1.

Some Manifolds Arising in the General Theory of Relativity (GTR)

What we care about is the manifold with the Riemannian curvature tensor R^i_{jkl} and the metric g_{ab} s.t. the Einstein equation holds:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab} . \quad \text{(EE)}$$

where $R_{ab} = R^i_{aib}$, $R = R_{ab} g^{ab}$, T_{ab} is the energy-momentum tensor of the medium and $\frac{8\pi G}{c^4}$ is a constant. We'll call the Einstein equation (EE) in the following text.

1. Spherically Symmetric Solution.

Def. A manifold (M, g) satisfying (EE) is said to be spherically symmetric if it admits the isometry group $SO(3)$.

Now we solve the spherically symmetric static solution in the vacuum space, (outside the source).
in the sense that the metric is indep. of time and $T_{ab} \equiv 0$, outside some r_0.

Let (M, g) be such a solution, with $(x_1, x_2, x_3, x_4) = (t, r, \theta, \varphi)$.

- static: the metric is invariant under the transf. $(t, r, \theta, \varphi) \mapsto (-t, r, \theta, \varphi)$,
  so for $i \neq 1$, $\cancel{g_{i4} \mapsto g_{i4}}$ $g_{i1} \longmapsto \frac{\partial x^\mu}{\partial x^i} \frac{\partial x^\nu}{\partial x^1} g_{\mu\nu} = -g_{i1} \Rightarrow g_{i1} = 0$ for $i \neq 1$.
- spherically sym.: the metric is invariant under the transf. $\begin{cases} (t, r, \theta, \varphi) \mapsto (t, r, \theta, -\varphi), \\ (t, r, \theta, \varphi) \mapsto (t, r, -\theta, \varphi). \end{cases}$

  so likewise we can get $g_{i3} = 0$ for $i \neq 3$, and $g_{j4} = 0$ for $j \neq 4$.

Hence the metric is of the form $dl^2 = g_{11} dt^2 + g_{22} dr^2 + g_{33} d\theta^2 + g_{44} d\varphi^2$.

In fact the spherical symmetry implies more. Given fixed $r = r_0$, the metric is invariant under any rotation through θ and φ. Hence we have

$r_0^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) = g_{33} d\theta^2 + g_{44} d\varphi^2$. (let the signatures be $(-, +, +, +)$).

Thus the metric should be $\underline{dl^2 = g_{00} dt^2 + g_{11} dr^2 + r^2 (d\theta^2 + \sin^2 d\varphi^2)}$

With $T_{ab} \equiv 0$, we get from (EE): $R_{ab} = 0$. Let $g_{00} = -e^{\nu(r)} c^2$, $g_{11} = +e^{\lambda(r)}$. (indep. of t). for $r > r_0$.

By the formula $R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab} \Gamma^d_{cd} - \Gamma^c_{ad} \Gamma^d_{bc}$ and $\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$.,

$$e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = k T^1_1 \quad ① \qquad \text{(see P. 5).}$$

We have

$$e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = k T^0_0 \quad ② \qquad \text{where } k := \frac{8\pi G}{c^4}, \ (\cdot)' := \partial_r, \ \dot{(\cdot)} := \partial_t .$$

$$e^{-\lambda} \cdot \frac{\dot{\lambda}}{r} = k T^1_0 \quad ③$$

$$\frac{1}{2} e^{-\lambda} \left(\nu'' + \frac{1}{2} \nu'^2 + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) - \frac{1}{2} e^{-\nu} \left(\ddot{\lambda} + \frac{1}{2} \dot{\lambda}^2 - \frac{\dot{\nu} \dot{\lambda}}{2} \right) = k T^2_2 = k T^3_3 . \quad ④$$

2. For $r>r_0$, we have $\dot{\lambda}=0$ ($\because$ ③). Hence even dropping the static condition, we can still have λ is indep. of the time outside the source.

For $r>r_0$, we have $\nu' = -\lambda' + kre^{\lambda}(T_1^1 - T_0^0) = -\lambda'$ from ①, ②. Hence $\nu = -\lambda$.

(assuming M is asymptotically flat in the sense that it's equivalent to the Minkowski's metric for $r\to\infty$).

② gives: $(e^{-\lambda})' + \frac{1}{r}e^{-\lambda} - \frac{1}{r} - krT_0^0 = 0$. Solve the ODE:

$$e^{-\lambda} = 1 + \frac{k}{r}\int_0^r s^2 T_0^0\,ds = 1 + \frac{k}{r}\int_0^{r_0} s^2 T_0^0\,ds.$$

Put $M := -\frac{4\pi}{c^2}\int_0^{r_0} s^2 T_0^0\,ds$, and we have $e^{-\lambda} = 1 - \frac{2MG}{c^2 r}$ and get:

$$d\ell^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2dt^2 + \frac{1}{1-\frac{2GM}{c^2 r}}dr^2 + r^2\underbrace{(d\theta^2+\sin^2\theta\, d\varphi^2)}_{= d\Omega^2}. \quad \text{for } r>r_0,$$

which is called the external Schwarzschild solution.

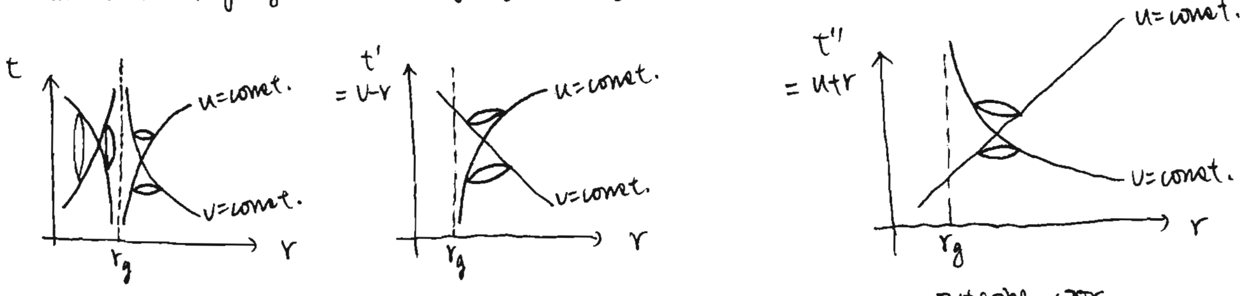
Observe that the metric has singularities at $r=r_g := \frac{2GM}{c^2}$ and $r=0$, where the latter is a true singularity, while the former is just the result of the bad choice of the coordinates. To see this, consider

$$\begin{cases} u = t - r - r_g\ln\left|\frac{r}{r_g}-1\right| \\ v = t + r + r_g\ln\left|\frac{r}{r_g}-1\right| \end{cases} \Rightarrow \begin{cases} du = dt - \frac{r}{r-r_g}dr \\ dv = dt + \frac{r}{r-r_g}dr \end{cases}.$$

Then we have

$$d\ell^2 = -\left(1-\frac{r_g}{r}\right)dv^2 + 2dvdr + r^2d\Omega^2 \quad \text{and} \quad d\ell^2 = -\left(1-\frac{r_g}{r}\right)du^2 - 2dudr + r^2d\Omega^2.$$

called the ingoing and outgoing Eddington-Finkelstein coor. resp..



Schwarzschild sol'n. singular at $r=r_g$.

ingoing coor. regular for ingoing light at $r=r_g$

outgoing coor. regular for outgoing light at $r=r_g$.

However this is still not the desired result (not matching totally).

If consider the coordinate (u,v,θ,φ), we then have

$$d\ell^2 = -\left(1-\frac{r_g}{r}\right)dudv + r^2d\Omega^2$$ where $r(u,v)$ is the sol'n of $\frac{1}{2}(v-u) = r + r_g\ln\left|\frac{r}{r_g}-1\right|$. (by def. of u,v).

To avoid the logarithmic divergence of $v-u$ as a function of r at r_g,

consider $$\begin{cases} u' := \begin{cases} -e^{-u/2r_g} & \text{if } r>r_g \\ e^{-u/2r_g} & \text{if } r<r_g \end{cases} \\ v' := e^{v/2r_g} \end{cases},$$ resulting in $-u'v' = \left(\frac{r}{r_g}-1\right)e^{r/r_g}$.

Such the equation determines $r=r(u',v')$ uniquely and the metric becomes

$$dl^2 = -\frac{4r_g^3}{r} e^{-r/r_g} du'dv' + r^2 d\Omega^2$$

with no divergence at $r=r_g$.

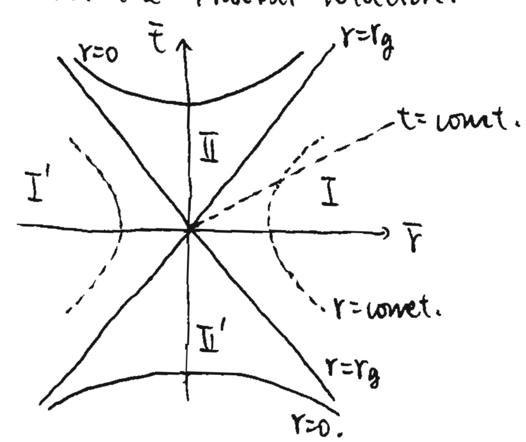
If we let $\bar{t} := \frac{v'+u'}{2}$ and $\bar{r} := \frac{v'-u'}{2}$, we have the coor. transf.

$$\begin{cases} \bar{r} = \sqrt{\frac{r}{r_g}-1}\; e^{r/2r_g} \cosh\frac{t}{2r_g} \\ \bar{t} = \sqrt{\frac{r}{r_g}-1}\; e^{r/2r_g} \sinh\frac{t}{2r_g} \end{cases} \text{ if } r>r_g, \quad \begin{cases} \bar{r} = \sqrt{1-\frac{r}{r_g}}\; e^{r/2r_g} \sinh\frac{t}{2r_g} \\ \bar{t} = \sqrt{1-\frac{r}{r_g}}\; e^{r/2r_g} \cosh\frac{t}{2r_g} \end{cases} \text{ if } r<r_g$$

and we have the extended Schwarzschild metric

$$dl^2 = -\frac{4r_g^3}{r} e^{-r/r_g} (d\bar{t}^2 - d\bar{r}^2) + r^2 d\Omega^2,$$

called the Kruskal solution.



