$f: X \to Y$, X nonsingular, $\dim_{\mathbb{C}} X = n$. Recall: ${}^pH^i: D^b_c(Y) \to \mathrm{Perv}(Y)$

(proj, proj)

Def: $U \underset{\text{Zar. open}}{\hookrightarrow} Y$, $L \in \mathrm{Loc}(U) \rightsquigarrow IC_Y(L) := {}^p j_{!*}(L[d_Y]) \in \mathrm{Perv}(Y)$ the intersection cohomology cpx of L.

Thm. (decomposition thm) $\exists$ iso in $D(Y)$

$$f_* \mathbb{Q}_X[n] \simeq \bigoplus_i {}^pH^i(f_*\mathbb{Q}_X[n])[-i].$$

Thm. (semisimplicity thm) Given a stratification $Y = \coprod_\alpha Y_\alpha$ for f, $\exists$ iso in $\mathrm{Perv}(Y)$

$$ {}^pH^i(f_*\mathbb{Q}_X[n]) = \bigoplus_{l=0}^{d_Y} i_{l,*} IC_{S_l}(L_{i,l})$$

let $S_l := \coprod_{\dim Y_\alpha = l} Y_\alpha$ ← loc. closed

$\forall i$.

Cor. $f_* \mathbb{Q}_X$ is semisimple.

§ idea of proof for "semismall" case

Def. $f: X \to Y$ as above and f surj. f is semismall if $\forall l, \forall y \in S_l$,

$$\dim f^{-1}(y) \le \frac{1}{2}(\dim Y - l).$$

Rmk. ① Take $l = \dim Y \Rightarrow f$ is generically finite. $\Rightarrow \dim X = \dim Y$.

② semismall $\Leftrightarrow f_*\mathbb{Q}_X[n] \in \mathrm{Perv}(Y)$.

In this case, the thm is:

$$f_*\mathbb{Q}_X[n] = \bigoplus_l i_{l,*} IC_{S_l}(L_l),$$

where $L_l: y \mapsto H^{BM}_{\dim Y - l}(\underbrace{f^{-1}(y)}_{F})$ $\quad (H^{BM}_i(F) := H^{-i}(F, \omega_F) = H^i_c(F)^*)$

Now, assume $n = 2m$, and $\dim \underset{\shortparallel}{f^{-1}(y)} = m$. $\Rightarrow \{y\}$ is a point stratum. The decomp. thm $\Rightarrow$

$$f_*\mathbb{Q}_X[n] = \underset{\text{some perv sheaf}}{\mathcal{F}} \oplus i_{y,*}(H^n(F)) \qquad (*)$$

Fact. Only need to show $(*)$ for these y's and n even!

Consider the pairing

$$\mathrm{Hom}(i_*\mathbb{Q}_y, f_*\mathbb{Q}_X[n]) \times \mathrm{Hom}(f_*\mathbb{Q}_X[n], i_*\mathbb{Q}_y) \to \mathrm{End}(i_*\mathbb{Q}_y) = \mathbb{Q}.$$

Then $(*)$ holds $\Leftrightarrow$ the pairing has $\mathrm{rk} = \dim H_n(F)$.

Lemma. $\mathrm{Hom}(i_*\mathbb{Q}_y, f_*\mathbb{Q}_X[n]) = H_n(F) = \mathrm{Hom}(f_*\mathbb{Q}_X[n], i_*\mathbb{Q}_y)$.

$$\begin{array}{ccc} F & \xrightarrow{f} & y \\ {\scriptstyle i}\downarrow & \square & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y \end{array} \Rightarrow i^! f_* = f_* i^!$$

Pf. $LHS \underset{i_! = i_*}{=} \mathrm{Hom}(\mathbb{Q}_y, i^! f_*\mathbb{Q}_X[n]) = \mathrm{Hom}(\mathbb{Q}_y, f_* i^! \underset{\omega_X[-n]}{\mathbb{Q}_X[n]})$

$\underset{\text{[dCM05,] p.24}}{=} \mathrm{Hom}(\mathbb{Q}_y, f_*\omega_F[-n])$

$= \mathrm{Hom}(\mathbb{Q}_F, \omega_F[-n]) \;\overset{?}{=}\; H_n(F)$

(ex C.2.16 $\Rightarrow H^{-i}(X, \mathbb{D}_X(M^\cdot)) = H^i_c(X, M^\cdot)^*$ for $M^\cdot \in D^b(\mathbb{Q}_X)$. Take $M^\cdot = \mathbb{Q}_X$ and $i = n$.)

Similarly, $RHS = H_n(F)$.

Fact. The pairing is the intersection form $H_n(F)\times H_n(F)\to H_0(X)=\mathbb{Q}$.

want: nondegenerate.

Thm. (semismall index thm) it's $(-1)^m$-definite.

This is implied by

Thm. (hard Lefschetz) L: some "semismall" l.b., $w=c_1(L)$. Then $\forall i\geq 0$,

$$w^i : H^{-i}(X)\xrightarrow{\sim} H^i(X).$$

$\max_{y\in S_\ell}\{2\dim f^{-1}(y)+\ell-\dim X\}$

{idea of pf for general case $\quad \| \quad \geq 0$

Def. $r(f):=\max\{i\in\mathbb{Z} \mid {}^pH^i(f_*\mathbb{Q}_X[n])\neq 0\}$ the defect of semismallness.

Define $\mathcal{X}:=\{(x,s)\in X\times \mathbb{P}^\vee \mid s(x)=0\}$, $\mathcal{Y}:=Y\times\mathbb{P}^\vee$.

$\mathbb{P}^\vee$: dual projective space; set of hyperplane section

$$\leadsto \begin{array}{ccccc} \mathcal{X} & \longrightarrow & X\times\mathbb{P}^\vee & \xrightarrow{p} & X \\ \downarrow{\scriptstyle g=f\times\iota} & \circlearrowright & \downarrow{\scriptstyle f\times id} & \circlearrowright & \downarrow{\scriptstyle f} \\ \mathcal{Y} & = & \mathcal{Y} & \xrightarrow{p} & Y \end{array}, \text{ and } g^{-1}(y,s)=f^{-1}(y)\cap Z(s).$$

$(y,s)\in\mathcal{Y}$

Lemma. ① $r(f)>0 \Rightarrow r(g)>r(f)$ $\quad\to$ induction.

