof of 1. here. We'll go back to this proposition later.

2. comes from

$$\pi_i(X \times Y) \cong \pi_i(X) \times \pi_i(Y)$$

3. comes from the long exact sequence of homotopy group of fibration. Fix a base point $x_0 \in X$. Let E be the space of paths in X start from x_0 . Let $\pi : E \to X$ sends each path to its endpoint. Then this becomes a fibration with fiber being loop space. Since E is contractible, $\pi_i(E) = 0$.

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(E) \arrow[r] \arrow[d, equal] & \pi_k(B) \arrow[r, "\cong"] & \pi_{k-1}(F) \arrow[r] & \pi_{k-1}(E) \arrow[r] \arrow[d, equal] & \cdots \\ & 0 & & & 0 & \end{tikzcd}$$

□

We are interested in cohomology algebra of $K(D, n)$.

Theorem 2. (Hurewicz isomorphism theorem)
For an $n-$connected cell complex K, where $n > 0$, the groups $\pi_{n+1}(K)$ and $H_{n+1}(K; \mathbb{Z})$ are naturally isomorphic.

Proof. Step1: If K is $n-$connected, we may assume K has only one $0-$cell and no $k-$cell for $0 < k \leq n$.
We have proved that K is homotopic to another cell complex which has only one $0-$cell. Now prove the following claim and step1. follows by using induction.

Claim 2. If K is a $n-$connected cell complex with only one cell and no $k-$cell for $0 < k \leq n-1$. Then K is homotopic to a space with only one $0-$cell and no $k-$cell for $0 < k \leq n$.

By hypothesis, the $n-$th skeleton of K is a bouquet of sphere, each sphere corresponds to a $n-$cell. Since $\pi_n(K) = 0$, there are maps $F_i : D^{n+1} \to K$ with $F_i \mid_{S^n} = \sigma_i^n$. The image of F_i contains the $0-$cell. And their union is contractible to this $0-$cell. We deduce that

$$K \sim K / \cup_i \operatorname{Im}(F_i)$$

While the latter space has no $n-$cell.

Step2: Construct the isomorphism.

Let u be the generator of $H_{n+1}(S^{n+1};\mathbb{Z})$. Define the Hurewicz homomorphism as follows

$$H : \pi_i(K) \to H_i(K;\mathbb{Z})$$
$$[f] \mapsto f_* u$$

This homomorphism well-defined for general K. We check it is bijective in our case.
Injectivity:
Under our assumption, $K_{n+1} \cong \bigvee S_i^{n+1}$. f is determined by its degree to each sphere. Write $f \sim (a_1, \cdots, a_k)$ if the degree of f composes with projection to S_i^{n+1} is a_i. Then by definition, $f_*(u) = [\sum_{i=1}^k a_i \sigma_i^{n+1}]$. $f_*(u) = 0$ if and only if $\sum_{i=1}^k a_i \sigma_i^{n+1} = \partial(\sigma^{i+2})$. Then the characteristic map of σ^{n+2} is $F : D^{n+2} \to K$, $F\mid_{s^{n+1}} = f$, hence $[f] = 0$.

Surjectivity:
As in the argument above, $f \sim (a_1, \cdots, a_k)$ then $f_*(u) = [\sum_{i=1}^k a_i \sigma_i^{n+1}]$. The characteristic map of σ_i^{n+1} gives $f \sim (0, \cdots, 1, \cdots 0)$ with 1 is in the $i-$th position. $\square$

Corollary 1.

$$H_n(K(D,n);\mathbb{Z}) \cong D$$

Next we want to compute $H^n(K(D,n);\mathbb{Q})$.

Lemma 1. For G is an abelian group, we have

$$H^n(K(D,n);G) \cong \mathrm{Hom}(D,G).$$

Proof. Let $\alpha \in C^n(K(D,n);G)$, α sends $C_n(K(D,n);\mathbb{Z})$ to G. If furthermore $\delta\alpha = 0$, then α sends $H_n(K(D,n);\mathbb{Z}) \cong D$ to G. If $\alpha = \delta\beta$, then α sends every element to zero since $\partial(C_n(K(D,n);\mathbb{Z})) = 0$ in any case. So this gives the map from LHS to RHS. Check it is bijective.

Injectivity:
If $[\alpha]$ sends every element in $H_n(K(D,n);\mathbb{Z})$ to zero, then α sends every element in $C_n(K(D,n);G)$ to zero, hence $[\alpha] = 0$.

Surjectivity:

Given $f \in \mathrm{Hom}(D,G)$, define $\alpha_f \in C^n(K(D,n);G)$ as follows.

$$C_n(K(D,n);\mathbb{Z}) \to H_n(K(D,n);\mathbb{Z}) \cong D \xrightarrow{f} G$$

It is obvious that α_f is sent to f under our map from LHS to RHS. $\square$

Corollary 2.

$$H^n(K(D,n);\mathbb{Q}) \cong \mathrm{Hom}(D,\mathbb{Q})$$

Finally we compute the cohomology algebra for $K(D,n)$ by using spectral sequence.

Theorem 3. For any finitely generated abelian group D, the cohomology algebra $H^*(K(D,n);\mathbb{Q})$ is a free skew-commutative algebra, generated by elements of the vector space $H^n(K(D,n);\mathbb{Q}) \cong \mathrm{Hom}(D,\mathbb{Q}) = D^*$.

Proof. By fundamental theorem of finitely generated abelian group and the fact $K(D_1,n) \times K(D_2,n) = K(D_1 \times D_2, n)$. It suffices to prove that

1. $H^*(K(\mathbb{Z}_m,n);\mathbb{Q}) = 0$
2. $H^*(K(\mathbb{Z},2n+1);\mathbb{Q}) \cong \bigwedge[u_{2n+1}]$
3. $H^*(K(\mathbb{Z},2n);\mathbb{Q}) \cong \mathbb{Q}[u_{2n}]$

For 1, we use induction on n. When $n=1$, $K(\mathbb{Z}/m\mathbb{Z},n)$ is $S^\infty/\mathbb{Z}_m$, whose cellular decomposition can be written down and $H^*(S^\infty/\mathbb{Z}_m,\mathbb{Q}) = 0$ follows.