I, I': $|\bar{r}|>|\bar{t}|$, i.e., $r>r_g$.

II, II': $|\bar{r}|<|\bar{t}|$ and $\bar{t}^2-\bar{r}^2<1$. ($\bar{t}^2-\bar{r}^2=1 \Leftrightarrow r=0$).

Then I+II is isometric to the ingoing coor.,

and I+II' is isometric to the outgoing coor..

The surface $r=r_g$ separating I, II: future event horizon,

and separating I, II': past event horizon.

II: black holes ; II': white holes.

Now we back to study the internal Schwarzschild sol'n. (i.e. $T_{ab}\neq 0$)

in the so-called "perfect fluid" with the hydrodynamical energy-momentum tensor:

$$T_{ab} = (\rho+p)\, u_a u_b + p\, g_{ab}$$

where ρ and p are the energy density and the pressure as functions of r,

and u is the four velocity of the fluid.

Let $u = e^{-\nu/2}\partial_t$. Then we have $T^0_0 = -\rho$, $T^i_i = p$ $(i=1\sim3)$. and other components vanish.

Recall the equations derived in the page 1.. By ③, we have $\dot{\lambda}=0 \Rightarrow \lambda=\lambda(r)$.

and again by ①, ② we have $\nu' = -\lambda' + kre^{\lambda}(T^1_1 - T^0_0)$. Integrate from 0 to $r<r_0$:

$$\nu(r,t)+\lambda(r) = \nu(0,t)+\lambda(0)+k\int_0^r se^{\lambda(s)}(p(s)+\rho(s))\,ds$$

By the same integration when $r\geq r_0$, we have $\nu(0,t)+\lambda(0) = -k\int_0^{r_0} se^{\lambda(s)}(p(s)+\rho(s))\,ds$, so

$\nu(r,t)=\nu(r) = -\lambda(r) - k\int_r^{r_0} se^{\lambda(s)}(p(s)+\rho(s))\,ds$. for $r<r_0$.

The ODE by ② gives $e^{-\lambda} = 1+\frac{k}{r}\int_0^r s^2 T^0_0\, ds = 1-\frac{k}{r}\int_0^r s^2\rho(s)\,ds$.

4. To get a complete solution, we need more conditions of p, ρ.

We knew there's a kind of conservation law for the energy-momentum tensor: $D_i T^i_j = 0$.

For $j=1$, we have $0 = D_i T^i_1 = \partial_i T^i_1 + \Gamma^i_{li} T^l_1 - \Gamma^l_{1i} T^i_l$

$$= p' + T^1_1(\Gamma^0_{10} + \cancel{\Gamma^1_{11}} + \cancel{\Gamma^2_{12}} + \cancel{\Gamma^3_{13}}) - T^0_0 \Gamma^0_{10} - \cancel{T^1_1 \Gamma^1_{11}} - \cancel{T^2_2 \Gamma^2_{12}} - \cancel{T^3_3 \Gamma^3_{13}}$$

$$= p' + p \cdot \frac{\nu'}{2} + \rho \cdot \frac{\nu'}{2}.$$

Then we have
$$\begin{cases} \nu = -\lambda - k\int_r^{r_0} s e^{\lambda(s)}(p(s)+\rho(s))\,ds \quad -(1) \\ e^{-\lambda} = 1 - \frac{k}{r}\int_0^r s^2 \rho(s)\,ds \quad -(2) \\ (p+\rho)\frac{\nu'}{2} + p' = 0 \quad -(3) \end{cases}$$
, with some initial conditions and assumptions:
$$(*)\begin{cases} \rho = \rho_0 = \text{const.} \\ p = p(r),\ p(r_0) = 0. \end{cases}$$

$(2) \Rightarrow \underline{e^{-\lambda}} = 1 - \frac{k}{r}\int_0^r s^2 \rho_0 ds^2 = \underline{1 - \frac{k\rho_0}{3} r^2}. \Rightarrow -\lambda' = e^{\lambda} \cdot \left(\frac{-2k\rho_0}{3} r\right)$

$\partial_r(1) \Rightarrow \nu' e^{-\lambda} + \lambda' e^{-\lambda} = kr(p+\rho_0)$

$\Rightarrow \nu'(1 - \frac{k\rho_0}{3}r^2) + \frac{2k\rho_0}{3} r = kr(p+\rho_0) \quad -(4)$

(3) with $(*) \overset{\text{ODE}}{\Rightarrow} p + \rho_0 = A e^{-\nu/2}$, where A (a const.) is to be determined to make the internal and external sol'n connected continuously.:

At $r = r_0$, $1 - \frac{2GM}{c^2 r_0} = e^{-\lambda(r_0)} = 1 - \frac{k\rho_0}{3} r_0^2 \Rightarrow \frac{k\rho_0}{3} r_0^2 = \frac{2GM}{c^2 r_0}$. (by the external sol'n)

By (4) $\Rightarrow \frac{(e^{\nu})'}{e^{\nu}}(1 - \frac{k\rho_0}{3} r^2) + \frac{2k\rho_0}{3} r = krAe^{-\nu/2}$

$\overset{\text{at } r_0}{\Longrightarrow} \frac{\frac{2GM}{c^2 r_0^2}}{1 - \frac{2GM}{c^2 r_0}}(1 - \frac{k\rho_0}{3} r_0^2) + \frac{2k\rho_0}{3} r_0 = kr_0 A e^{-\nu/2} \Rightarrow A = \rho_0 \sqrt{1 - \frac{k\rho_0}{3} r_0^2}.$

Combining above: $\nu'(1 - \frac{k\rho_0}{3} r^2) + \frac{2}{3} k\rho_0 r - krAe^{-\nu/2} = 0.$

$$\overset{\text{ODE}}{\Longrightarrow} \underline{e^{\nu/2} = \sqrt{1 - \frac{k\rho_0}{3} r^2} \cdot B + \frac{3A}{2\rho_0}}$$
where B is to be determined to make the sol'n conti.

At $r = r_0$, $\sqrt{1 - \frac{k\rho_0}{3} r_0^2} = e^{\nu(r_0)/2} = \sqrt{1 - \frac{k\rho_0}{3} r_0^2} \cdot B + \frac{3}{2}\sqrt{1 - \frac{k\rho_0}{3} r_0^2} \Rightarrow B = \frac{-1}{2}.$

Then we got $e^{\nu/2}$ and $e^{-\lambda}$ and can write down the explicit internal sol'n:

$$dl^2 = -\left(\frac{-1}{2}\sqrt{1 - \frac{k\rho_0}{3} r^2} + \frac{3}{2}\sqrt{1 - \frac{k\rho_0}{3} r_0^2}\right)^2 c^2 dt^2 + \frac{1}{1 - \frac{k\rho_0 r^2}{3}} dr^2 + r^2 d\Omega^2.$$

rmk The pressure $p(r)$ $(r < r_0)$ can be found by above:

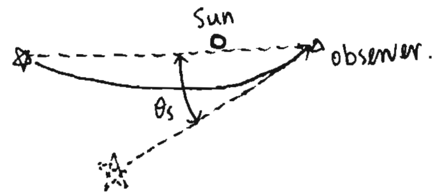
$$p = -\rho_0 + Ae^{-\nu/2} = \rho_0\left(-1 + \frac{-2\sqrt{1 - \frac{k\rho_0}{3} r_0^2}}{\sqrt{1 - \frac{k\rho_0}{3} r^2} - 3\sqrt{1 - \frac{k\rho_0}{3} r_0^2}}\right).$$

which would diverge at the center of the fluid $(r = 0)$ if $r_0 = \sqrt{\frac{8}{3k\rho_0}}$.

5. Now we study a well-known result in the Schwarzschild's model:

the bending of light rays, which gave Einstein worldwide fame in around 1918 A.D.

Einstein's ideas were that regarding a gravitational field acting on a light ray as a non-uniform optical medium.



The difference θ_s may be very small, for $\frac{r_g}{R_s} = \frac{2GM_s}{c^2 R_s} \approx 10^{-6}$, (the famous solar eclipse).
but if the star is far away from us (it's often in this case), the difference is measurable.

First, consider the coor. transf. $r = (1+\frac{r_g}{4r'})^2 r'$. Then the external Schwarzschild metric becomes

$$dl^2 = -(1-\frac{r_g}{4r'})^2 (1+\frac{r_g}{4r'})^{-2} c^2 dt^2 + (1+\frac{r_g}{4r'})^4 (dr'^2 + r'^2 d\Omega^2).$$

For $r \gg r_g$, we have $r' \approx r$ and $dl^2 \approx -c^2 dt^2 + (1+\frac{r_g}{r})(dx^2+dy^2+dz^2)$, with $\frac{r_g}{r} \ll 1$.

Then here the speed of the light is $\frac{c}{1+\frac{r_g}{r}}$ $\Rightarrow$ refractive index: $1+\frac{r_g}{r}$. $\Rightarrow \frac{1}{n} \approx 1 - \frac{r_g}{r}$.