② $r(f)=0\Rightarrow r(g)=0$.

Problem ① $\mathrm{Hom}(i_*\mathbb{Q}_y, \underline{f_*\mathbb{Q}_X[n]})$? — not perv, so Hom isn't a gp!

②. $H_n(F)\times H_n(F)\not\to\mathbb{Q}$!

③. semismall index thm doesn't hold!

use spectral seq. (Hodge filt. on $Rf_*\mathbb{Q}$)

Sol'n ③ perverse filt. $\cdots\to {}^p\tau^{\leq m}f_*\mathbb{Q}_X[n]\to {}^p\tau^{\leq m+1}f_*\mathbb{Q}_X[n]\to\cdots$

reduce to abs. hard Lefschetz when $Y=pt$!

② modify ($\therefore$ need "relative" hard Lefschetz: ${}^pH^{-i}(f_*\mathbb{Q}_X[n])\xrightarrow{\sim}{}^pH^i(f_*\mathbb{Q}_X[n])$ $\forall i\geq 0$)

① only do for ${}^pH^0(f_*\mathbb{Q}_X[n])$. Some extra work for simplicity for $i\neq 0$. (relative weak Lefschetz)

Ref.

De Cataldo, Luca Migliorini, The Hard Lefschetz Thm and the Top. of S.S. Maps

——, The Hodge Theory of Alg. Maps

Geordie Williamson, The Hodge Theory of the Decomposition Thm

R05221009 郭子郎
D-module final report, I

Decomposition Theorem: Semismall Case

We try to follow the method in De Cataldo and Luca Migliorini's paper [1] to prove the Decomposition Thm in the very special semismall case. The technical detail is included as much as possible.

Assumption: $f: \underset{\text{proj}}{X} \to \underset{\text{proj}}{Y}$ surjective holomorphic, X nonsingular, $\dim_{\mathbb{C}} X = n$. By [2], we choose stratifications $X = \coprod_{\alpha \in I} X_\alpha$, $Y = \coprod_{\beta \in J} Y_\beta$, st. $\forall$ conn. cpn $S \subseteq Y_\beta$,

(1) $f^{-1}(S) = \coprod_{\alpha \in I'} X_\alpha$ for some $I' \subseteq I$, and $\forall \alpha \in I'$, $f|_{X_\alpha}$ is a submersion.

($\Rightarrow \forall y$, $f^{-1}(y) = \coprod f^{-1}(y) \cap X_\alpha$ is a stratification)

(2) $\forall y \in S$, $\exists U \underset{\text{open}}{\subseteq} S$ and $h: U \times f^{-1}(y) \underset{\substack{\text{homeo} \\ \text{stratum-preserving}}}{\simeq} f^{-1}(U)$ st. $f \circ h: U \times f^{-1}(y) \to U$ is projection.

Let $S_\ell := \coprod_{\dim Y_\beta = \ell} Y_\beta$.

Def.[3] f is semismall if $\forall \ell$, $\forall y \in S_\ell$, $\dim f^{-1}(y) \le \frac{1}{2}(\dim Y - \ell)$.

(Take $\ell = \dim Y \Rightarrow f$ is generically finite $\Rightarrow \dim X = \dim Y$!)

Def. ([1], def 2.1.3.) $M \in Pic(X)$ is semismall (or "lef" in [1]) if for $k >> 0$, $M^{\otimes k}$ is bpf and $M^{\otimes k}: X \to \mathbb{P}$ is semismall onto its image.

Let $H^r(X) = H^r(X, \mathbb{Q})$ (deRham), $w \in H^2(X)$. $\forall 0 \le r \le n$, def

$$\Psi: H^{n-r}(X) \times H^{n-r}(X) \to \mathbb{R}$$
$$(\alpha, \beta) \mapsto (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X w^r \wedge \alpha \wedge \beta$$

Rmk. Ψ is nondegenerate $\Leftrightarrow$ $w^r: H^{n-r}(X) \to H^{n+r}(X)$ is iso.

($\Rightarrow$) if $w^r \wedge \alpha = 0 \Rightarrow \Psi(\alpha, \beta) = 0$ $\forall \beta \Rightarrow \alpha = 0$. $\therefore w^r$ is inj. $\Rightarrow$ surj. $\Rightarrow$ iso.

($\Leftarrow$) Take $\alpha \ne 0$. $\Rightarrow w^r \wedge \alpha \ne 0$. $\therefore$ can choose suitable β so that $\int_X w^r \wedge \alpha \wedge \beta \ne 0$.

Def. ([1], def 2.1.1.) w satisfies property HL (hard Lefschetz) if $w^r: H^{n-r}(X) \xrightarrow{\sim} H^{n+r}(X)$ $\forall 0 \le r \le n$.

Define $P_w^{n-r} := \ker\left(H^{n-r}(X) \xrightarrow{w^{r+1}} H^{n+r+2}(X)\right)$ (primitive elements)

Prop. ([1], prop 2.2.2.) w is HL $\Rightarrow \forall 0 \le r \le n$,

① $H^{n-r}(X) = P_w^{n-r} \oplus w(H^{n-r-2}(X))$

② $H^{n-r}(X) = \bigoplus w^i(P_w^{n-r-2i})$, the summands being pairwise orthogonal wrt. Ψ.

pf. ① Consider

$$H^{n-r-2}(X) \to H^{n-r}(X) \to H^{n+r}(X) \to H^{n+r+2}(X).$$

dim. thm on $H^{n-r}(X)$ w/ w^{r+1}: $\dim H^{n-r}(X) = \dim P_w^{n-r} + \underline{\dim \operatorname{im} w^{n-r}}$

$\dim H^{n+r+2}(X) = \dim H^{n-r-2}(X) = \dim w(H^{n-r-2}(X))$

because $w^{r+1}: H^{n-r-2}(X) \to H^{n+r+2}(X)$ is bij.

②: apply ①.

Def. ① a rational Hodge str. of weight r is H: v.s. $/\mathbb{Q}$, w/ $H_{\mathbb{C}} = \bigoplus_{p+q=r} H^{p,q}$ st. $H^{p,q} = \overline{H^{q,p}}$

② a polarization of the weight r Hodge str. H is $\Psi: H\times H \to \mathbb{Q}$, bilin, (anti) symm. for r even (odd), st. $\Psi := \Psi_{\mathbb{C}}$ satisfies

(a) $(p,q) \neq (s,t) \Rightarrow H^{p,q}$ and $H^{s,t}$ are o.g.