Assume the hypothesis holds for $n<k$. Then for $n=k$, consider the fiber space as in the proof of proposition 1.3.

$$\begin{tikzcd} K(\mathbb{Z}_m,n-1) \cong \Omega(K(\mathbb{Z}_m,n)) \arrow[r] & E \arrow[d] \\ & K(\mathbb{Z}_m,n) \end{tikzcd}$$

See the spectral sequence. $E_2^{p,q} \cong H^p(K(\mathbb{Z}_m,n)) \otimes H^q(K(\mathbb{Z}_m,n-1))$. It is zero when $q>0$, hence $d_r^* = 0$ for $r \geq 2$. Hence $E_2^{p,q} = E_\infty^{p,q}$.

$$0 = H^k(E) = \sum_{p+q=k} E_\infty^{p,q} = \sum_{p+q=k} E_2^{p,q} = E_2^{k,0} = H^k(K(\mathbb{Z}_m,n))$$

For 2.3. Use induction on n. When $n = 1$, $K(\mathbb{Z}, 1) = S^1$. $H^*(S^1) = \mathbb{Q}[u_1]$ holds.

For n=2k+1, assume the proposition holds for all $n = 2k$. Still consider the fiber bundle as in 1.

$$E_2^{p,q} \cong H^p(K(\mathbb{Z}, 2n+1)) \otimes H^q(K(\mathbb{Z}_m, 2n))$$

It is zero when q is not divided by $2n$. Thus $d_r^* = 0$ for $1 < r < 2n+1$.

$$E_2^{0,2kn} = \langle u_{2n}^k \rangle_{\mathbb{Q}}$$

The same argument as before, $E_2^{2n+1,0} \ni v \triangleq d_{2n+1}^*(u_2 n) \neq 0$, and $H^{2n+1}(K(\mathbb{Z}, 2n+1); \mathbb{Q}) = \langle v \rangle_{\mathbb{Q}}$. Hence

$$E_2^{2n+1,2kn} = \langle u_{2n}^k v \rangle_{\mathbb{Q}}$$

and $2v^2 = (d_{2n+1}^*)^2 u^2 = 0$. So define $u_{2n+1} = v$,

$$H^*(K(\mathbb{Z}, 2n+1); \mathbb{Q}) \cong \bigwedge [u_{2n+1}]$$

The case $n = 2k$ is similar. $\square$

The goal of today's note is to prove the Cartan-Serre theorem.

Theorem 1. (Cartan-Serre)
Let X be a simply-connected complex. Suppose that there is a dimension preserving isomorphism from the subgroup of the rational cohomology algebra of X generated by the elements of dimension $< k$, to the corresponding subgroup of some free skew-commutative algebra A (whose algebra generators of dimension $< k$ occur in dimensions say). Then the following assertions are valid:

(i) The Hurewicz homomorphism

$$H : \pi_i(X) \otimes \mathbb{Q} \to H_i(X; \mathbb{Q})$$

has trivial kernel for all $i < k$.

(ii) The image $H(\pi_i(X) \otimes \mathbb{Q})$ consists precisely of those elements of $H_i(X; \mathbb{Q})$ which have zero scalar product with every element x of $H^*(X; \mathbb{Q})$ decomposing non-trivially as a product $x = yz$ with $\deg y > 0, \deg z > 0$. Consequently, for $i < k - 1$ the group is isomorphism (in fact its image is dual) to the quotient group $H^i(X; \mathbb{Q})/\Gamma$, where Γ is generated by all elements of $H^i(X; \mathbb{Q})$ which decompose non-trivially as products.

Recall the proposition we proved last time

$$H^*(K(D, n); \mathbb{Q}) = \bigwedge(H^n(K; \mathbb{Q}))$$

The conditions of the theorem are fulfilled for $K(D, n)$ for $k = \infty$. The Hurewicz homomorphism is isomorphism for $i = n$ and zero for others, so the first statement holds. The second statement also holds by the proposition. The theorem is also true for

$$K = K(D_1, \alpha_1) \times K(D_2, \alpha_2) \times \cdots$$

To prove the general case X, we try to construct a map $f : X \to K$. Here K is as above, where D_j is free abelian group of rank equal to the free generators of $H^*(X; \mathbb{Q})$ of dimension α_j. We require this map induces isomorphism of rational cohomology group of dimension $< k$.

We construct this map as follows. First define on X_l to be constant map, and then try to extend this map to be defined on X_{l+1}, X_{l+2}, and so on. Now the question is when can we extend such map? we postpone the proof of the theorem and give some examples of this kind of extension problem.

Example 1. If we have defined $f : X_k \to Y$, we want to extend f to be defined on each $(k+1)-$cell. Given any $(k+1)-$cell $\sigma^{k+1} : (D^{k+1}, S^k) \to (X^{k+1}, X_k)$, since f is defined on $\sigma^{k+1}(S^k)$, $f \circ \sigma^{k+1}|_{S^k}$ is an element in $\pi_k(Y)$. It is clear that f can be extend to be defined on σ^{k+1} if and only if this element homologous to a constant map. Especially if $\pi_k(Y) = 0$.

Example 2. We give a geometric meaning of $H^n(X; D)$.

Proposition 1. Give any (connected) cell complex X, each homotopy class of maps $f : X \to K(D, n)$ is fully and canonically determined by some element of the cohomology group $H^n(X; D)$, and vice versa. There is thus a natural one-to-one correspondence

$$[X, K(D, n)] \leftrightarrow H^n(X; D)$$

Proof. We first describe the correspondence from LHS to RHS. Given $[f] \in [X, K]$, it induces

$$f^* : H^n(K; D) \to H^n(X; D)$$

Recall last time we have proved that for all abelian group G,

$$H^n(K(D, n), G) \cong \mathrm{Hom}(D, G)$$

Thus

$$H^n(K; D) \cong \mathrm{Hom}(D, D)$$

which admits a ring structure. Let u denotes the element in $H^n(K; D)$ corresponding to identity in $\mathrm{Hom}(D, D)$. Then the correspondence is given by

$$[X, K] \to H^n(X; D)$$
$$[f] \mapsto f^* u$$

We check this map is bijective.