Now put the model on the hyperplane $z=0$. By the formula of refractive error:

$$\theta_s = \int_{-\infty}^{\infty} \frac{\partial(\frac{1}{n})}{\partial y} dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial y}(1-\frac{r_g}{r}) dx = \int_{-\infty}^{\infty} -r_g \cdot \frac{-\frac{\partial r}{\partial y}}{r^2} dx \quad \left(\begin{array}{l} y = R_s \\ r^2 = x^2+y^2 \ (z=0) \end{array} \right.$$

$$= r_g R_s \int_{-\infty}^{\infty} \frac{dx}{(x^2+y^2)^{3/2}}$$

$$= r_g R_s \cdot \frac{2}{R_s^2} = \frac{4GM_s}{c^2 R_s} \approx 1''.75.$$

the experiment observation of which was conducted by Eddington in Australia in 1922 A.D..

rmk. Christoffel symbols when calculating the Schwarzschild metric:

$$\Gamma^1_{11} = \frac{\lambda'}{2},\ \Gamma^0_{01} = \frac{\nu'}{2},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^0_{11} = \frac{\dot\lambda}{2} e^{\lambda-\nu},\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{00} = \frac{\nu'}{2} e^{\nu-\lambda},$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{23} = \cot\theta,\ \Gamma^1_{01} = \frac{\dot\lambda}{2},\ \Gamma^0_{00} = \frac{\dot\nu}{2},\ \Gamma^1_{33} = -r\sin^2\theta\, e^{-\lambda}$$

and others are zero.

2. Axially Symmetric Solutions

Def. A Manifold (M, g) satisfying (EE) is said to be axially symmetric and stationary if it's indep. of an angular space-like coor. φ and the time t ($\varphi \in [0, 2\pi)$).

i.e., there is given an action on M with the abelian group $\mathbb{R} \times S^1$ leaving the metric invariant, and we further require the orbits of the subgroup $\mathbb{R} \times \{s_0\}$ are time-like and those of $\{t_0\} \times S_1$ are space-like.

We still solve it in the vacuum space, that is, $T_{ab} \equiv 0$, and (EE) becomes $R_{ab} \equiv 0$.

By the definition there exists time-like k and space-like m s.t. $[k, m] = 0$.

Then by the Frobenius thm., we can write with the coor. t, φ, x_1, x_2 :

$$ds^2 = -f dt^2 + 2m\, dt\, d\varphi + l\, d\varphi^2 + e^{\nu}(dx^{1\,2} + dx^{2\,2}) \qquad \text{(see P. 10).}$$

Now we let $\rho^2 := fl + m^2$, where f, m, l are all functions of x^1, x^2.

Then by $R_{ab} \equiv 0$, we have the non-vanishing Ricci components: (let $(\varphi, t) = (x^3, x^4)$)

$$R_{11} = \tfrac{1}{2}(\nu_{11} + \nu_{22}) + \tfrac{1}{\rho}\left(\rho_{11} - \tfrac{1}{2}\nu_1\rho_1 + \tfrac{1}{2}\nu_2\rho_2 - \tfrac{1}{2\rho}(f_1 l_1 + m_1^2)\right) = 0;$$

$$R_{22} = \tfrac{1}{2}(\nu_{11} + \nu_{22}) + \tfrac{1}{\rho}\left(\rho_{22} + \tfrac{1}{2}\nu_1\rho_1 - \tfrac{1}{2}\nu_2\rho_2 - \tfrac{1}{2\rho}(f_2 l_2 + m_2^2)\right) = 0;$$

$$R_{12} = \tfrac{1}{\rho}\left(\rho_{12} - \tfrac{1}{2}\nu_2\rho_1 - \tfrac{1}{2}\nu_1\rho_2 - \tfrac{1}{4\rho}(f_1 l_2 + f_2 l_1 + 2m_1 m_2)\right) = 0;$$

$$R_{33} = \tfrac{1}{2}e^{-\nu}\left(l_{11} + l_{22} + \tfrac{l}{4\rho^2}(f_1 l_1 + f_2 l_2 + m_1^2 + m_2^2) - \tfrac{l_1\rho_1}{\rho} - \tfrac{l_2\rho_2}{\rho}\right) = 0;$$

$$R_{34} = \tfrac{1}{2}e^{-\nu}\left(m_{11} + m_{22} + \tfrac{m}{\rho^2}(f_1 l_1 + f_2 l_2 + m_1^2 + m_2^2) - \tfrac{m_1\rho_1}{\rho} - \tfrac{m_2\rho_2}{\rho}\right) = 0;$$

$$R_{44} = \tfrac{1}{2}e^{-\nu}\left(-f_{11} - f_{22} - \tfrac{f}{\rho^2}(f_1 l_1 + f_2 l_2 + m_1^2 + m_2^2) + \tfrac{f_1\rho_1}{\rho} + \tfrac{f_2\rho_2}{\rho}\right) = 0.$$

Let $c := \dfrac{2\rho_{12}\rho_2 + (\rho_{11} - \rho_{22})\rho_1}{\rho_\alpha^2}$, $d := \dfrac{2\rho_{12}\rho_1 - (\rho_{11} - \rho_{22})\rho_2}{\rho_\alpha^2}$, $\Sigma' := f_1 l_1 - f_2 l_2 + m_1^2 - m_2^2$, $\Pi' := f_1 l_2 + f_2 l_1 + 2m_1 m_2$

and we get from above (in this discussion α takes sum over 1, 2):

$$\nu_1 = -\frac{\Sigma'\rho_1 + \Pi'\rho_2}{2\rho\,\rho_\alpha^2} + c; \quad \nu_2 = \frac{\Sigma'\rho_2 - \Pi^2\rho_1}{2\rho\,\rho_\alpha^2} + d;$$

$$\tilde{\delta} f + \frac{f}{\rho^2}(f_\alpha l_\alpha + m_\alpha^2) = 0; \quad \tilde{\delta} l + \frac{l}{\rho^2}(f_\alpha l_\alpha + m_\alpha^2) = 0; \quad \tilde{\delta} m + \frac{m}{\rho^2}(f_\alpha l_\alpha + m_\alpha^2) = 0$$

where $\tilde{\delta}$ is defined as $\partial_\alpha^2 - \frac{\rho_\alpha}{\rho}\partial_\alpha$.

Now consider $\gamma := \frac{1}{2}\ln f e^{\nu}$ and $w := \frac{m}{f}$. Then the metric becomes

$$ds^2 = \frac{1}{f}\left(e^{2\gamma}(dx^{1\,2} + dx^{2\,2}) + \rho^2 d\varphi^2\right) - f(dt - w\,d\varphi)^2.$$

and from above we have

$$\gamma_1 = -\frac{\Sigma\rho_1 + \Pi\rho_2}{4\rho\,\rho_\alpha^2} + \frac{c}{2}; \quad \gamma_2 = \frac{\Sigma\rho_2 - \Pi\rho_1}{4\rho\,\rho_\alpha^2} + \frac{d}{2}; \quad \delta f + \frac{f}{\rho^2}\left(w_\alpha^2 f^2 - \frac{\rho^2}{f^2} f_\alpha^2\right) = 0; \quad \tilde{\delta} w + 2w_\alpha \frac{f_\alpha}{f} = 0$$

where $\Sigma := -\frac{\rho^2}{f^2}(f_1^2 - f_2^2) + f^2(w_1^2 - w_2^2)$, $\Pi := -2\frac{\rho^2}{f^2} f_1 f_2 + 2f^2 w_1 w_2$ and $\delta := \partial_\alpha^2 + \frac{\rho_\alpha}{\rho}\partial_\alpha$.

7. Now from the eq'n $\tilde{\delta} w + 2 w_\alpha \frac{f_\alpha}{f} = 0$, we have $(\frac{w_1 f^2}{\rho})_1 + (\frac{w_2 f^2}{\rho})_2 = 0$.

Then $\exists$ a map g s.t. $w_1 = \frac{-\rho}{f^2} g_2$ and $w_2 = \frac{\rho}{f^2} g_1$. (called the Ernst equation).

And with this, the eq'n of f become $f \delta f + (g_\alpha^2 - f_\alpha^2) = 0$

Besides, from $g_{12} = g_{21}$ and $w_{12} = w_{21}$, we have $f \delta g - 2 f_\alpha g_\alpha = 0$.

Hence we're going to solve $\begin{cases} \delta f - \frac{1}{f}(f_\alpha^2 - g_\alpha^2) = 0 \\ \delta g - \frac{2}{f}(f_\alpha g_\alpha) = 0 \end{cases}$, and γ, w, ρ can be got.

We try to solve it in complex function. By the equations, let $\varepsilon := f + ig$,

and we have

$$
\begin{aligned}
(\mathrm{Re}\,\varepsilon)\cdot \delta\varepsilon &= f(\varepsilon_{11} + \varepsilon_{22} + \tfrac{\rho_1}{\rho}\varepsilon_1 + \tfrac{\rho_2}{\rho}\varepsilon_2) \\
&= f(f_{11} + f_{22} + \tfrac{\rho_1}{\rho} f_1 + \tfrac{\rho_2}{\rho} f_2 + i(g_{11} + g_{22} + \tfrac{\rho_1}{\rho} g_1 + \tfrac{\rho_2}{\rho} g_2) \\
&= f\cdot \tfrac{1}{f}(f_\alpha^2 - g_\alpha^2) + f\cdot i\cdot \tfrac{2}{f} f_\alpha g_\alpha = \varepsilon_\alpha^2.
\end{aligned}
$$

Then let $\xi := \frac{1+\varepsilon}{1-\varepsilon}$, producing

$$
\begin{cases} f = \dfrac{\xi\bar{\xi} - 1}{(\xi+1)(\bar{\xi}+1)} \\ g = -i\,\dfrac{\xi - \bar{\xi}}{(\xi+1)(\bar{\xi}+1)} \end{cases}
$$

$$
\begin{aligned}
(\xi\bar{\xi} - 1)\delta\xi &= (\xi\bar{\xi} - 1)(\xi_{\alpha\alpha} + \tfrac{\rho_\alpha}{\rho}\xi_\alpha) \\
&= \frac{2(\varepsilon + \bar{\varepsilon})}{(1-\varepsilon)(1-\bar{\varepsilon})}\cdot\left(2\left(\frac{\varepsilon_{\alpha\alpha}}{(1-\varepsilon)^2} + \frac{2\varepsilon_\alpha^2}{(1+\varepsilon)^3}\right) + \frac{\rho_\alpha}{\rho}\cdot\frac{2\varepsilon_\alpha}{(1-\varepsilon)^2}\right) \qquad f\delta\varepsilon = \varepsilon_\alpha^2 \\
&= \frac{4f}{(1-\varepsilon)(1-\bar{\varepsilon})}\cdot 2\cdot\left(\frac{\varepsilon_\alpha^2}{f(1-\varepsilon)^2} + \frac{2\varepsilon_\alpha^2}{(1-\varepsilon)^3}\right) \\
&= \frac{4f}{(1-\varepsilon)(1-\bar{\varepsilon})}\cdot\frac{2\varepsilon_\alpha^2}{(1-\varepsilon)^2}\cdot\frac{1-\varepsilon+2f}{f(1-\varepsilon)} \qquad 1-\varepsilon+2f = 1+\bar{\varepsilon} \\
&= 2\cdot\frac{4\varepsilon_\alpha^2}{(1-\varepsilon)^4}\cdot\frac{1+\bar{\varepsilon}}{1-\bar{\varepsilon}} = 2\bar{\xi}\xi_\alpha^2
\end{aligned}
$$

we'll use these to see the Kerr's solutions.