(b) $\forall\, 0 \neq \alpha \in H^{p,q} \Rightarrow i^{p-q}\Psi(\alpha,\bar{\alpha}) > 0$.

Thm. (Hard Lefschetz for ample l.b.) $M \in Pic(X)$ ample $\Rightarrow$ ① $c_1(M)$ is HL, and ② $\Psi_{c_1(M)}: H^{n-r}(X)\times H^{n-r}(X) \to \mathbb{R}$ is a polarization of the weight $(n-r)$ Hodge str. $P^{n-r}_{c_1(M)} \subseteq H^{n-r}(X)$. ↖defined in P.1.

Pf. See [8], or [9] p.90.

Thm ([1] 2.2.6) Assume $M \in Pic(X)$ is semismall, and $c_1(M)$ is HL. $\Rightarrow \Psi_{c_1(M)}: H^{n-r}(X)\times H^{n-r}(X) \to \mathbb{R}$ is a polarization of the weight $(n-r)$ Hodge str. $P^{n-r}_{c_1(M)} \subseteq H^{n-r}(X)$.

Pf. Let $w := c_1(M) \in H^2(X)$. ∵ semiample $\Rightarrow$ nef and nef cone $\subseteq \overline{\text{ample cone}}$ $\subset H^{n+r}(X)$ ∴ can choose $w = \lim w_i$, where $w_i = c_1(M_i)$, M_i ample.

For $\alpha \in P^{n-r}_w \cap H^{p,q}(X)$, check: $\pi_i(\alpha) := \alpha - w_i \circ (\underbrace{w_i^{r+2}}_{H^{n-r-2}\to H^{n+r+2}})^{-1} \circ w_i^{r+1}(\alpha)$ is the projection to $P^{n-r}_{w_i}$.

w is invertible $\Rightarrow w^{-1} = \lim w_i^{-1} \Rightarrow \lim \pi_i(\alpha) = \alpha$. ∴ $i^{p-q}\Psi_w(\alpha,\bar{\alpha}) = \lim i^{p-q}\Psi_{w_i}(\pi_i(\alpha), \overline{\pi_i(\alpha)}) \geq 0$.
and w is HL $\Rightarrow \Psi_w$ nondeg $\Rightarrow > 0$. ∴ (b) ok.

(a) holds similarly. ※

Next, we prove that $c_1(M)$ is HL for M semismall.

§ some algebraic topology

Define $H^{BM}_*(X)$:= homology w/ loc. finite supp. If X: cpt oriented sm conn., then $H^{BM}_*(X) = H_*(X)$ (sing), and $M_X \in H^{BM}_n(X)$ its fundamental class.

Def. [10] For $f: X^a \to Y^b$ (cpt oriented sm conn.), def $f_!: H^k(X) \to H^{b-a+k}(Y)$ st.

$$\begin{tikzcd} H^*(X) \arrow[r, "\cap M_X", "\sim"'] \arrow[d, "f_!"'] & H_{a-*}(X) \arrow[d, "f_*"] \\ H^{b-a+*}(Y) \arrow[r, "\cap M_Y", "\sim"'] & H_{a-*}(Y) \end{tikzcd}$$

↖Poincaré duality

eg. $i: D \hookrightarrow \mathbb{P}$ (hyperplane), $D \rightsquigarrow M \in Pic(\mathbb{P}) \Rightarrow i_!(1) = c_1(M) =: w \in H^2(\mathbb{P})$.

Prop. ① $i_!(\beta \wedge i^*(\alpha)) = i_!(\beta)\wedge\alpha \quad \forall \beta \in H^*(D), \alpha \in H^*(X)$ (projection formula) ([10])

② $i_! \circ i^*(\alpha) = w \wedge \alpha \quad \forall \alpha \in H^*(X)$ ([10])

③ $i^* \circ i_!(\beta) = w|_D \wedge \beta \quad \forall \beta \in H^*(D)$ ([3]) ④ $(w\wedge)^r = i_! \circ (w|_D \wedge)^{r-1} \circ i^*$. ([3],[1])

that is, $S = (\beta \wedge i^*(\alpha)) \cap M_D$!

Pf. ① want: $(i_!(\beta)\wedge\alpha) \cap M_{\mathbb{P}} = i_*((\beta \wedge i^*(\alpha)) \cap M_D)$. Let $S \xrightarrow{j_1} D$ be st. $\int_S j_1^*(\gamma) = \int_D \beta \wedge i^*(\alpha) \wedge \gamma \quad \forall \gamma \in H^*(D)$, and $R \xrightarrow{j_2} \mathbb{P}$ be st. $\int_R j_2^*(\gamma) = \int_D \beta \wedge \gamma \quad \forall \gamma \in H^*(D)$. Then $\int_R j_2^* \circ i^*(\gamma) = \int_{\mathbb{P}} i_!(\beta)\wedge\gamma \quad \forall \gamma \in H^*(X)$, by the definition of $i_!(\beta)$. Now for $\gamma \in H^*(X)$,

$$\int_S j_1^* \circ i^*(\gamma) = \int_D \beta \wedge i^*(\alpha\wedge\gamma) = \int_R j_2^* \circ i^*(\alpha\wedge\gamma) = \int_{\mathbb{P}} i_!(\beta)\wedge\alpha\wedge\gamma,$$ ie. $i_*(S) = (i_!(\beta)\wedge\alpha) \cap M_{\mathbb{P}}$.

② just take $\beta = 1$ in ①.

④ we do the case $r=2$: $i_!(w|_D \wedge i^*(\alpha)) = i_!(i^* \circ i_!(1) \wedge i^*(\alpha)) \overset{①}{=} (i_! i^* i_!(1)) \wedge \alpha \overset{②}{=} i_!(1) \wedge i_!(1) \wedge \alpha = w \wedge w \wedge \alpha$

Rmk. ① The formula ③ appears only in [3]! It's not used in DeCataldo and Migliorini's papers, nor can I find a similar statement in [6] or [10].