Surjectivity:

Given $\alpha \in H^n(X; D)$, we construct f as follows:

Step1.

Define f on X_{n-1} to be constant map, sending each point in X_{k-1} to the only one zero cell in $K(D, n)$.

Step2.

Define f on $n-$cells. Given $n-$cell $\sigma^n : (D^n, S^{n-1}) \to (X_n, X_{n-1})$, $\alpha(\sigma^n) \in D \cong \pi_n(K)$ can be seen as a map from D^n to K which sends S^{n-1} to a single point. Define f on σ^n as follows:

$$\begin{tikzcd}
(D^n, S^{n-1}) \arrow[r, "\sigma^n"] \arrow[dr, "\alpha(\sigma^n)"'] & \sigma^n \subset (X_n, X_{n-1}) \arrow[d, "f"] \\
& (K(D,n), *)
\end{tikzcd}$$

Thus we define f on each $n-$cell, hence on X_n.

Step3.

Define f on X_{n+1}. Given $(n+1)-$cell $\sigma^{n+1} = (D^{n+1}, g)$, $g : S^n \to X_n$. By definition we have

$$\alpha(\partial\sigma^{n+1}) = f \circ g$$

since α is cocycle we have $f \circ g = 0 \in \pi_n(K)$. Hence we can extend f to be defined on X_{n+1}.

Step4.

Define f on $X_k, k > n+1$. Note that $\pi_k(K) = 0$ for $k > n$. If we have defined f on X_k, given $(k+1)-$cell $\sigma^{k+1} = (D^{k+1}, h)$, $0 = \pi_k(K) \ni h \circ f = 0$. Thus f can be extended to X_{k+1}.

By construction we have $\alpha = f^* u$.

Injectivity:

If $f_0^*u = f_1^*u$, we assume f_0, f_1 sends X_{n-1} to the same single point. Given any singular $n-$cycle $\sigma^n : I^n \to X$, $u(f_0 \circ \sigma^n) = u(f_1 \circ \sigma^n)$. This means $f_0 \circ \sigma^n$ and $f_1 \circ \sigma^n$ represent the same element in $H_n(K;\mathbb{Z}) \cong D$ by the definition of u. So there exists $\sigma^{n+1} : I^n \times I \to K$ such that

$$\sigma^{n+1}(x,0) = f_0 \circ \sigma^n(x), \quad \sigma^{n+1}(x,1) = f_1 \circ \sigma^n(x), \quad \sigma^{n+1}(\partial I^n, t) = *$$

I.e. a homotopy between $f_0 \circ \sigma$ and $f_1 \circ \sigma$. Finding σ^{n+1} for each σ^n gives a homotopy between $f_0|_{X_n}$ and $f_1|_{X_n}$. We want to extend this homotopy to $F : X \times I \to K$ between f_0, f_1.

Step1.

Define F on $X \times \{0,1\}$ to be f_0, f_1 respectively. Define F on $X_{n-1} \times I$ to be constant.

Step2.

Define F on $X_n \times I$ to be

$$\begin{tikzcd}
(I^n, \partial I^n) \times I \arrow[r, "\sigma^n \times \mathrm{id}"] \arrow[dr, "\sigma^{n+1}"] & (X_n, X_{n-1}) \times I \arrow[d, "F"] \\
& (K(D,n), *)
\end{tikzcd}$$

Step3.

Extend F on $X_{n+1} \times I$. Given $(n+1)-$cell τ, F restricts on boundary of $\tau \times I$ gives an element in $\pi_{n+1}(K) = 0$. So We can extend F on $X_{n+1} \times I$. And similarly on $X_{k+1} \times I$ for $k > n$. □

Corollary 1. If X, Y are cell complexes and $K(D,n)$, then $X \sim Y$.

This can be derived from the proof above. Note that

$$[X,Y] = [Y,X] = \mathrm{Hom}(D,D)$$

Composition of elements in $[X,Y]$ and $[Y,X]$ gives composition of $\mathrm{Hom}(D,D)$. Let the elements correspond to identity in $[X,Y],[Y,X]$ are f, g respectively. By the injectivity of the correspondence, we see $f \circ g \sim \mathrm{id}_Y$, $g \circ f \sim \mathrm{id}_X$.

Definition 1. (Cohomology operation)

A cohomology operation θ is a natural transformation of the functors $H^k(-;G_1)$

and $H^l(-;G_2)$. In other words, it is a map

$$\theta : H^k(X;G_1) \to H^l(X;G_2)$$

defined for all cell complex X. And given any continuous map $f : X \to Y$

$$f^*\theta = \theta f^*$$

Cohomology operation is an importance tool to study homotopy group. We will see its usage next time. By the proposition above we have the following corollary.

Corollary 2. The cohomology operations $\theta : H^n(-,D) \to H^p(-,G)$ are in natural one-to-one correspondence with the elements of the group $H^p(K(D,n);G)$.

$$\{\theta : H^n(-,D) \to H^p(-,G)\} \leftrightarrow H^p(K(D,n);G)$$

Proof. we state the correspondence and omit the check.
Given such cohomology operation, let $u \in H^n(K(D,n);D)$ as before, the correspondence from LHS to RHS is given by

$$\theta \mapsto \theta(u)$$

Given $y \in H^p(K(D,n);G)$. By the proposition

$$H^n(X;D) \leftrightarrow [X,K(D,n)]$$
$$f^*u \mapsto [f]$$

The correspondence from RHS to LHS is

$$y \to [\theta : f^*u \to f^*y]$$

□

Example 3. Given $f : X \to Y$, we can convert it to a fiber bundle projection as following (and thus can use spectral sequence). If f is injective, we may consider this map is an inclusion. Let $E(X,Y)$ be the path space consists of all paths start in X end in Y. The projection map sends each path to its endpoint. It is clear that

$E \sim X$. If f is not injective, define

$$\tilde{f} : X \times \{1\} \to Y$$
$$(x,1) \mapsto f(x)$$

The mapping cylinder is defined by

$$C_f = X \times I \cup_{\tilde{f}} Y$$

It is clear that $X \times I \sim X$, $C_f \sim Y$, and now the map $X \times I \to C_f$ is an inclusion and we can apply the argument above.