- Under the coor. $(\mu, \theta, \varphi, t)$, consider $\rho = \sinh\mu\sin\theta$, $z = \cosh\mu\cos\theta$. (in terms of cylindrical coor.)

  Then the equation above becomes $(\xi\bar{\xi} - 1)(\xi_{\mu\mu} + \xi_{\theta\theta} + \coth\mu\,\xi_\mu + \cot\theta\,\xi_\theta) = 2\bar{\xi}(\xi_\mu^2 + \xi_\theta^2)$

  which can be solved by $p\cosh\mu + iq\cos\theta = \xi$ where $p^2 + q^2 = 1$.

  Then we have
  $$f = \frac{p^2\cosh^2\mu + q^2\cos^2\theta - 1}{(p\cosh\mu + 1)^2 + q^2\cos^2\theta}$$
  $$w = \frac{q(p\cosh\mu + 1)\sin^2\theta}{2p(p^2\cosh^2\mu - 1 + q^2\cos^2\theta)}$$
  $$e^{2\gamma} = \frac{p^2\cosh^2\mu - 1 + q^2\cos\theta}{k^2} \quad \text{for some constant } k.$$

  Take $k = p$ and introduce $p = \frac{1}{m}$, $q = \frac{a}{m}$ ($\Rightarrow m^2 - a^2 = 1$)

  and $\rho = \sqrt{r^2 + a^2 - 2mr}\,\sin\theta$, $z = (r-m)\cos\theta$ ($\Rightarrow r = \cosh\mu\sqrt{m^2 - a^2} + m$). We have

$$ds^2 = (r^2 + a^2\cos\theta)\left(d\theta^2 + \frac{dr^2}{r^2 + a^2 - 2mr}\right) - dt^2 + (r^2 + a^2)\sin^2\theta\,d\varphi^2 + \frac{2mr}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta\,d\varphi)^2.$$

- Under the coor. (μ,θ,φ,t) consider $\rho=\cosh\mu\cos\theta$, $z=\sinh\mu\sin\theta$,

  and then likewise we have $(\xi\bar{\xi}-1)(\xi_{\mu\mu}+\xi_{\theta\theta}+\tanh\mu\,\xi_\mu-\tan\theta\,\xi_\theta)=2\bar{\xi}(\xi_\mu^2+\xi_\theta^2)$

  which can be solved by $\xi=p\sinh\mu+iq\sin\theta$ with $q^2-p^2=1$.

  and can based on these to get corresponding $f, \omega, e^{2\gamma}$ and dl^2 under $r=\sinh\mu\sqrt{a^2-m^2}+m$.

  (the metric got from above is in the same form as the previous one with $p=\frac{1}{m}$, $q=\frac{a}{m}$).

In the physical sense, m is the mass and ma is the angular momentum of the black hole.

Now we let $\Delta:=r^2+a^2-2mr$ and $\Sigma:=r^2+a^2\cos\theta$ and see some properties of the Kerr metric:

$$dl^2=-dt^2+\Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right)+(r^2+a^2)\sin^2\theta\,d\varphi^2+\frac{2mr}{\Sigma}(a\sin^2\theta\,d\varphi-dt)^2.$$

- When $m=0$, the metric reduce to the Minkowski metric:

  Under the spheroidal coor. $\begin{cases} x=\sqrt{r^2+a^2}\sin\theta\cos\varphi \\ y=\sqrt{r^2+a^2}\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$, we have

$$dx^2+dy^2+dz^2\overset{(m=0)}{=}\left(\frac{r}{\sqrt{\Delta}}\sin\theta\cos\varphi\,dr+\sqrt{\Delta}\cos\theta\cos\varphi\,d\theta-\sqrt{\Delta}\sin\theta\sin\varphi\,d\varphi\right)^2+\left(\frac{r}{\sqrt{\Delta}}\sin\theta\sin\varphi\,dr+\sqrt{\Delta}\cos\theta\sin\varphi\,d\theta+\sqrt{\Delta}\sin\theta\cos\varphi\,d\varphi\right)^2+(\cos\theta\,dr-r\sin\theta\,d\theta)^2$$

$$=\left(\frac{r^2}{\Delta}\sin^2\theta+\cos^2\theta\right)dr^2+(\Delta\cos^2\theta+r^2\sin^2\theta)\,d\theta^2+\Delta\sin^2\theta\,d\varphi^2$$

Then $-dt^2+dx^2+dy^2+dz^2=-dt^2+\frac{\Sigma}{\Delta}dr^2+\Sigma\,d\theta^2+(r^2+a^2)\sin^2\theta\,d\varphi^2:=dl^2$.

- When $a=0$, the metric reduce to the Schwarzschild metric:

$$dl^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2(d\theta^2+\sin^2\theta\,d\varphi^2).\quad\text{(non-rotating case.)}$$

- When $r\to\infty$, the metric reduces to the Minkowski metric.

  i.e., it's asymptotically flat:

$$dl^2\approx-dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2\quad\left(\frac{\Sigma}{\Delta}\to1,\ \frac{2mr}{\Sigma}\to0 \text{ as } r\to\infty\right).$$

- The metric under this coor. (r,θ,φ,t), called the Boyer-Lindquist coor.,

  has singularities at $\Delta=0$ and $\Sigma=0$, while the former is a coor. singularity:

  Consider $d\bar{t}=dt+\frac{r^2+a^2}{\Delta}dr$, and $d\bar{\varphi}=d\varphi+\frac{a}{\Delta}dr$, and we have

$$-dt^2=-d\bar{t}^2-\frac{(r^2+a^2)^2}{\Delta^2}dr^2+2\cdot\frac{r^2+a^2}{\Delta}d\bar{t}\,dr$$

$$(r^2+a^2)\sin^2\theta\,d\varphi^2=(r^2+a^2)\sin^2\theta\,d\bar{\varphi}^2+(r^2+a^2)\frac{a^2}{\Delta^2}\sin^2\theta\,dr^2-2(r^2+a^2)\frac{a}{\Delta}\sin^2\theta\,dr\,d\bar{\varphi}$$

$$(dt-a\sin^2\theta\,d\varphi)^2=\left(d\bar{t}-a\sin^2\theta\,d\bar{\varphi}-\frac{\Sigma}{\Delta}dr\right)^2=d\bar{t}^2+a^2\sin^4\theta\,d\bar{\varphi}^2+\frac{\Sigma^2}{\Delta^2}dr^2-2a\sin^2\theta\,d\bar{t}\,d\bar{\varphi}$$

$$\Rightarrow dl^2=-\left(1-\frac{2mr}{\Sigma}\right)d\bar{t}^2+2\,d\bar{t}\,dr+\Sigma\,d\theta^2+\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\,d\bar{\varphi}^2-2a\sin^2\theta\,dr\,d\bar{\varphi}-\frac{4mra}{\Sigma}\sin^2\theta\,d\bar{t}\,d\bar{\varphi}.$$

In this Kerr coor. $(r,\theta,\bar{\varphi},\bar{t})$, the singularity $\Delta=0$ in the B-L coor. is removed.

9. To see the structure the Kerr Black Hole, we continue using the spheroidal coor.:

$$\begin{cases} x = \sqrt{r^2+a^2}\sin\theta\cos(\varphi - \tan^{-1}\frac{a}{r}) \\ y = \sqrt{r^2+a^2}\sin\theta\sin(\varphi - \tan^{-1}\frac{a}{r}) \\ z = r\cos\theta \end{cases}$$

in which the metric becomes

$$dl^2 = \left(\eta_{ij} + \frac{2mr^3}{r^4+a^2z^2} k_i k_j\right) dx^i dx^j \quad — \circledast$$

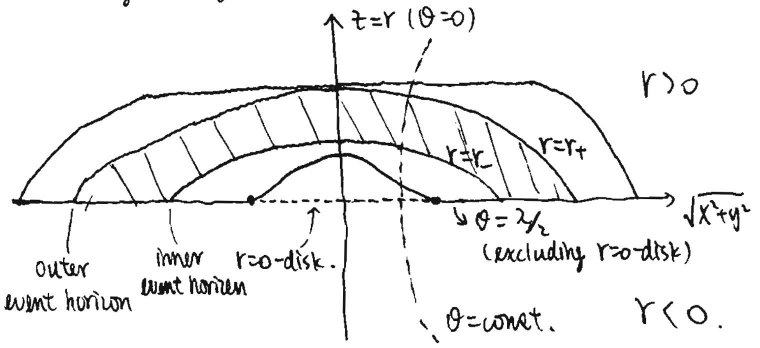
where $\eta_{ij} = \begin{cases} -1 & \text{if } i=j=0 \\ 1 & \text{if } i=j\neq 0 \\ 0 & \text{otherwise} \end{cases}$ and $k = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right)$.