② The "$\cap M_X$" isomorphism holds in a more general setup. If X: oriented, connected real n-mfd, $Y \underset{\text{closed}}{\subseteq}$, then $\exists$ cap product

$$H^j(Y, Y-X) \otimes H_k^{BM}(Y) \xrightarrow{\cap} H_{k-j}^{BM}(X),$$

and "$\cap M_X$" induces $H^{n-i}(Y, Y-X) \overset{\cap M_X}{\underset{\sim}{\to}} H_i^{BM}(X)$. See [6], section 19.1. For a definition of cap product in terms of derived categories, see [10], section IX.3.

§ Hard Lefschetz for semismall maps

Now we back to prove the HL property for $\overset{w}{\overset{\shortparallel}{c_1(M)}}$, M being semismall. Let $f = M^{\otimes k}: X \to Y \overset{\mathbb{P}}{\subseteq}$.

Lemma. If $(X_D, w|_{X_D})$ is HL, then (X, w) is HL $\Leftrightarrow$ $\Psi_{w|_{X_D}}|_{i^*H^{n-1}(X)}$ is nondegenerate.

Pf. See [11], proposition (3.5).

see the upcoming thm for its definition

the choice of this involves "Lefschetz pencil"

Thm. (weak Lefschetz theorem) $X \subseteq \mathbb{P}$ sm. $\Rightarrow$ for a general hyperplane section D, letting $X_D := X \cap D$, we have $H_k(X_D) \xrightarrow{i_*} H_k(X)$: $\begin{cases} \text{iso}, & k < n-1 \\ \text{surj}, & k = n-1 \end{cases}$.

Pf. See [11], theorem (2.1).

Pf of HL for $c_1(M)$: Bertini $\Rightarrow$ choose $D \in |M|$ sm. Note we may choose D st. $M|_{X_D}$ is semismall. By induction, assume $(X_D, w|_{X_D})$ is HL ($\Rightarrow$ polarization, by [1] thm 2.2.6.). Then $\forall k > 1$,

$$\begin{tikzcd} H^{n-k}(X) \arrow[r, "i^*", "\sim"'] \arrow[rrr, bend right, "w^k"'] & H^{n-k}(X_D) \arrow[r, "w|_{X_D}^{k-1}", "\sim"'] & H^{n+k-2}(X_D) \arrow[r, "i_!", "\sim"'] & H^{n+k}(X) \end{tikzcd}$$

weak Lefschetz

$\therefore$ It remains to deal with $k=1$, i.e. $H^{n-1}(X) \xrightarrow{w} H^{n+1}(X)$. By lemma, want: $\Psi_{w|_{X_D}}|_{i^*H^{n-1}(X)}$ is nondeg.

Observe: ① $\Psi_{w|_{X_D}}(i^*(\alpha), \beta) = \pm \Psi_w(\alpha, i_!(\beta))$ $\quad (\Rightarrow i^*H^{n-1}(X)^\perp = \ker(H^{n-1}(X_D) \xrightarrow{i_!} H^{n+1}(X)))$

reason: let $R = (i^*(\alpha) \wedge \beta) \cap M_{X_D} \Rightarrow i_* R = i_!(i^*(\alpha) \wedge \beta) \cap M_X = (\alpha \wedge i_!(\beta)) \cap M_X$.

$\because \int_{X_D} w|_{X_D}^k \wedge i^*(\alpha) \wedge \beta = \int_R j^*(w_{X_D}^k) = \int_X w^k \wedge \alpha \wedge i_!(\beta)$. $\Rightarrow \Psi_{w|_{X_D}}(i^*(\alpha), \beta) = \pm \Psi_w(\alpha, i_!(\beta))$.

$(j: R \hookrightarrow X_D)$

② $\ker(H^{n-1}(X_D) \xrightarrow{i_!} H^{n+1}(X)) \subseteq P^{n-1}_{w|_{X_D}}(X_D)$.

reason: $w|_{X_D} \wedge \beta = i^* \circ i_!(\beta)$.

polarization $\Rightarrow$ $\Psi_{w|_{X_D}}|_{P^{n-1}_{w|_{X_D}}(X_D)}$ is nondeg $\Rightarrow$ $\Psi_{w|_{X_D}}|_{i^*H^{n-1}(X)}$ is nondeg. #

by prop 2.2.2

$H^{n-1}(X_D) = \bigoplus w^i(P_{w|_{X_D}}^{n-1-2i})$, the summands being pairwise o.g.

§ about fundamental class

Let $\dim_{\mathbb{C}} X = n$, $X = \bigsqcup X_i$ irre cpn.

<u>Prop.</u> $H_i^{BM}(X) = 0 \ \forall i > 2n$, and $H_{2n}^{BM}(X)$ is free abelian, w/. one generator for each X_i.

<u>Pf</u> ([6], lemma 19.1.1.) Let $U = X_{reg}$, $S = X - U$. $U \xrightarrow{j} X \xleftarrow{i} S$.

$$\rightsquigarrow \cdots \to H_{i+1}^{BM}(U) \to H_i^{BM}(S) \xrightarrow{i_*} H_i^{BM}(X) \xrightarrow{j^*} H_i^{BM}(U) \to H_{i-1}^{BM}(X) \to \cdots$$

The statement holds for U. For S, use induction (let $U' := S_{reg}$, $S' := S - U'$, ...).

Thus, every irreducible variety Z has a "fundamental class" $[Z] \in H_{2\dim Z}(Z)$, whose restriction to any component U_i of Z_{reg} is $\mu_{U_i} \in H_{2\dim Z}(Z_{reg})$.

For $Z_1^p, Z_2^q \underset{\text{subvar}}{\subseteq} X_{\mathbb{C}\text{ sm, cpt}}$ w/. $\dim_{\mathbb{C}} X = n = p+q$, $\exists$ intersection form ((anti) symm. if p even (odd))

$$H_p^{BM}(Z_1) \times H_q^{BM}(Z_2) \xrightarrow{i_{1*} \times i_{2*}} H_p^{BM}(X) \times H_q^{BM}(X) \underset{\sim}{\xrightarrow{P.D.}} H_p(X) \times H^{2n-q}(X) \to \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \alpha \cdot \beta.$$

§ Hodge index theorem for semismall maps

Now let $f: X \to Y$ satisfy the assumption in p.1. and is semismall. Assume $\dim_{\mathbb{C}} X = n \in 2\mathbb{Z}$, and $y \in Y$ is st. $\dim_{\mathbb{C}} f^{-1}(y) = n/2$. Let $f^{-1}(y) = \coprod_{\ell=1}^{r} Z_\ell$ irre. cpn.