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "F"] & E(X \times I, C_f) \arrow[d] \\ Y & C_f \arrow[l, "\sim"'] \end{tikzcd}$$

It can be checked that $f \sim F$. We can prove the following proposition by this construction.

Proposition 2. Assume X, Y is simply-connected. If $f : X \to Y$ induces isomorphism of rational cohomology group of dimension $< k$, and isomorphism for $\pi_2(X) \to \pi_2(Y)$. Then it induces isomorphism of rational homotopy group $< k-1$.

Proof. Step1. We convert this map to fiber bundle projection.

$$\begin{tikzcd} F \arrow[r] & \tilde{X} \arrow[d] \\ & \tilde{Y} \end{tikzcd}$$

Claim 1. $H^i(F; \mathbb{Q}) = 0$ for $0 < i < k-1$.

Recall the proposition we state last time. $p_* : H_n(E) \to H_n(B)$ may be computed as follows:

$$H_n(E) = E^\infty_{n,0} = E^r_{n,0} \subset E^2_{n,0} \subset H_n(B)$$

Now the projection map induces isomorphism $H_i(\tilde{X}) \cong H_i(\tilde{Y})$ for $i < k$. So

$$E^2_{i,0} = E^\infty_{i,0}, \quad E^\infty_{i-p,p} = 0 \text{ for } p \neq 0,\ i < k$$

From this we see $E^2_{i,0}$ is in kernel of d_r for all r. See $E^2_{0,1}$, we have

$$0 = E^{\infty}_{0,1} = E^3_{0,1}$$

So every element in $E^2_{0,1}$ is in image of $d_2 : E^2_{2,0} \to E^2_{0,1}$, which is zero map. So

$$0 = E^2_{0,1} \cong H_1(F;\mathbb{Q}), \qquad E^2_{p,1} \cong H_p(\tilde{Y}) \otimes H_1(F) = 0 \forall p$$

Similarly we get $H^i(F;\mathbb{Q}) \cong H_i(F;\mathbb{Q}) = 0$ for $0 < i < k-1$.

Step2. By assumption and long exact sequence for homotopy group, we see F is simply-connected.

Claim 2. If the cohomology groups of a simply connected space F are trivial for $i < k-1$, then the groups $\pi_i F \otimes \mathbb{Q}$ are also trivial for $i < k-1$.

If this claim is true, then we have

$$\cdots \to \pi_i(F) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to \cdots$$

tensor with $\mathbb{Q}$ and note that tensor is right exact, we have

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_i(F) \otimes \mathbb{Q} & \longrightarrow & \pi_i(X) \otimes \mathbb{Q} & \xrightarrow{\cong} & \pi_i(Y) \otimes \mathbb{Q} & \longrightarrow & \pi_{i-1}(F) \otimes \mathbb{Q} & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

Thus we are done. Now prove this claim.

Assume the first nontrivial homotopy group is $\pi_s(F)$ for $s > 1$. Since the Hurewicz homomorphism is isomorphism for the first nontrivial homotopy group, this gives

$$\pi_s(F) \otimes \mathbb{Q} \cong H_s(F;\mathbb{Q}) = 0.$$

I.e. $\pi_s(F)$ is finite. We construct $f : F \to K(\pi_s(F), s)$ such that it induces isomorphism for the $s-$th homotopy group. This is possible since we may assume $F_{s-1} = *, F_s = \bigvee_{i=1}^{l} S^s_i$, and $K_s = \bigvee_{i=1}^{l} S^s_i$. f is defined on $s-$skeleton, it can extend to be defined on $(s+1)-$skeleton by Hurewicz isomorphism theorem, and extend to higher dimensional skeleton by the higher homotopy groups of K is trivial.

We convert f to fiber bundle projection.

$$\begin{tikzcd} F_1 \arrow[r] & F \arrow[d] \\ & K(\pi_s(F), s) \end{tikzcd}$$

See long exact sequence of homotopy groups gives

$$\pi_s(F_1) = 0, \quad \pi_i(F_1) = \pi_i(F) \text{ for } i \neq s$$

Also see the cohomology spectral sequence of this fiber bundle, since $H^*(K(\pi_s(F), s), \mathbb{Q})$ is generated by

$$H^s(K, \mathbb{Q}) \cong H_s(K, \mathbb{Q}) \cong \pi_s(K) \otimes \mathbb{Q} = 0$$

We have $H^i(F_1; \mathbb{Q}) = 0$ for $i < k - 1$.

Since F_1 satisfies the condition in claim, we may apply the same trick to F_1 until $s = k - 2$. □

After these examples we start our proof of Cartan-Serre theorem.

Proof. of Cartan-Serre theorem.
As in the beginning, now we know it is possible to construct the map $f : X \to K$ with the induced homomorphism $f^* : H^i(K; \mathbb{Q}) \to H^i(X; \mathbb{Q})$ is isomorphism for $i < k$. We may take D_1 to be the first nontrivial homotopy group of X rather than the free abelian group, then f induces isomorphism of the second homotopy group. Now apply proposition 2 above, we get

$$\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(K) \otimes \mathbb{Q}$$

And the commutative diagram completes the proof.

$$\begin{tikzcd} \pi_i(X) \otimes \mathbb{Q} \arrow[r, "f_*"] \arrow[d, "H"] & \pi_i(K) \otimes \mathbb{Q} \arrow[d, "H"] \\ H_i(X; \mathbb{Q}) \arrow[r, "f_*"] & H_i(K; \mathbb{Q}) \end{tikzcd}$$

□

The goal of today's note is to state some corollaries and applications of Cartan-Serre theorem, and to compute first few nontrivial stable homotopy group of sphere.