(Then some of the properties mentioned can be directly seen in the relation, for the η_{ij} represents the Minkowski's metric).

Then r is implicitly determined by x, y, z: $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} - 1 = 0$.

Now we let $\Delta = 0$ at $r = r_\pm$ ($\Leftrightarrow r_\pm = m \pm \sqrt{m^2-a^2}$).

Then this coor. singularity doesn't exist in the case $a>m$. For $m \geq a$:



The "r=0-disk" denotes the area $r=0$, $x^2+y^2 \leq a$, which is exactly the singularity of $\circledast$.

However, for the genuine singularity $\Sigma = 0$, it is located at $r=0$, $\theta = \frac{\pi}{2}$, i.e., $\partial(r=0\text{-disk})$, called the ring singularity. Then in the r=0-disk with $\theta < \frac{\pi}{2}$, the metric is still regular.

• zero angular momentum observer (ZAMO)

Consider an object falling toward the black hole with zero angular momentum with velocity u ($u_\varphi = 0$).

Since the Kerr metric is asymptotically flat, we have $u^\varphi = \eta^{\varphi\eta} u_\eta = \eta^{\varphi\varphi} u_\varphi = 0$. when $r \to \infty$.

(The angular velocity of the ZAMO is defined as $\Omega := \frac{d\varphi}{dt} = \frac{u^\varphi}{u^t}$.)

But, it vanish only at infinity, to compute which we have: $0 = u_\varphi = g_{\varphi\varphi} u^\varphi + g_{\varphi t} u^t$.

Then $\Omega = \frac{u^\varphi}{u^t} = -\frac{g_{\varphi t}}{g_{\varphi\varphi}} = -\frac{2mra\sin^2\theta}{\Sigma} \Big/ \left((r^2+a^2)\sin\theta + \frac{2mra^2\sin^4\theta}{\Sigma}\right) = \frac{2mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}$.

Since $(r^2+a^2)^2 > a^2(r^2+a^2-2mr)\sin^2\theta$, we have $\Omega/(ma) > 0$ always!

i.e., the angular velocity of the ZAMO has the same sign as the angular momentum of the black hole. In other words, the ZAMO always "corotates" with the black hole.

Details of the argument applying the Frobenius thm.:

First by $[k,m]=0$, we know $\exists$ cor. (t,φ). Now we claim: $(\mathrm{span}\langle k,m\rangle)^{\perp}$ is also integrable.

To show the claim, I apply the Cartan's lemma:

Let $v_1,\dots,v_n$ be linearly indep. vectors and $w_1,\dots,w_n$ s.t. $v_1\wedge w_1+\dots+v_n\wedge w_n=0$ in a v.s. V over K.

Then $\exists\, a_{ij}\in K$ s.t. $\forall i=1\sim n$, $w_i=\sum_{j=1}^{n}a_{ij}v_j$.

Recall that the definition of the volume form: $\varepsilon:=\sqrt{|g|}\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3$. and we had established

the relation: $\varepsilon^{a_1a_2a_3a_4}\varepsilon_{b_1b_2b_3b_4}=(-1)\,4!\,\delta^{[a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4]}_{b_4}$ where $[\ ]$ means the alternating sum.

(pf. of the claim). Let ξ and ψ be the 1-form dual to k and m. i.e., $g(k,X)=\xi(X)$, $g(m,X)=\psi(X)$.

Now to show $\xi=0$ and $\psi=0$ is integrable, we have to verify $\begin{cases} d\xi=\mu_1\wedge\xi+\mu_2\wedge\psi \\ d\psi=\lambda_1\wedge\xi+\lambda_2\wedge\psi\end{cases}$.

By the Cartan's lemma, it suffices to show $\psi\wedge\xi\wedge d\xi=0$ and $\xi\wedge\psi\wedge d\psi=0$.

Consider 1-form w with $w_a=\varepsilon_{abcd}\,\xi^b\underbrace{g^{ce}\nabla_e\xi^d}_{(=:\nabla^c\xi^d)}$. Then we have $\psi\wedge\xi\wedge d\xi=w(m)\,\varepsilon$.

To show $w(m)=0$, for any vector field X, by the Cartan's homotopy formula:

$dw(X,m)=-dw(m,X)=-i(m)dw(X)=di(m)w(X)-L_m w(X)=\underline{dw(m)}(X)-L_m w(X)=X(w(m))-L_m w(X)$.

Hence $X(w(m))=dw(X,m)+L_m w(X)$.

- $L_m w=0$:

  $w_a=\varepsilon_{abcd}\,\xi^b g^{ce}\nabla_e\xi^d$ and $[m,k]=0$ and $L_m g=0$ (Killing condition),

  we have that the components of ξ and g are independent of m.

- $dw=0$:

  Calculate

$$\begin{aligned}\varepsilon^{abcd}\nabla_c w_d&=\varepsilon^{abcd}\nabla_c(\varepsilon_{defg}\xi^e\nabla^f\xi^g) \qquad \leftarrow \varepsilon^{abcd}\varepsilon_{defg}=-3!\,\delta^{[a}_{e}\delta^{b}_{f}\delta^{c]}_{g}\\ &=-3!\,\nabla_c(\xi^{[a}\nabla^{b}\xi^{c]}) \qquad\qquad \leftarrow \nabla^b\xi^c+\nabla^c\xi^b=0\ (\because \text{Killing})\\ &=-3!\cdot\tfrac{1}{3!}\cdot 2\nabla_c(\xi^a\nabla^b\xi^c+\xi^b\nabla^c\xi^a+\xi^c\nabla^a\xi^b)\end{aligned}$$

By $\begin{cases}\nabla_a\nabla_b\xi_c+\nabla_b\nabla_c\xi_a=-R_{cab}{}^{d}\xi_d\\ \nabla_b\nabla_c\xi_a+\nabla_c\nabla_a\xi_b=-R_{abc}{}^{d}\xi_d\\ -\nabla_c\nabla_a\xi_b-\nabla_a\nabla_b\xi_c=R_{bca}{}^{d}\xi_d\end{cases}\Rightarrow \nabla_a\nabla_b\xi_c=R_{abc}{}^{d}\xi_d.$

$$\begin{aligned}\Rightarrow\ \nabla_c(\xi^a\nabla^b\xi^c+\xi^b\nabla^c\xi^a)&=(\nabla_c\xi^a)(\nabla^b\xi^c)+(\nabla_c\xi^b)(\nabla^c\xi^a)+\xi^a\nabla_c\nabla^b\xi^c+\xi^b\nabla_c\nabla^c\xi^a\\ &=(\nabla_c\xi^a)(\nabla^b\xi^c+\nabla^c\xi^b)+\xi^a\xi^b R_{db}-\xi^b\xi^d R_{da}=0\end{aligned}$$

and $\nabla_c(\xi^c\nabla^a\xi^b)=(\nabla_c\xi^c)(\nabla^a\xi^b)+\xi^c\nabla_c\nabla^a\xi^b=\xi^c\xi^d R^{ab}{}_{dc}=0$.

Hence $*dw=0\Rightarrow dw=0$, and the claim is proven. ✡

3. Homogeneous and Isotropic Cosmology

The idea of the homogeneity of the cosmology come from optical and radio surveys of the sky, as well as the very accurate uniformity of the microwave background radiation, suggesting that the distribution of galaxies is the same in all directions.

The assumption that the universe is homogeneous and isotropic is termed the cosmological principle.

To study it, we start from seeing the symmetry of a manifold.

In the following text we'll apply the notation:

symmetric sum : $(\cdot)_{(1\cdots n)} := \frac{1}{n!}\sum_{\sigma\in S_n}(\cdot)_{\sigma(1)\cdots\sigma(n)}$.

alternating sum : $(\cdot)_{[1\cdots n]} := \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)(\cdot)_{\sigma(1)\cdots\sigma(n)}$ ($(\cdot)$ is often a tensor.)

Now on a manifold (M,g), let ξ be a killing vector, i.e., $\mathcal{L}_\xi g_{ij}=0$.

To see "how symmetric" the manifold M is, the properties of ξ is essential.

First by definition, we have

$$\xi^k\partial_k g_{ij} + g_{kj}\partial_i\xi^k + g_{ik}\partial_j\xi^k = 0.$$

Since the connection is compatible with the metric, we have

$$0 = \nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^n_{ik}g_{nj} - \Gamma^n_{jk}g_{in}$$

From two above we obtain

$$\begin{aligned} 0 &= \xi^k(\Gamma^k_{i\underline{k}}g_{kj} + \Gamma^k_{j\underline{k}}g_{ik}) + g_{kj}\partial_i\xi^k + g_{ik}\partial_j\xi^k \\ &= g_{kj}(\partial_i\xi^k + \Gamma^k_{in}\xi^n) + g_{ik}(\partial_j\xi^k + \Gamma^k_{jn}\xi^n) \\ &= \nabla_i\xi_j + \nabla_j\xi_i \quad — \star \end{aligned}$$

By definition we have $(\nabla_k\nabla_j - \nabla_j\nabla_k)\xi_i = R^m_{ijk}\xi_m$.