<u>Thm.</u> [17, 2.4.1.] (Hodge index theorem) $(-1)^{n/2} (Z_i \cdot Z_j)_{i,j}$ is positive definite.

<u>Pf</u> step 1: cl: $H_n^{BM}(f^{-1}(y)) \to H_n(X)$ is inj.

Pf: We show the dual $r: H^n(X) \to H^n(f^{-1}(y))$, is surj. Take $Y^\circ \ni y$ st. y is the only point stratum in Y°.

- $\forall \tilde{y} \in Y^\circ$, $(R^n f_* \mathbb{Q}_X)_{\tilde{y}} = \varinjlim_{Y \supseteq V \ni \tilde{y}} H^n(f^{-1}(V), \mathbb{Q}_X) \underset{\uparrow}{=} \varinjlim_{U \supseteq f^{-1}(\tilde{y})} H^n(U, \mathbb{Q}_X) \underset{\uparrow}{=} H^n(f^{-1}(\tilde{y}))$.

  in analytic top. direct limit of inj. mod is inj. over Noe. ring, and the restriction of an injective coherent sheaf to an open subset is still injective

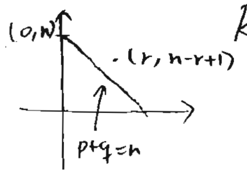
  $\therefore \operatorname{supp}(R^n f_* \mathbb{Q}_X|_{Y^\circ}) = \{y\}$.

- Recall: Leray spectral sequence: $E_2^{p,q} = H^p(Y^\circ, R^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(f^{-1}(Y^\circ), \mathbb{Q}_X)$

  If $S_q := \operatorname{supp}(R^q f_* \mathbb{Q}_X|_{Y^\circ})$, then $2(\dim f^{-1}(S_q) - \dim S_q) \geq q$. (?) (See [7], lemma 1.2.)

  $\therefore n \underset{\text{semismall}}{\geq} 2\dim f^{-1}(S_q) - \dim S_q \geq q + \dim S_q$

  $\therefore$ If $p+q > n$, then $H^p(Y^\circ, R^q f_* \mathbb{Q}_X|_{Y_0}) = 0$.



Recall, the differential at E_2 page is: $d_2^{pq}: E_2^{pq} \to E_2^{p+2, q-2+1}$

$$\therefore E_2^{0n} = E_\infty^{0n} = H^n(f^{-1}(Y^\circ), \mathbb{Q}_X) / F^1 H^n(f^{-1}(Y^\circ), \mathbb{Q}_X) \twoheadleftarrow H^n(f^{-1}(Y^\circ), \mathbb{Q}_X)$$

$$\therefore H^n(f^{-1}(Y^\circ)) \twoheadrightarrow H^0(Y^\circ, R^n f_* \mathbb{Q}_X) = (R^n f_* \mathbb{Q}_X)_y = H^n(f^{-1}(y)).$$

To conclude the surjectivity of r, use [12], proposition (8.2.6).

step 2: Take $A \in \operatorname{Pic}(Y)$ ample, $M := f^*(A)$. The image of each Z_i in $H_n(X) \cong H^n(X)$ belongs to $P_M^n(X) = \ker(w \wedge : H^n(X) \to H^{n+2}(X))$. ($w = c_1(M)$)

Pf: ([3], p.13) We have

$$\begin{array}{ccccc} Z_i \in H_n(X) & \times & H_{2n-2}(X) & \to & H_{n-2}(X) \ni Z_i \cap H \\ \text{PD}\| \wr & & \text{PD}\| \wr \ \ H & & \wr\ \text{PD}\| \\ \eta_{Z_i} \in H^n(X) & \times & H^2(X) \ni c_1(M) & \to & H^{n+2}(X) \end{array}$$

(recall: $\eta_{Z_i \cap H} = \eta_{Z_i} \wedge \eta_H$; [13], p.69.)

$H = f^*(\tilde{H})$ for some $\tilde{H}$; but we can choose $\tilde{H}$ so that $\tilde{H} \cap \{y\} = \phi$!

$0 < i^{\frac{n}{2} - \frac{n}{2}} \underbrace{\Psi(\eta_{Z_i}, \bar{\eta}_{Z_i})}_{\Psi(\eta_{Z_i}, \bar{\eta}_{Z_i})} \Leftarrow H^{\frac{n}{2}, \frac{n}{2}}(X) \ni \eta_{Z_i}$

§ the relevant strata

Recall: $Y = \coprod_{\beta \in J} Y_\beta$ and $S_i := \coprod_{\dim Y_\beta = i} Y_\beta$

([1], 3.2.1.)

Def. S_i is relevant if $2\dim f^{-1}(S_i) - \dim S_i = n$. For all S_i, def $\mathcal{L}_i := (R^{n-i} f_* \mathbb{Q}_X)|_{S_i} \in Loc(S_i)$.

Rmk. ([4]) semismall implies, in general we have $2\dim f^{-1}(S_i) - \dim S_i \leq n$. Recall the topological triviality on the strata in p.1: $\forall y \in S_i$, $\exists U \underset{open}{\subseteq} S_i$ st. $f^{-1}(U) \underset{homeo}{\simeq} f^{-1}(y) \times U$

$\therefore \dim f^{-1}(S_i) = \dim f^{-1}(y) + \dim S_i$.

$\therefore$ not relevant $\Rightarrow n - \dim S_i > 2(\dim f^{-1}(S_i) - \dim S_i) = 2\dim f^{-1}(y)$ $\forall y \in S_i \Rightarrow \mathcal{L}_i$ is trivial

relevant $\Rightarrow (\mathcal{L}_i)_y = H^{n-i}(f^{-1}(y))$ $\forall y \in S_i$.