1 corollary of Cartan-Serre theorem

Corollary 1. For any Lie group G the nontrivial rational homotopy groups $\pi_i(G)\otimes \mathbb{Q}$ correspond one-toone with the free generators of the cohomology algebra $H^*(G;\mathbb{Q}) \cong \bigwedge[x_{i_1}, \dots x_{i_k}]$ of the same dimension. In particular,

$$\pi_{2k}(G) \otimes \mathbb{Q} = 0$$

Corollary 2. If X is an $(n-1)$-connected complex, then for all $q < 2n-1$ the rational Hurewicz homomorphism is isomorphism.

$$H : \pi_q(X) \otimes \mathbb{Q} \to H_q(X) \otimes \mathbb{Q}$$

These two corollaries are immediate. Next we compute rational homotopy of sphere. If $n = 2k+1$, then $H^*(S^n;\mathbb{Q}) = \bigwedge[u_{2n+1}]$ is a free skew-commutative algebra, so by Cartan-Serre theorem

$$\pi_j(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & ,\text{for } j = n \\ 0 & , \text{ otherwise} \end{cases}$$

If n us even, then its cohomological algebra is not free skew-commutative. In this case we consider its loop space. By simply computation using spectral sequence we see

$$H^*(\Omega(S^n);\mathbb{Q}) \cong \bigwedge[x_{n-1}] \otimes [x_{2n-2}]$$

So by Cartan-Serre theorem

$$\pi_j(\Omega(S^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & ,\text{for } j = n-1, 2n-2 \\ 0 & , \text{ otherwise} \end{cases}$$

So

$$\pi_j(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & ,\text{for } j = n, 2n-1 \\ 0 & , \text{ otherwise} \end{cases}$$

2 First few nontrivial stable homotopy group of sphere

2.1 Suspension homomorphism

We want to compute $\pi_{n+k}(S^n)$ when k is small. It makes sense to say "the homotopy group of S^n" since for n large enough, we have

$$\pi_{n+k}(S^n) \cong \pi_{(n+1)+k}(S^{n+1})$$

via the suspension homomorphism defined as follows.

Definition 1. (reduced suspension, suspension homomorphism)

1. Given topological space X, we define the reduced suspension ΣX as follows

$$\Sigma X \triangleq X \times I/(X \times \{0\}) \cup (X \times \{0\}) \cup (* \times I)$$

   for some base point $*$.

2. Define a natural map $\phi : X \to \Omega\Sigma X$ by sending x to the loop $(t \mapsto (x,t) \in \Sigma X)$. The suspension homomorphism E is defined as follows

$$\begin{tikzcd} \pi_i(X) \arrow[r, "\phi_*"] \arrow[dr, "E"'] & \pi_i(\Omega\Sigma X) \arrow[d, "\cong"] \\ & \pi_{i+1}(\Sigma X) \end{tikzcd}$$

   Note that $\Sigma S^n \sim S^{n+1}$.

Proposition 1. If X is $(n-1)-$connected, then the suspension homomorphism is isomorphism for $i < 2n - 1$.

Proof. By using Mayer-Vietoris sequence of reduced homology group, it can be shown that

$$\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)$$

Consider the fiber bundle

$$\begin{tikzcd} \Omega\Sigma X \arrow[r] & E \arrow[d] \\ & \Sigma X \end{tikzcd}$$

Lemma 1. homolgy long exact sequence of fiber bundle

If we have a fiber bundle whose total space denoted by E, base space denoted by B and fiber denoted by F. Provided that $H_i B = 0$ for $0 < i < p$, $H_j(F) = 0$ for $0 < j < q$, then we have the long exact sequence of homology group. There is a similar version for cohomology.

Proof. of lemma.

By assumption we have $E^2_{i,j} = 0$ for $0 < i < p$ or $0 < j < q$. Thus the only nonzero terms in the triangle $\{E^2_{i,j} \mid i + j \leq p + q - 1\}$ are $\{E^2_{0,j}\}$ and $\{E^2_{i,0}\}$. And we have for $k \leq p + q - 1$

$$H_k(E) = \sum_{i+j=k} E^\infty_{i,j} = E^\infty_{i,0} + E^\infty_{0,j}$$

Due to the position of $E^2_{i,0}$ and $E^2_{0,j}$ we have

$$E^\infty_{i,0} = E^i_{i,0} = \ker d_{i-1} : E_{i,0} \to E_{0,i-1}, \quad E^\infty_{0,j} = E^{j+1}_{0,j} = E^2_{0,j} / \operatorname{Im} d_{j+1} : E^2_{j+1,0} \to E^2_{0,j}$$

So it means the short exact sequence splits.

$$0 \to E^{k+1}_{0,k} \to H_k(E) \to E^k_{k,0} \to 0$$

Hence the long exact sequence

$$H_{p+q-1}(F) \xrightarrow{i_*} H_{p+q-1}(E) \xrightarrow{p_*} H_{p+q-1}(B) \xrightarrow{d_{p+q-2}} H_{p+q-2}(F) \longrightarrow \cdots \longrightarrow 0$$

$\square$

Where E is the space of path. Note that the base space and the fiber space are $n, (n-1)-$connected respectively. From the homology spectral sequence we see for $k < 2n - 1$

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(E) \arrow[r] & H_k(\Sigma X) \arrow[r, "\cong"] & H_{k-1}(\Omega\Sigma X) \arrow[r] & H_{k-1}(E) \arrow[r] & \cdots \\
 & & H_{k-1}(X) \arrow[u, "\cong"] \arrow[ur, "\phi_*"] & & &
\end{tikzcd}$$

It can be shown that the triangle commutes. $\square$

So $\pi_{n+k}(S^n)$ is independent of n provided $n \geq k + 2$. And these are the cases we are interested in.

2.2 Bockstein homomorphism and Steenrod squares

We give some definitions, propositions and facts.

Definition 2. A cohomology operation is said to be stable if it is defined on mod p cohomology (i.e. $H^q(-;\mathbb{Z}_p) \to H^{q+i}(-;\mathbb{Z}_p)$) and commutes with the suspension homomorphism.