Then by the relation $\star$ above and the first Bianchi identity:

$$R^m_{ijk} + R^m_{jki} + R^m_{kij} = 0,$$

We obtain:

$$\nabla_{[k}\nabla_j\xi_{i]} = 0.$$

Then

$$\begin{aligned} \nabla_{[k}\nabla_{j]}\xi_i &= (\nabla_i\nabla_j\xi_k - \nabla_j\nabla_i\xi_k + \nabla_k\nabla_i\xi_j - \nabla_i\nabla_k\xi_j)/2 \\ &= (-\nabla_i\nabla_k\xi_j + \nabla_j\nabla_k\xi_i - \nabla_k\nabla_j\xi_i - \nabla_i\nabla_k\xi_j)/2 \end{aligned}$$

Hence
$$\tfrac{1}{2}R^{m}_{ijk}\xi_m = -\nabla_i\nabla_k\xi_j + \tfrac{1}{2}R^{m}_{ikj}\xi_m$$
$$\Rightarrow \nabla_i\nabla_k\xi_j = R^{m}_{ikj}\xi_m .$$

Above is the general properties of killing vectors. Now we start:

Def. (M,g) is called homogeneous if it's homogeneous at every $p\in M$, in the sense that $\forall q\in M$, $\exists$ an isometry φ st. $\varphi(p)=q$.
i.e., $\forall p\in M$, there's a killing vector with any given direction.

Def. (M,g) is called isotropic if it's isotropic at every $p\in M$, in the sense that $\forall\, v, w\in T_pM$, $\exists$ an isometry φ st. $\varphi(p)=p$, $\varphi_*(v)=w$.
i.e., $\forall p\in M$, $\exists$ a killing form ξ st. $\xi^i(p)=0$ with $\nabla_i\xi_j(p)$ taking arbitrary values (with the constraint we just got: $\nabla_i\xi_j(p)+\nabla_j\xi_i(p)=0$).

Now by above $\nabla_m\nabla_j\xi_k = R^{n}_{mjk}\xi_n$, we have
$$\nabla_i\nabla_m\nabla_j\xi_k = (\nabla_i R^{n}_{mjk})\xi_n + R^{n}_{mjk}\nabla_{[i}\xi_{n]} .$$
and by definition
$$2\nabla_{[i}\nabla_{m]}\nabla_j\xi_k = R^{n}_{kmi}\nabla_j\xi_n + R^{n}_{jmi}\xi_n$$

Hence
$$(\nabla_i R^{n}_{mjk} - \nabla_m R^{n}_{ijk})\xi_n = R^{n}_{kmi}\nabla_j\xi_n + R^{n}_{jmi}\nabla_n\xi_k + R^{n}_{ijk}\nabla_m\xi_n - R^{n}_{mjk}\nabla_i\xi_n$$
$$= (R^{n}_{jim}\delta^{s}_{k} - R^{n}_{kim}\delta^{s}_{j} + R^{n}_{ijk}\delta^{s}_{m} - R^{n}_{mjk}\delta^{s}_{i})\nabla_s\xi_n$$

By the condition of isotropy, for any $p\in M$, we can choose ξ st. $\xi(p)=0$ and $\nabla_s\xi_n(p)$ is an arbitrary antisymmetric matrix. (every entry is a vector)

Then we obtain
$$\begin{cases} \nabla_i R^{l}_{mjk} - \nabla_m R^{l}_{ijk} = 0 & -① \\ R^{[l}_{jim}\delta^{s]}_{k} - R^{[l}_{kim}\delta^{s]}_{j} + R^{[l}_{ijk}\delta^{s]}_{m} - R^{[l}_{mjk}\delta^{s]}_{i} = 0. & -② \end{cases}$$

Expand ② and contract s, k:
$$nR^{l}_{jim} - R^{l}_{jim} - R^{l}_{jim} + R^{k}_{kim}\delta^{l}_{j} + R^{l}_{ijm} - R^{k}_{ijk}\delta^{l}_{m} - R^{l}_{mji} + R^{k}_{mjk}\delta^{l}_{i} = 0.$$

$$\xrightarrow[\text{Bianchi Identity}]{\text{1st.}} \quad (n-1)R^{l}_{jim} + R_{ij}\delta^{l}_{m} - R_{mj}\delta^{l}_{i} = 0.$$

Sum over j, m $\overset{\times g^{jm}}{\Rightarrow}$ $nR^{l}_{i} - R\delta^{l}_{i} = 0. \Rightarrow nR_{li} - Rg_{li} = 0.$

Coming two above:

$$(n-1)R_{kjim} = R_{mj}g_{ki} - R_{ij}g_{km} = \frac{R}{n}(g_{mj}g_{ki} - g_{mk}g_{ji})$$

$$\Rightarrow R_{kjim} = \frac{R}{n(n-1)}(g_{mj}g_{ki} - g_{mk}g_{ji})$$

We knew that:

Lemma (M,g) has constant sectional curvature at $p \in M \Leftrightarrow R_{ijkl} = K(p)(g_{ik}g_{jl} - g_{il}g_{jk})$.

Hence we conclude that (M,g) has constant curvature at p $\forall p \in M$.

Moreover by ①, contract l, j: $\nabla_i R_{mk} - \nabla_m R_{ik} = 0$.

Sum over m, k $\xRightarrow{\times g^{mk}}$ $\nabla_i R - \nabla_m R^m_{\ i} = 0$. $\Rightarrow \nabla_i R - \frac{1}{2}\nabla_i R = 0 \Rightarrow \nabla_i R = 0$.

Hence R is a constant on M, implying that M has constant sectional curvature too!

Lemma. If (M,g) is complete and simply-connected with constant sectional curvature,

(1) $K=-1 \Leftrightarrow M \simeq \mathbb{H}^n$; (2) $K=0 \Leftrightarrow M \simeq \mathbb{R}^n$; (3) $K=1 \Leftrightarrow M \simeq \mathbb{S}^n$.

(Both lemmas above can be found in the Differential Geometry by Chin-Lung Wang.).

Hence we have:

Thm. If (M,g) is homogeneous, isotropic, complete and simply-connected, then it has constant curvature K and is isotropic to $\mathbb{H}^n$, $\mathbb{R}^n$ or $\mathbb{S}^n$ up to K.

Now let $\varepsilon := \text{sign}(K)$ and $a := \sqrt{\frac{1}{|K|}}$, and we can realize M as

$$\{(x^1, x^2, x^3, z) \mid \varepsilon \delta_{\alpha\beta} x^\alpha x^\beta + z^2 = a^2\}. \Rightarrow \varepsilon \delta_{\alpha\beta} x^\alpha dx^\beta + z\,dz = 0. \quad (\{x^1, x^2, x^3\}: \text{Euclidean}).$$

Then if the time is fixed, by eliminating z, we get the metric of the space:

$$ds^2 = \delta_{\alpha\beta}\,dx^\alpha dx^\beta + \varepsilon\,dz^2 = \left(\delta_{\alpha\beta} + \frac{\varepsilon x^\alpha x^\beta}{a^2 - \varepsilon \delta_{ij} x^i x^j}\right) dx^\alpha dx^\beta.$$

Introducing the spherical coor. $\{r, \theta, \varphi\}$, we have:

$$ds^2 = \frac{a^2\,dr^2}{a^2 - \varepsilon r^2} + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{=: d\Omega^2 \text{ later.}}$$

If redefine r as $\sqrt{|K|}\,r$ when $K \neq 0$ (in the case $K=0$ all is flat), then

$$ds^2 = \frac{1}{|K|^{-1}}\left(\frac{dr^2}{1-\varepsilon r^2} + r^2 d\Omega^2\right).$$

Then we may take our metric : $dl^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-\varepsilon r^2} + r^2 d\Omega^2\right)$. for some function a.

conventionally set positive.

Under this metric we solve the (EE) in the perfect fluid:

$$R_{ab} - \frac{1}{2} R g_{ab} = k T_{ab} \quad , \quad T_{ab} = (\rho + p) u_a u_b + p\, g_{ab}$$

with again $u = (1, 0, 0, 0)$. Then we calculate R_{ab} and Γ^a_{bc} in P.15. and get:

$$R_{00} - \frac{1}{2} R g_{00} = k T_{00} \Rightarrow -3\frac{\ddot{a}}{a} + \frac{1}{2} \cdot \frac{b}{a^2}(a\ddot{a} + \dot{a}^2 + \varepsilon) = k\rho .$$

$$R_{11} - \frac{1}{2} R g_{11} = k T_{11} \Rightarrow \frac{a\ddot{a} + 2\dot{a}^2 + 2\varepsilon}{1 - \varepsilon r^2} - \frac{1}{2} \cdot \frac{b}{a^2}(a\ddot{a} + \dot{a}^2 + \varepsilon) \cdot \frac{a^2}{1 - \varepsilon r^2} = k \cdot \frac{a^2}{1 - \varepsilon r^2} \cdot p .$$

Hence we obtain
$$\begin{cases} 3(\varepsilon + \dot{a}^2) = k a^2 \rho \text{ --- } (1) \\ \varepsilon + 2a\ddot{a} + \dot{a}^2 = -k a^2 p \text{ --- } (2) \end{cases}$$

The metric is called the Robertson-Walker's metric and above is called the Friedmann eq'n.

Now we start to see what it implies. By the eq'n, we have

$$\ddot{a} = \frac{-k}{6}(\rho + 3p) < 0 .$$

Hence for any fixed t_0, we have $\forall t$, $a(t) \leq \dot{a}_0 (t - t_0) + a_0$.

Set $t_0 := \frac{a_0}{\dot{a}_0}$, the origin of time in some sense. Then above implies $a(t) \leq \dot{a}_0 t$, so there's some $\bar{t} < t_0$ st. $a(\bar{t}) = 0$. (corresponding to the Big Bang).

(The value t_0^{-1} is the well-known Hubble constant.)

For some reasons of physics we write $\rho = \rho_m + \rho_r$ and $p = p_r$ with $p_r = \frac{1}{3}\rho_r$, where "m" means matter and "r" means radiation.