Now, fix a particular $S := S_i$, $\mathcal{L}_S := \mathcal{L}_i$. For $s \in S$, choose $U \underset{open}{\subseteq} Y$ st. $S' := S \cap U$ is contractible and $i^*: H^{n-d}(f^{-1}(U)) \to H^{n-d}(f^{-1}(s))$ is iso. Let $[F_1], \dots, [F_r]$: basis of $H^{BM}_{n+i}(f^{-1}(S'))$.

$$\rightsquigarrow \quad i_s^!: H^{BM}_{n+i}(f^{-1}(S')) \xrightarrow{\sim} H^{BM}_{n-i}(f^{-1}(s))$$
$$[F_\ell] \mapsto [f_\ell := f^{-1}(s) \cap F_\ell]$$

Rmk. In [6] 6.2, $i^!$ is defined in terms of Chow ring. For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ g\downarrow & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array},$$

where i is regular embedding of codim. d. def $i^!: A_k Y' \to A_{k-d} X'$ by $i^!(\sum n_i [V_i]) = \sum n_i [X \cdot V_i]$

This map is well-defined and called the refined Gysin homomorphism.

Consider the composition

$$\ell_{S,s}: H^{BM}_{n-i}(f^{-1}(s)) \underset{\sim}{\xrightarrow{(i_s^!)^{-1}}} H^{BM}_{n+i}(f^{-1}(S')) \underset{\sim}{\xrightarrow{(\cap M_{f^{-1}(U)})^{-1}}} H^{n-i}(f^{-1}(U), f^{-1}(U - S'))$$
$$\to H^{n-i}(f^{-1}(U)) \underset{\sim}{\xrightarrow{i^*}} H^{n-i}(f^{-1}(s)) \underset{\sim}{\to} H^{BM}_{n-i}(f^{-1}(s))^*.$$

(this is indep. of U) topological triviality $\rightsquigarrow \ell_S: \mathcal{L}_S^* \to \mathcal{L}_S$.

([1], 3.2.4)

Prop. ① S: not relevant $\Rightarrow \ell_S = 0$.

② $s \in S$. Then $\ell_{S,s}$ is iso. $\Leftrightarrow H^k(f^{-1}(U)) \to H^k(f^{-1}(U - S'))$ is iso $\forall k \leq n-i-1$

$\Leftrightarrow H^k(f^{-1}(U - S')) \to H^{k+1}(f^{-1}(U), f^{-1}(U - S'))$ is iso $\forall k \geq n-i$.

Pf: ② For example, assume $\ell_{S,s}$ is iso. We have M-V seq:

$$\cdots \to H^{n-i-1}(f^{-1}(U), f^{-1}(U - S')) \to H^{n-i-1}(f^{-1}(U)) \underset{\sim}{\xrightarrow{!}} H^{n-i-1}(f^{-1}(U - S')) \to H^{n-i}(f^{-1}(U), f^{-1}(U - S')) \underset{assumption}{\xrightarrow{\sim}} H^{n-i}(f^{-1}(U)) \to \cdots$$

$H^{BM}_{n+i+1}(f^{-1}(S')) = 0$, since $\dim f^{-1}(S') \leq \frac{n+i}{2}$

As for the other isomorphism, use $H^k(f^{-1}(U)) \simeq H^k(f^{-1}(s))$ and that $\dim f^{-1}(s) \leq \frac{n-i}{2}$.

<u>Lemma</u>. ([1], 3.2.5.) $\rho_s : \mathcal{L}_s^* \to \mathcal{L}_s$ is symmetric (that is, $\rho_{s,s} : H_{n-d}^{BM}(f^{-1}(s)) \to H_{n-d}^{BM}(f^{-1}(s))^*$ is symmetric $\forall s \in S$).

<u>Pf.</u> Need more knowledge about "refined intersection product" from [6].

§ inductive study on semismall maps

We first prove a direct sum criterion in derived categories, which is crucial to accomplish our inductive study of semismall maps.

<u>Def.</u> ([1], 3.1.) For A: cochain complex,

$$\tau_{\le t} A := [\cdots \to A^{t-1} \to \ker d^t \to 0 \to 0 \to \cdots]$$

$$\tau_{\ge t} A := [\cdots \to 0 \to \operatorname{coker} d^{t-1} \to A^t \to A^{t+1} \to \cdots]$$

<u>Rmk.</u> The definition for $\tau_{\ge t} A$ differs from that in our textbook [5]! But here, we still have $H^i(\tau_{\ge t} A) = \begin{cases} H^i(A), & i \ge t \\ 0, & i < t \end{cases}$.

<u>Lemma.</u> ([1], 3.1.1.) $A \simeq \tau_{\le t} A$, $B \simeq \tau_{\ge t} B \Rightarrow \operatorname{Hom}_{D(\mathcal{C})}(A, B) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(H^t(A), H^t(B))$.

<u>Pf.</u> Clear.

<u>Lemma.</u> ([1], section 3.1.) $A \xrightarrow{h} B \to C \to A[1]$ dist. Δ. If $h = 0 \Rightarrow C \simeq A[1] \oplus B$.

<u>Prop.</u> ([1], 3.1.2) $C \xrightarrow{u} A \xrightarrow{v} B \to C[1]$ dist Δ, $A \simeq \tau_{\le t} A$, $C \simeq \tau_{\ge t} C$. Then
$\left(H^t(u) : H^t(C) \to H^t(A) \text{ is iso}\right) \Leftrightarrow \left(A \simeq \tau_{\le t-1} B \oplus H^t(A)[-t]\text{, and the map } v \text{ is the direct sum of } \tau_{\le t-1} B \to B \text{ and zero map}\right)$.

<u>pf.</u> ($\Leftarrow$) Clear.

($\Rightarrow$) Some easy observations first: ① $H^l(A) \xrightarrow{H^l(\tau_{\le t}(v))} H^l(\tau_{\le t} B)$ is iso $\forall l \le t-1$; ② $H^t(v) = 0$.

$\exists$ dist Δ: $\tau_{\le t-1} B \xrightarrow{v_{t-1}} \tau_{\le t} B \to H^t(B)[-t] \to \tau_{\le t-1} B[1]$

Apply $\operatorname{Hom}(\underset{\tau_{\le t}A}{A}, -) \Rightarrow \cdots \to \operatorname{Hom}^{-1}(\tau_{\le t} A, H^t(B)[-t]) \to \operatorname{Hom}^0(\tau_{\le t} A, \tau_{\le t-1} B) \to \operatorname{Hom}^0(\tau_{\le t} A, \tau_{\le t} B) \to \operatorname{Hom}(\tau_{\le t} A, H^t(B)[-t]) \to \cdots$

$\tau_{\le t} v \mapsto 0$ (by ②)

$\therefore \exists$ lifting
$$\begin{tikzcd} \tau_{\le t}A \simeq A \arrow[r, dashed, "v'"] \arrow[d, "\tau_{\le t}(v)"'] & \tau_{\le t-1} B \arrow[dl, "v_{t-1}"] \\ \tau_{\le t} B & \end{tikzcd}$$
$\leftarrow$ observe: ③ by the trivial statement $H^l(\tau_{\le t-1} B) \xrightarrow{\sim} H^l(\tau_{\le t} B)$ $\forall l \le t-1$ and ①, we have $H^l(v') = \begin{cases} \text{iso}, & l \le t-1 \\ 0, & l \ge t \end{cases}$.