From the discussion above, we are interested in cohomology which commutes with suspension homomorphism. Furthermore we require it is defined on mod p cohomology since $\mathbb{Z}_p$ is a field so we can apply spectral sequence. And we can recover $H^s(F;\mathbb{Z})$ by $H^s(F;\mathbb{Z})$ in the following sense.

Definition 3. $\forall [y] \in H^q(X;\mathbb{Z}_m)$, choose a representative $y \in C^q(X)$. We have

$$\delta y = mu$$

for some $u \in C^{q+1}(X;\mathbb{Z})$. Define the Bockstein homomorphism

$$\begin{aligned} \delta_1 : H^q(X;\mathbb{Z}_m) &\to H^{q+1}(X;\mathbb{Z}_m) \\ [y] &\mapsto [\frac{1}{m}\delta y] = [u] \end{aligned}$$

This homomorphism is well-defined.

We may define $\delta_2 : \ker\delta_1 \to H^{q+1}(X;\mathbb{Z})/\operatorname{Im}\delta_1$ as follows: If $[x] \in \ker\delta_1$, that is $\frac{1}{m}(\delta x) = my + \delta z$, then

$$\begin{aligned} \delta_2 : \ker\delta_1 &\to H^{q+1}(X;\mathbb{Z})/\operatorname{Im}\delta_1 \\ [x] &\mapsto [y] = [\frac{1}{m}(\frac{1}{m}\delta x - \delta z)] \end{aligned}$$

It is again well-defined. The image can be seen as $[\frac{1}{m}\delta_1(x)]$ in $H^{q+1}(X;\mathbb{Z})/\operatorname{Im}\delta_1$. We may similarly define $\delta_k : \cap_{i<k}\ker\delta_i \to H^{q+1}(X;\mathbb{Z}_m)/\cup_{i<k}\operatorname{Im}\delta_i$ iteratively

$$\delta_k([x]) = [\frac{1}{m}(\delta_{k-1}x)]$$

Proposition 2. If $[\alpha] \in H^q(X;\mathbb{Z}_p)$ is an image under δ_k but not an image under δ_i for $i < k$, then x represents a basic generator $[\tilde{x}]$ of $H^q(X;\mathbb{Z})$ of order p^k.

Proof. Assume $[\alpha] = \delta_k[x]$, then $p[\alpha] = [\delta_{k-1}x]$ so by induction the proposition holds. $\square$

Note conversely, if $[\alpha]$ is an element in $H^q(X;\mathbb{Z})$ with order p^k, then $p^k\alpha = \delta(\beta)$, then $\delta_k[(\beta]) = [\alpha]$ in Z_p cohomology. If we define

$$\partial_1 : H_q(X;\mathbb{Z}_p) \to H_{q-1}(X;\mathbb{Z}_p)$$
$$[\sigma] \mapsto [\frac{1}{p}\partial\sigma]$$

We may define ∂_k similarly. And similar proposition holds hence if we know the structure $H_{n+1}(F;\mathbb{Z}_p)$ and the action of δ_k, then we can determine the $p-$component of $\pi_{n+1}(F)$.

Fact 1.

1. Corresponding to each integer $i \geq 0$, there is a stable cohomology operation θ, a "Steenrod square", denoted by Sq^i, which is a homomorphism on each H^q:

$$Sq^i : H^q(X,\mathbb{Z}_2) \to H^{q+i}(X,\mathbb{Z}_2)$$

and which has the following properties:

   1. $Sq^i(x) = 0 \qquad$ for $q < i$;
   2. $Sq^0 \equiv 1$;
   3. $Sq^i(x) = x \smile x \qquad$ for $q = i$;
   4. $Sq^i(xy) = \sum_{j+k=i} Sq^j(x)Sq^k(y)$;
   5. $Sq^1(x) = \delta_1(x)$.

2. For each odd prime p and each integer $i \geq 0$, there is stable cohomology operations

$$St_p^i : H^q(X,\mathbb{Z}_p) \to H^{q+i}(X,\mathbb{Z}_p)$$

such that

   1. $St_p^i(x) = 0 \qquad$ for $q < 2i$;
   2. $St_p^0 \equiv 1$;

3. $St_p^i(x) = x^p \quad$ for $q = 2i$;

4. $St_p^i(xy) = \sum_{j+k=i} St_p^j(x) St_p^k(y)$;

3. In mod p cohomology all stable cohomology operations are composite of the "Steenrod operations".

4. (Adam's relation) For $a < 2b$,

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

where the binomial coefficient is taken mod 2. Hence

$$
\begin{aligned}
&Sq^1 Sq^1 = \delta_1^2 = 0 \\
&Sq^1 Sq^2 = Sq^3 \\
&Sq^1 Sq^3 = 0, \quad Sq^1 Sq^{2q} = Sq^{2q+1} \\
&Sq^2 Sq^2 = Sq^3 Sq^1, \quad Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1, \quad Sq^2 Sq^4 = Sq^5 Sq^1 + Sq^6 \\
&Sq^3 Sq^3 = Sq^5 Sq^1
\end{aligned}
$$

We skip the proof of these facts. There is another property of stable cohomology operation that we should know. In a spectral sequence, there is a many-valued homomorphism

$$
\begin{aligned}
\tau : H^q(F) \supset E_n^{0,q} A^q \triangleq \cap_{r<q} \ker d_r^* &\to H^{q+1}(B) \\
\alpha &\mapsto d_r^* \alpha
\end{aligned}
$$

In fact this map can be equivalently defined as

$$\tau = (p^*)^{-1} \delta^*$$

where p is the fiber projection, $\delta^* : H^q(F) \to H^{q+1}(E, F)$ is the coboundary homomorphism. From this we know τ and Sq^i, δ_k commute. Since if $y \in \tau(x)$, it means $p^* y = \delta^* x$, and by property of $\theta = Sq^i, \delta_k$ we have

$$p^* \theta y = \theta p^* y = \theta \delta^* x = \delta^* \theta x$$

This homomorphism is called the transgression homomorphism. Later we will see in our case τ is one-valued. It helps us a lot in the computation.