Now by the conservation law $D_i T^i_j = 0$, we have:

$$0 = D_i T^i_0 = \partial_i T^i_0 + \Gamma^i_{\ell i} T^\ell_0 - \Gamma^\ell_{0i} T^i_\ell = \partial_0 T^0_0 + T^0_0 (\Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}) - T^1_1 (\Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03})$$
$$= -\dot{\rho} + (-\rho)\left(\frac{3\dot{a}}{a}\right) - p \cdot \left(\frac{3\dot{a}}{a}\right) .$$

$\Rightarrow \dot{\rho} a + 3(p + \rho)\dot{a} = 0$. Now we discuss:

- $p = 0$, $\rho = \rho_m$ (matter-dominated):

$$\dot{\rho} a^3 + 3\rho \dot{a} a^2 = 0 \Rightarrow \frac{d}{dt}(\rho a^3) = 0 \Rightarrow \rho a^3 = \text{const. indep. of } t.$$

Differentiate (1): $b\dot{a}\ddot{a} = 2k a\dot{a}\rho + k a^2 \dot{\rho} = 2k a\dot{a}\rho - 3k\rho\dot{a}a = -k\rho\dot{a}a$

$$\Rightarrow \dot{a}^2 = -\varepsilon - 2a\ddot{a} = \frac{k\rho a^3}{3a} - \varepsilon = \frac{k\rho_0 a_0^3}{3a} - \varepsilon .$$

Now we see the cases $\varepsilon = 1, 0, -1$.

• $\varepsilon=1$:

$\dot a$ decreases until it vanishes at $a=\frac{k}{3}\rho_0 a_0^3 =: a_{max}$.

Hence the expansion of the universe slow down until it stops and turns into a contraction.

If we rescale t s.t. $t=0$ at $R=0$, then $\dot a^2=\frac{k}{3a}\rho_0 a_0^3-1$ admits a sol'n (solving ODEs)

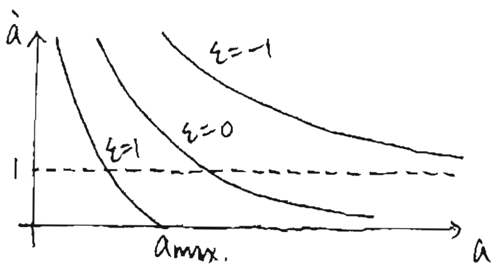
$$t=\frac{1}{2}a_{max}\left(\cos^{-1}\left(1-\frac{2a}{a_{max}}\right)-2\frac{a}{a_{max}}\sqrt{1-\frac{a}{a_{max}}}\right).$$

This kind of universe is termed "closed" since its space is finite.

When $\varepsilon=0,-1$, the space may be infinite, and has sol'ns:

• $\varepsilon=0$: $t=\frac{2}{3}\sqrt{\frac{3a^3}{k\rho_0 a_0^3}}$.

• $\varepsilon=-1$: $t=\frac{k\rho_0 a_0^3}{6}\left(\sqrt{1+\left(1+\frac{6a}{k\rho_0 a_0^3}\right)^2}-\cosh^{-1}\left(1+\frac{6a}{k\rho_0 a_0^3}\right)\right)$.



• $\rho=\rho_r$, $p=p_r=\frac{1}{3}\rho_r$ (radiation-dominated):

$\dot\rho a^4+4\rho\dot a a^3=0 \Rightarrow \frac{d}{dt}(\rho a^4)=0 \Rightarrow \rho a^4=$ const, indep. of t.

Then $\dot a^2=\frac{k\rho a^4}{3a^2}-\varepsilon=\frac{k\rho_0 a_0^4}{3a^2}-\varepsilon$.

in which we see that the expansion goes faster than that in the previous case.

One of the unsolved problems in cosmology is to find the model fitting with the observed universe.

rmk In fact, (M,g) is isotropic only if (M,g) is homogeneous;
(M,g) is homogeneous only if (M,g) is complete.

Hence in our discussion we may just call (M,g) an isotropic and simply-connected mfd.

rmk The Christoffel symbols and some curvature of the Robertson-Walker metric:

$\Gamma^0_{11}=\frac{a\dot a}{1-\varepsilon r^2}$, $\Gamma^0_{22}=a\dot a r^2$, $\Gamma^0_{33}=a\dot a r^2\sin^2\theta$,

$\Gamma^1_{01}=\Gamma^2_{02}=\Gamma^3_{03}=\frac{\dot a}{a}$, $\Gamma^1_{22}=-r(1-\varepsilon r^2)$, $\Gamma^1_{33}=-r(1-\varepsilon r^2)\sin^2\theta$,

$\Gamma^2_{12}=\Gamma^3_{13}=\frac{1}{r}$, $\Gamma^2_{33}=-\sin\theta\cos\theta$, $\Gamma^3_{23}=\cot\theta$ and others vanish.

$R_{00}=-3\frac{\ddot a}{a}$, $R_{11}=\frac{a\ddot a+2\dot a^2+2\varepsilon}{1-\varepsilon r^2}$, $R_{22}=r^2(a\ddot a+2\dot a^2+2\varepsilon)$, $R_{33}=r^2(a\ddot a+2\dot a^2+2\varepsilon)\sin\theta$

and others vanish, and $R=\frac{6}{a^2}(a\ddot a+\dot a^2+\varepsilon)$.

4. More General Models

Last time we obtained some results of homogeneous models given some conditions. by seeing some special properties of killing vectors. This time we construct the universe model by the 3-dimentional Lie algebra: Bianchi models.

Now assume the mnf. M can be foliated with 3-dimensional spatial section $M \simeq \mathbb{R} \times \Sigma_t$. and say Σ_t admits a 3-dimensional Lie algebra of killing vectors $\langle \xi_1, \xi_2, \xi_3 \rangle$ with structure constants $\{C^k_{ij}\}_{i,j,k=1\sim 3}$, i.e., $\forall i,j = 1\sim 3$, $[\xi_i, \xi_j] = C^k_{ij} \xi_k$.

Fix $p_0 \in M$ and find a basis $\{e_i\}_{i=1\sim 3}$ of its tangent space, and extend them to be vector fields by Lie transporting, i.e., requiring $\forall i,j$, $L_{\xi_i} e_j = 0$.

Since $\{e_i\}$ and $\{\xi_i\}$ both span the tangent spaces at every point, ξ_i should coincide with e_i at some point $p \in M$, say $\xi_i|_{p_0} = e_i|_{p_0}$, and generally they are different up to an intervible transformation, say $e_i = \alpha^j_i \xi_j$.

Now for i,j, we define D^k_{ij} by $[e_i, e_j] = D^k_{ij} e_k$. Then by the invariant condition above:

$$0 = [\xi_i, e_j] = [\xi_i, \alpha^k_j \xi_k] = \xi_i(\alpha^k_j)\xi_k + \alpha^k_j[\xi_i, \xi_k] = \xi_i(\alpha^k_j)\xi_k + \alpha^l_j C^k_{il} \xi_k .$$

Hence at p_0, since $\alpha^i_j = \delta^i_j$, we have $\xi_i(\alpha^k_j) = -C^k_{ij}$. Then

$$D^k_{ij} e_k = [e_i, e_j] = [\alpha^l_i \xi_l, \alpha^m_j \xi_m] = \alpha^l_i \xi_l(\alpha^m_j)\xi_m - \alpha^m_j \xi_m(\alpha^l_i)\xi_l + \alpha^l_i \alpha^m_j [\xi_l, \xi_m]$$

$$\Rightarrow D^k_{ij} e_k|_{p_0} = \left(-C^m_{ij}\xi_m + C^l_{ji}\xi_l + C^n_{ij}\xi_n\right)\Big|_{p_0} \Rightarrow D^k_{ij} = C^k_{ji} = -C^k_{ij} \text{ at } p_0 .$$

Since C^k_{ij}. D^k_{ij} just depend on t, we have $[e_i, e_j] = -C^k_{ij} e_k$.

Now let $\{w^i\}_{i=1\sim 3}$ be the dual 1-form of $\{e_i\}_{i=1\sim 3}$. By the Cartan's intrinsic formula:

$$dw^k(e_i, e_j) = e_i(w^k(e_j)) - e_j(w^k(e_i)) - w^k([e_i, e_j]) = -w^k(D^l_{ij} e_l) = C^k_{ij}.$$

Hence we can get $dw^k = \frac{1}{2} C^k_{ij} w^i \wedge w^j$. Then we can have a metric admitting the killing vector fields ξ_1. ξ_2. ξ_3: $dl^2 = g_{ij} w^i \otimes w^j$

where g_{ij} is a constant for any given time t.

Recall that in the reference [1] we have learned that there are roughly 9 types of 3-dimensional Lie algebra, called Bianchi models of type I ~ IX.

Based on their structure constants we can find some invariant basis $\{e_i\}$ and use them to find some types of metrics with undetermined g_{ij}, to be solved by the Einstein's eqn.

17. To see the detailed list of the 9 types' Killing vector fields and their corresponding invariant bases, one can consult the reference [4], supplying 9 types' killing vector fields, invariant basis and their dual one-forms.

rmk. The Friedmann-Robertson-Walker models are corresponding to the Bianchi's models I when $\varepsilon=0$, V when $\varepsilon=-1$ and IX when $\varepsilon=1$. (special cases).

There's no general way to solve a solution for all types, so we see some.

- Type-I model.

For the type-I model, $C^i_{jk}=0$ for all i,j,k. We have killing vectors $\xi_1=\partial_1$, $\xi_2=\partial_2$, $\xi_3=\partial_3$ and invariant bases $e_1=\partial_1$, $e_2=\partial_2$, $e_3=\partial_3$.

Then $w^i=dx^i$ for $i=1\sim3$. Then we have the metric $dl^2=-dt^2+\delta_{ij}g_{ij}dx^idx^j$.