Complete v' into dist Δ: $\tau_{\le t} A \xrightarrow{v'} \tau_{\le t-1} B \xrightarrow{v''} M(v') \to \tau_{\le t} A[1]$. want: $v'' = 0$ and apply lemma above.

By ③ $\Rightarrow M(v') \simeq H^t(A)[-t+1]$, and moreover $H^{t-1}(v'') = 0$. Lemma 3.1.1 applied on $\tau_{\le t-1} B$ and $M(v') \Rightarrow v'' = 0$.

$\therefore$ Lemma $\Rightarrow \underset{A[1]}{\tau_{\le t} A[1]} \simeq \tau_{\le t-1} B[1] \oplus \underset{H^t(A)[-t+1]}{M(v')}$. #

Recall: $j: U \underset{\text{Zar. open}}{\hookrightarrow} Y$, $L \in Loc(U) \to IC_Y(L) := {}^p j_{!*}(L[\dim Y]) \in Perv(Y)$.

([1], 3.3.1.)

Def. $f: X \to Y$ satisfies the assumption in p.1 (ie, surjective, holomorphic, X nonsingular, both X and Y projective) and is semismall. Let $I :=$ index set for S_i's and $I' \subseteq I$ is the index set for relevant S_i's. We say the decomposition thm holds for f if $\exists$ iso

$$Rf_* \mathbb{Q}_X[n] \simeq \bigoplus_{k \in I'} i_* IC_{S_k}(L_k).$$

For fixed i, let $V := \coprod_{l > i} S_l$, $V' := \coprod_{l \geq i} S_l$, $S := S_i$. $\to V \underset{\text{open}}{\overset{\alpha}{\hookrightarrow}} V' \underset{\text{closed}}{\overset{\beta}{\hookleftarrow}} S$

Thm ([1], 3.3.3.) Assume the decomp. thm holds for $f|_{f^{-1}(V)}$. Then

$$\left(\begin{matrix}\text{the decomp. thm holds for} \\ f|_{f^{-1}(V')}, \text{ and the iso retricts} \\ \text{to that over } V\end{matrix}\right) \Leftrightarrow \left(P_S: L_S^* \to L_S \text{ is iso}\right).$$

Pf. ($\Leftarrow$) We have dist. Δ

$$\beta_*\beta^!(Rf_*\mathbb{Q}_X[n]|_{V'}) \xrightarrow{u} (Rf_*\mathbb{Q}_X[n])|_{V'} \xrightarrow{v} \underset{\substack{\| \\ (R\alpha_*\alpha^{-1}(Rf_*\mathbb{Q}_X[n]|_{V'}))}}{R\alpha_*(Rf_*\mathbb{Q}_X[n]|_V)} \xrightarrow{+1} \qquad (*)$$

For $s \in S$, the cohomology of $(*)$ of deg-i at s is:

$$\cdots \to \underset{\substack{\| \\ H^{BM}_{n+i}(f^{-1}(S')) \underset{\text{iii}}{\simeq} H^{BM}_{n-i}(f^{-1}(s))}}{H^{n-i}(f^{-1}(U), f^{-1}(U-S'))} \overset{P_{S,s}}{\xrightarrow{\ \sim\ }} \underset{\substack{\| \\ H^{n-i}(f^{-1}(s)) = (H^{BM}_{n-i}(f^{-1}(s)))^*}}{H^{n-i}(f^{-1}(U))} \to H^{n-i}(f^{-1}(U-S')) \to H^{n-i+1}(f^{-1}(U), f^{-1}(U-S')) \to \cdots$$

$\therefore$ When retricting $(*)$ to S, $H^i(u|_S)$ is iso. And, $(Rf_*\mathbb{Q}_X[n])|_S \simeq \tau_{\leq i}(Rf_*\mathbb{Q}_X[n])|_S$ ($\because \dim f^{-1}(s) \leq \frac{n-i}{2}$) and $\beta_*\beta^!(Rf_*\mathbb{Q}_X[n]|_{V'})|_S \simeq \tau_{\geq i}\beta_*\beta^!(Rf_*\mathbb{Q}_X[n]|_{V'})|_S$ $\therefore$ prop 3.1.2 yields

$$(Rf_*\mathbb{Q}_X[n])|_S \simeq \tau_{\leq i-1} R\alpha_*(Rf_*\mathbb{Q}_X[n]|_V)|_S \oplus \underline{H^i(Rf_*\mathbb{Q}_X[n]|_S)[i]}.$$

$$\text{“}L_S[i] = IC_{\bar{S} \cap V'}(L_S) \quad (\because S \underset{\text{closed}}{\subseteq} V') \quad \therefore \bar{S} \cap V' = S$$

Applying β_*, noting $\beta^{-1} \circ \beta_* = id$ since S is closed,

$$Rf_*\mathbb{Q}_{f^{-1}(V')}[n] \simeq \tau_{\leq i-1} R\alpha_*(Rf_*\mathbb{Q}_X[n]|_V) \oplus \beta_* IC_{\bar{S} \cap V'}(L_S).$$

$$\| \quad {}^p\alpha_{!*}(Rf_*\mathbb{Q}_X[n]|_V)$$

$\therefore$ the decomp. thm holds for V'.

recall: ${}^p j_{!*}(L[d_X]) \simeq (\tau^{\leq -1} Rj_{1,*}) \circ \cdots \circ (\tau^{\leq -d_X} Rj_{d_X *})(L[d_X])$ ([5], prop 8.2.11.)

Finally, recall from [5], cor 8.1.44 that $\beta_* IC_{\bar{S} \cap V'}(L_S)$ is supported in S $\therefore$ the iso restricts to that over V.