2.3 Computation of $\pi_{n+1}(S^n), \pi_{n+2}(S^n, \pi_{n+3}(S^n))$

Assume we have already gotten the following tables of generators of several cohomology groups.

$H^{n+q}(K(\mathbb{Z}, n); \mathbb{Z}_2)$:

$q=0$	1	2	3	4	5	6	7	8	9	$\cdots$
u	0	Sq^2u	Sq^3u	Sq^4u	Sq^5u	Sq^6u Sq^4Sq^2u	Sq^7u Sq^5Sq^2u	Sq^8u Sq^6Sq^2u	Sq^9u Sq^7Sq^2u Sq^6Sq^3u	$\cdots$

$H^{n+q}(K(\mathbb{Z}_{2p^h}, n); \mathbb{Z}_2)$,

$q=0$	1	2	3	4	
u	$\delta_h u$	Sq^2u	Sq^3u $Sq^2\delta_h u$	Sq^4u $Sq^3\delta_h u$	

$H^{n+q}(K(\mathbb{Z}, n); \mathbb{Z}_p)$,

$q=0$	1	2	$\cdots$	$2p-2$	$2p-1$	
u	0	0	0	St_p^1u	$\delta_1 St_p^1 u$	

$H^{n+q}(K(\mathbb{Z}_{p^h}, n); \mathbb{Z}_p)$

$q=0$	1	2	$\cdots$	$2p-2$	$2p-1$	
u	$\delta_h u$	0	0	St_p^1u	$\delta_1 St_p^1 u$ $St_p^1\delta_h u$	

Recall in the proof of Cartan-Serre theorem last time. For a $(s-1)-$connected F, we construct a fiber bundle

$$\begin{tikzcd} F_1 \arrow[r] & F \arrow[d] \\ & K(\pi_s(F), s) \end{tikzcd}$$

Then we have shown that

$$\pi_s(F_1) = 0, \quad \pi_i(F_1) = \pi_i(F) \text{ for } i \neq s$$

This time we take $F = S^n$, $\pi_n(S^n) = \mathbb{Z}$.

$$\begin{tikzcd} F_1 \arrow[r] & S^n \arrow[d] \\ & K(\mathbb{Z}, n) \end{tikzcd}$$

Here F_1 is $n-$connected. We want to compute $H_{n+1}(F_1; \mathbb{Z})$, then by Hurewicz theorem

$$H_{n+1}(F_1; \mathbb{Z}) \cong \pi_{n+1}(F_1) \cong \pi_{n+1}(S^n)$$

By the discussion in section 2.2, we have to compute $H^{n+1}(F_1; \mathbb{Z}_p)$ for all p prime. Since Z_p is a field we can use spectral sequence. In this case we know $H^q(S^n; \mathbb{Z}_p) = 0$ for $q \neq n$ and $H^i(F_1) = 0$ for $i \leq n$. So for $m < 2n$

$$0 = H^m(S^n) = \sum_{i+j=m} E_\infty^{i,j} = E_{m+2}^{m,0} + E_{m+2}^{0,m}$$

Hence $d_{m+1}^* = \tau : H^m(F_1; \mathbb{Z}_p) = E_{m+1}^{0,m} \to E_{m+1}^{m+1,0} = H^{m+1}(K; \mathbb{Z}_p)$ is bijective. So

$$H^{n+q}(F_1; \mathbb{Z}_p) \cong H^{n+q+1}(K; \mathbb{Z}_p) \quad 0 < q < n$$

Now use the tables of $H^{n+q}(K(\mathbb{Z}, n)\, \mathbb{Z}_p)$ we get the table of $H^{n+q}(F_1; \mathbb{Z}_p)$ where $p > 2$

$q = 0$	1	$\cdots$	$2p-3$	$2p-2$	
0	0	0	$St_p^1 \tau(u) = v$	$\delta_1 St_p^1 \tau(u) = \delta_1(v)$	

Let the dual of $\delta_1(v)$ in $H_{n+2p-2}(F_1; \mathbb{Z}_p)$ to be σ. Then $\partial_1 \sigma \in H_{n+2p-3}$ is the dual of v, hence nonzero. By the proposition of Bockstein homomorphism we see the $p-$component of $H^{n+2p-3}(F_1; \mathbb{Z})$ is $\mathbb{Z}_p$, and the $p-$component of $H^j(F_1; \mathbb{Z})$ is 0 for $0 < j < n + 2p - 3 < 2n$. So

$$\pi_{n+1}^{(p)}(F_1) = 0 = \pi_{n+1}^{(p)}(S^n) \quad , \ p > 2$$

$\pi_{n+q}^{(p)}(F_1) = 0 = \pi_{n+q}^{(p)}(S^n) \quad , \ p > 2, \ \ 0 < q < 2p - 3 < n$ in Modern Geometry(??)

When $p = 2$, from the table of $H^{n+q}(K(\mathbb{Z}, n), \mathbb{Z}_2)$, we get the table of $H^{n+q}(F; \mathbb{Z}_2)$

$q = 1$	2	3	4	5	6	
$v = Sq^2 \tau(u)$	$Sq^3 \tau(u)$ $= Sq^1 Sq^2(u)$ $= Sq^1 v$	$w = Sq^4 \tau(u)$ $(Sq^2 v = 0)$	$Sq^5 \tau(u)$ $= Sq^4 Sq^1 \tau(u) + Sq^2 Sq^3 \tau(u)$ $= Sq^2 w$	$Sq^4 v$ $Sq^2 w$	$Sq^3 w$ $Sq^5 v$	

We omit the computation of $q = 5, 6$. For the same reason we see

$$\pi^{(2)}_{n+1}(F_1) = \mathbb{Z}_2 = \pi^{(2)}_{n+1}(S^n)$$

$$\pi_{n+1}(S^n) = \mathbb{Z}_2$$

Next we see $\pi_{n+2}(S^n)$. We consider fiber bundle

$$\begin{tikzcd} F_2 \arrow[r] & F_1 \arrow[d, "f"] \\ & K(\mathbb{Z}_2, n+1) \end{tikzcd}$$