One of the suggested form is $dl^2=-dt^2+A^2dx^2+B^2(dy^2+dz^2)$ where A, B only depend on x, t and suppose $B=f(x)g(t)$. Then (EE) gives

$$\begin{cases} 2\frac{\ddot{B}}{B}-\frac{B'^2}{A^2B^2}+\frac{\dot{B}^2}{B^2}=-kp \\ \frac{\ddot{B}}{B}-\frac{B''}{A^2B}+\frac{A'B'}{A^3B}+\frac{\dot{A}\dot{B}}{AB}+\frac{\ddot{A}}{A}=-kp \\ 2\frac{B''}{A^2B}-2\frac{A'B'}{A^3B}-2\frac{\dot{A}\dot{B}}{AB}+\frac{B'^2}{A^2B^2}-\frac{\dot{B}^2}{B^2}=-kp \\ \dot{B}'-\frac{B'\dot{A}}{A}=0 \end{cases}$$

with $T_{ij}=(p+\rho)u_iu_j-pg_{ij}$ and $(\dot{\cdot}):=\frac{d}{dt}$, $(\cdot)':=\frac{d}{dx}$.

The conservation of energy $\nabla_i T^i_j=0$ implies $p'=0$. and $\frac{\dot{\rho}}{p+\rho}=-\left(\frac{\dot{A}}{A}+\frac{2\dot{B}}{B}\right)$. $(\Rightarrow p=p(t))$.

By the last eq'n, we have $A=B'h(x)$ for some function h. Then the first two:

$$\begin{cases} \frac{1}{h^2f^2}=kpg^2+2g\ddot{g}+\dot{g}^2 \\ \frac{h'}{h^3ff'}=-kpg^2-2g\ddot{g}-\dot{g}^2 \end{cases} \Rightarrow hf=\lambda \text{ for some constant } \lambda.$$

Hence $\frac{1}{\lambda^2}=kpg^2+2g\ddot{g}+\dot{g}^2$, and we have $A=\lambda g\frac{f'}{f}$, $B=gf$.

Then set $f(x)=e^{\frac{x}{\lambda}}$, and we have $dl^2=-dt^2+g(t)^2\left(dx^2+e^{\frac{2x}{\lambda}}(dy^2+dz^2)\right)$.

The conservation condition gives $\frac{\dot{\rho}}{p+\rho}=-3\frac{\dot{g}}{g}\xrightarrow{\text{ODE}}\rho=\frac{-3}{g^3}\int p(t)g^2g_t\,dt+M(x)$ for some functions M.

However, by the 3rd eq'n and $A = kg\frac{f'}{f}$, $B = gf$, we have

$-k\rho = \frac{3}{\lambda^2 g^2} - \frac{3\dot{g}^2}{g^2}$, along with the integral, we have $M(x)$ also depends on t.

Hence M is a constant.

Now we again study matter-dominated and radiation-dominated time.

- $p = 0$:

  By $\frac{\dot{\rho}}{\rho} = -3\frac{\dot{g}}{g}$, we have $\rho g^3 = \lambda_1$, constant. Then solving ODE $-k\rho = \frac{3}{\lambda^2 g^2} - \frac{3\dot{g}^2}{g^2}$

  we have $\lambda^3\left(\sqrt{\frac{g}{\lambda^2}\left(\frac{g}{\lambda^2} + \frac{k\lambda_1}{3g}\right)} - \frac{k\lambda_1}{3}\ln\left(\sqrt{\frac{g}{\lambda^2} + \frac{k\lambda_1}{3g}} + \sqrt{\frac{g}{\lambda^2}}\right)\right) = t + \lambda_2$ for some constant λ_2.

- $p = \frac{\rho}{3}$:

  By $\frac{\dot{\rho}}{\rho + \frac{\rho}{3}} = -3\frac{\dot{g}}{g}$, we have $\rho g^4 = \lambda_1$, constant. Then the ODE gives

  $$g = \frac{1}{\lambda}\sqrt{(t+\lambda_2)^2 - \frac{k\lambda_1\lambda^4}{3}}$$ for some constant λ_2.

  implying that the universe would expand as time increases.

- Type-V model:

For the type-V model, $C^1_{13} = C^2_{23} = 1$, $C^1_{31} = C^2_{32} = -1$ and others vanish, and we have killing vectors $\xi_1 = \partial_2$, $\xi_2 = \partial_3$, $\xi_3 = \partial_1 + x^2\partial_2 + x^3\partial_3$, and

invariant basis $e_1 = e^{mx^1}\partial_2$, $e_2 = e^{mx^1}\partial_3$, $e_3 = \partial_1$ for some constants m

and $w^1 = e^{-mx^1}dx^2$, $w^2 = e^{-mx^1}dx^3$ and $w^3 = dx^1$, so the metric is in the form

$ds^2 = -dt^2 + A^2dx^2 + e^{2mx}(B^2dy^2 + C^2dz^2)$ where A, B, C are functions of t.

Then with the energy-momentum tensor $T_{ij} = (p+\rho)u_iu_j - pg_{ij}$, (EE) gives

$$\begin{cases} \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = -kp \\ \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} = -kp \\ \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} = -kp \\ \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3m^2}{A^2} = k\rho \\ \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0 \end{cases}$$

and $\nabla_i T^i_j = 0$ gives

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0.$$

To see something, we introduce some parameters. Let $a := (ABC)^{1/3}$, $V = ABC = a^3$, and

$H := \frac{1}{3}(H_1 + H_2 + H_3) := \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)$. Hence we have $H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{\dot{a}}{a}$.

Since the metric is characterized by H, we give a constraint $\underline{H = l a^{-n}}$ for some constants $l>0$ and $n\geq 0$.

Let $q := -\dfrac{\ddot{a}a}{\dot{a}^2}$, the deceleration parameter. Then we have

$$q = -\frac{(-l^2(n-1)a^{-2n+1})a}{(l a^{-n+1})^2} = n-1,$$ a constant. and by $\dot{a} = l a^{-n+1}$, we have $a = \begin{cases} (nlt+c_1)^{\frac{1}{n}}, & n\neq 0 \\ c_2 e^{lt}, & n=0. \end{cases}$

(c_1, c_2 const.).

Finally, let $\sigma^2 := \frac{1}{2}\left(\frac{\dot{A}^2}{A^2}+\frac{\dot{B}^2}{B^2}+\frac{\dot{C}^2}{C^2}\right) - \frac{1}{6}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)^2$, the equations become

$$\begin{cases} kp = H^2(2q-1) - \sigma^2 + \dfrac{m^2}{A^2} \\ k\rho = 3H^2 - \sigma^2 + \dfrac{3m^2}{A^2} \\ \dot{\rho} + 3H(\rho+p) = 0 \\ \dfrac{\ddot{a}}{a} + \dfrac{2\dot{a}^2}{a^2} = \dfrac{1}{2}k(\rho - p) + \dfrac{2m^2}{a^2}. \end{cases}$$

rmk. For $m=0$, $A=B=C$, it's the flat FRW model, and we knew when $p=0$, $\rho>0$, the expansion decelerate.

Now by the original equations, we have

$$\begin{cases} \dfrac{A}{B} = d_1 e^{k_1\int \frac{dt}{a^3}} \\ \dfrac{B}{C} = d_2 e^{k_2\int \frac{dt}{a^3}} \\ \dfrac{C}{A} = d_3 e^{k_3\int \frac{dt}{a^3}} \\ A^2 = BC. \end{cases} \Rightarrow \begin{cases} A = l_1^{1/3} a\, e^{\frac{K_1}{3}\int \frac{dt}{a^3}} \\ B = l_2^{1/3} a\, e^{\frac{K_2}{3}\int \frac{dt}{a^3}} \\ C = l_3^{1/3} a\, e^{\frac{K_3}{3}\int \frac{dt}{a^3}} \end{cases}$$

where d_1, d_2, d_3, k_1, k_2 and k_3 are constants with $d_1 d_3 = d_2^{-2}$ and $k_1+k_2+k_3=0$, and $K_1 := k_1 - k_3$, $K_2 := -2k_1 - k_3$, $K_3 := k_1 + 2k_3$, $l_1 := (d_1/d_3)^{1/3}$, $l_2 := (1/d_1^2 d_3)^{1/3}$, $l_3 := (d_1 d_3^2)^{1/3}$,

leading to $K_1 = 0$, $K_2 = -K_3 =: K$, $l_1 = 1$, $l_2 = l_3^{-1} =: M^3$, K, M: constants.

Hence we get $dl^2 = -dt^2 + a^2 dx^2 + e^{2mx}\left((M a e^{\frac{K}{3}\int\frac{dt}{a^3}})^2 dy^2 + (M^{-1} a e^{-\frac{K}{3}\int\frac{dt}{a^3}})^2 dz^2\right)$.

and a is a function of t, explicitly given for $n=0$ and $n\neq 0$.

- $n\neq 0$: $a = (nlt+c_1)^{\frac{1}{n}}$.

Under the transformation $t \mapsto \dfrac{t-c_1}{nl}$, the metric takes the form

$$dl^2 = -\frac{1}{n^2 l^2}dt^2 + t^{\frac{2}{n}}dx^2 + e^{2mx}t^{\frac{2}{n}}\left(M^2 e^{\frac{2Kt^{\frac{n-3}{n}}}{3l(n-3)}}dy^2 + M^{-2}e^{-\frac{2Kt^{\frac{n-3}{n}}}{3l(n-3)}}dz^2\right).$$

Then we see $\sqrt{-g} \propto t^{\frac{4}{n}}$, so for $n>0$, we also get the expanding universe.

- $n=0$: $a = c_2 e^{lt}$.

Under the transformation $t \mapsto \dfrac{t}{l}$, we have the metric

$$dl^2 = -\frac{1}{l^2}dt^2 + c_2^2 e^{2t}dx^2 + c_2^2 e^{2mx}\left(M^2 e^{2(t - \frac{K}{3c_2^3 l}e^{-t})}dy^2 + M^{-2}e^{2(t+\frac{K}{3c_2^3 l}e^{-t})}dz^2\right).$$

Then $\sqrt{-g} \propto c_2^4 e^{4t}$

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