Assume (??) is true. Then $\pi^{(p)}_{n+2}(F_2) = \pi^{(p)}_{n+2}(F_1) = 0$ for $p \neq 2$. When $p = 2$, see the spectral sequence of coefficient $\mathbb{Z}_2$. Since the base is $n-$connected and the fiber is $(n+1)$-connected, we can use the long exact sequence of cohomology for $q < 2n$.

$$H^{n+q}(F_1; \mathbb{Z}_2) \xrightarrow{i^*} H^{n+q}(F_2; \mathbb{Z}_2) \xrightarrow{\tau} H^{n+q+1}(K; \mathbb{Z}_2) \xrightarrow{f^*} H^{n+q+1}(F_1; \mathbb{Z}_2)$$

We know $f^* : H^{n+1}(F_1) \to H^{n+1}(K)$ is isomorphism, so $f^*(u) = v$ where u, v are generator of each group. Thus $f^*(H^{n+q}(K; \mathbb{Z}_2))$ are precisely the image of v under Steenrod operations. Since $Sq^2 v = 0, Sq^2 u \neq 0$, there must be an $x \in H^{n+2}(F_2)$ with $\tau(x) = Sq^2 u$ by exactness.

Also, w is not in image f^*, so $i^* w \triangleq \tilde{w} \neq 0$. According to these We can write down the table of $H^{n+q}(F_2; \mathbb{Z}_2)$

$q = 2$	3	4	5	6	
$x = \tau^{-1}(Sq^2 v)$	$\tilde{w}$ $Sq^1 x$	$(Sq^1 \tilde{w} = 0)$ $Sq^2 x = \delta_2(\tilde{w})$	$Sq^2 \tilde{w}$ $(Sq^3 x = 0)$	$Sq^3 \tilde{w}$ $Sq^4 x$	

We explain the information in the table.
First $Sq^1(x) \neq 0$ since $\tau(Sq^1(x)) = Sq^1 Sq^2 v = Sq^3 v \neq 0$.
This give us

$$\pi_{n+2}(S^n) = \mathbb{Z}_2$$

Next $Sq^1 \tilde{w} = 0$ since

$$Sq^1 \tilde{w} = Sq^1 i^* w = i^* Sq^1 w = i^* Sq^2 Sq^1 v = Sq^2 Sq^1 i^*(v) = 0$$

To see $Sq^2 x = \delta_2(\tilde{w})$ and $Sq^3 x = 0$ we need the following lemma

Lemma 2. If $a = f^*(\bar{a}) = \delta_h w$ and $b = \tau^{-1}\delta_1(\bar{a})$, then the elements $b, \tilde{w} \in H^*(F_2; \mathbb{Z}_2)$ satisfy $b = \delta_{h+1}\tilde{w}$.

We omit the proof of this lemma. We take $a = Sq^1 w = Sq^2 Sq^1 v = f^*(Sq^2 Sq^1 u)$,

$$b = \tau^{-1}\delta_1(Sq^2 Sq^1 u) = \tau^{-1}(Sq^1 Sq^2)Sq^1 u = \tau^{-1}Sq^3 Sq^1 u = \tau^{-1}Sq^2 Sq^2 u = Sq^2 x$$

By the lemma $Sq^2 x = \delta_2 \tilde{w}$. To see$Sq^3 x = 0$,

$$Sq^3 x = Sq^1 Sq^2 x = \delta_1 \delta_2 \tilde{w} = 0$$

Finally to compute $\pi_{n+3}(S^n)$ we take the fiber bundle

$$\begin{array}{ccc} F_3 & \longrightarrow & F_2 \\ & & \downarrow f \\ & & K(\mathbb{Z}_3, n+1) \end{array}$$

Similarly we have the long exact sequence

$$H^{n+q}(F_2; \mathbb{Z}_2) \xrightarrow{i^*} H^{n+q}(F_3; \mathbb{Z}_2) \xrightarrow{\tau} H^{n+q+1}(K; \mathbb{Z}_2) \xrightarrow{f^*} H^{n+q+1}(F_2; \mathbb{Z}_2)$$

f^* is isomorphism at $q = 2$, it means image of $f^*(x) = u$ where x is as above, and u is generator of $H^{n+2}(K(\mathbb{Z}_2, n+2); \mathbb{Z}_2)$. Next see

$$H^{n+q}(K; \mathbb{Z}_2) \longrightarrow H^{n+q+1}(F_2; \mathbb{Z}_2) \longrightarrow H^{n+q+1}(F_3; \mathbb{Z}_2) \longrightarrow H^{n+q+1}(K; \mathbb{Z}_2)$$

We find that $\tilde{w} \in H^3(F_2, \mathbb{Z}_2)$ is not in image of f^*. So we let $\breve{w} = i^*(\tilde{w})$.

$$H^{n+3}(F_3; \mathbb{Z}_w) = \langle \breve{w} \rangle$$

Next see every element in $H^{n+4}(F_2)$ is image of f^*, so $i^* : H^{n+4})F_2 \to H^{n+4}(F_3)$ is zero. By $Sq^3 x = 0 = f^*(Sq^3 u)$, we have

$$H^4(F_2, \mathbb{Z}_2) = \langle \tau^{-1}(Sq^3 u) \rangle$$

By the lemma, take $\bar{a} = Sq^2 u$, $a = f^*(\bar{a}) = Sq^2 x = \delta_2 \tilde{w}$,

$$\tau^{-1}(Sq^3 u) = \tau^{-1}\delta_1 Sq^2 x = \delta_3 \breve{w}$$

So we get the table of $H^{n+q}(F_3; \mathbb{Z}_2)$.

$q = 2$	3	4	
0	$\breve{w}$	$\delta_3 \breve{w}$	

So

$$\pi_{n+3}(S^n) = \pi_{n+3}^{(2)}(S^n) \times \pi_{n+3}^{(3)}(S^n) = \mathbb{Z}_8 \times \mathbb{Z}_3 = \mathbb{Z}_{24}$$