The proof of ($\Rightarrow$) is similar. #

Thm. ([1], 3.4.1.) The decomposition theorem holds for f.

Pf. First, the decomp. thm holds for the largest strata, $f|_{f^{-1}(S_n)}$: $Rf_*\mathbb{Q}_X[n]|_{S_n} \simeq \underset{\substack{R^0 f_* \mathbb{Q}_{f^{-1}(S_n)}[n] \\ \|}}{Rf_*\mathbb{Q}_{f^{-1}(S_n)}[n]}$

Let $S = S_i$ be a relevant strata. By thm 3.3.3, want: P_S: iso. Let $s \in S$, and $U \underset{\text{open}}{\subseteq} Y$ be as in p.5. Let $A \in Pic(Y)$ be very ample. $\therefore L := f^*(A)$ is semismall, gbgs.

Bertini $\Rightarrow$ choose $H_1, \dots, H_i \in |A|$ st. $H := H_1 \cap \cdots \cap H_i$ satisfies:

① H is nonsingular, $\dim_{\mathbb{C}} H = n-i$; ② $f^{-1}(H) \to H$ is semismall; ③ $\#(H \cap (U \cap S)) < \infty$.

Pick $s' \in H \cap (U \cap S)$. $\therefore s'$: point stratum. By Hodge index thm for semismall maps (thm 2.4.1),

$$H^{BM}_{n-i}(f^{-1}(s')) \times H^{BM}_{n-i}(f^{-1}(s')) \to H_{n-i}(H) \times H^{n-i}(H) \to \mathbb{C}$$

is $(-1)^{\frac{n-i}{2}}$ definite. $\Rightarrow$ nondegenerate $\Rightarrow P_{S,s'}$ is iso. By prop 3.2.4, $P_{S,s}$ is iso as well. #

Decomposition Theorem: General Case

We explain the proof presented in De Cataldo and Luca Migliorini's paper [2]. One of the main difference from the semismall case is that we have to deal with those perverse cohomology ${}^pH^i(f_*\mathbb{Q}_X[n])$ for those $i\neq 0$; this is achieved by means of the relative Hard Lefschetz theorem, whose proof and its immediate consequence (i.e. the decomposition theorem without semisimplicity assertion) will be explained in detail. Another difference from the semismall case is in the proof of the semisimplicity of ${}^pH^0(f_*\mathbb{Q}_X[n])$. However, the spirit of the proof is in fact the same as what we have done before.

Assumption: same as p.1, except that f needs not be surjective now.

§ how the induction starts

The decomposition theorem relies on an inductive proof of several theorems at the same time. The first step toward this induction is that the "defect of semismallness" goes down by cutting a universal hyperplane section.

<u>Def.</u> ([2], def 4.7.2; [3], p.20) (the defect of semismallness) Let $Y^i := \{y\in Y \mid \dim f^{-1}(y) = i\}$.

$$r(f) := \max_i \{2i+\dim Y^i - \dim X \mid Y^i\neq\phi\} = \max_{\substack{l\\ y\in S_l}}\{2\dim f^{-1}(y) + l - \dim X\} = \max\{i\in\mathbb{Z} \mid {}^pH^i(f_*\mathbb{Q}_X[n])\neq 0\}.$$

Then $r(f)=0 \Leftrightarrow f$ is semismall.

Consider the universal hyperplane section diagram

$$\begin{array}{ccc} \mathcal{X} := \{(x,s)\in X\times\mathbb{P}^\vee \mid s(x)=0\} & \overset{i}{\hookrightarrow} & X\times\mathbb{P}^\vee \\ & \searrow^{g:=f\circ i} & \downarrow f':=f\times id \\ & & \mathcal{Y} := Y\times\mathbb{P}^\vee. \end{array}$$

<u>Thm.</u> ([2], thm 4.7.4.) ($r(f)$ goes down)

① $r(f)>0 \Rightarrow r(g)<r(f)$

② $r(f)=0 \Rightarrow r(g)=0$.

<u>pf.</u> For $s\in\mathbb{P}^\vee$, $X_s = X\cap Z(s)$. Def $\mathcal{Y}^i := \{(y,s)\in\mathcal{Y} \mid \dim \underbrace{g^{-1}(y,s)}_{= f^{-1}(y)\cap X_s} = i\}$, $\mathcal{Y}^{i''} := \{(y,s) \mid \dim f^{-1}(y) \underset{=i+1,\ \dim(f^{-1}(y)\cap X_s)=i}{=} i = \dim(f^{-1}(y)\cap X_s)\}$.

Then for $(y,s)\in\mathcal{Y}^i$, $(y,s)\in\mathcal{Y}^{i'} \Leftrightarrow X_s$ contains a top dim'l cpn of $f^{-1}(y)$. $\Rightarrow \mathcal{Y}^{i'} \underset{\text{closed}}{\subseteq} \mathcal{Y}^i$, and we have

$$\mathcal{Y}^i = \mathcal{Y}^{i'} \amalg \mathcal{Y}^{i''} \quad \therefore \text{ open}$$

Observe: $\dim \mathcal{Y}^{i''} = \dim Y^{i+1} + \dim\mathbb{P}^\vee \leq r(f) - 2(i+1) + \underbrace{\dim X + \dim\mathbb{P}^\vee}_{=\dim\mathcal{X}+1}$ $\quad\therefore 2i + \underbrace{\dim\mathcal{Y}^{i''}}_{=\dim \mathcal{Y}^i} - \dim\mathcal{X} \leq r(f)-1, \forall i\geq 0$

$\therefore r(g)\leq r(f)-1$. #

Recall our main theorems are:

<u>Thm.</u> (decomposition thm) $\exists$ iso in $D(Y)$.
([2], thm 2.1.1, (b))

$$f_*\mathbb{Q}_X[n] \simeq \oplus_i {}^pH^i(f_*\mathbb{Q}_X[n])[-i].$$

<u>Thm.</u> (semisimplicity thm) $\forall i$, $\exists$ iso in $Perv(Y)$
([2], thm 2.1.1, (c))

$${}^pH^i(f_*\mathbb{Q}_X[n]) \simeq \oplus_{l=0}^{\dim Y} i_{l,*} IC_{\overline{S_l}}(L_{i,l}),$$

where $\alpha_l : S_l \underset{\text{closed}}{\hookrightarrow} U_l := \coprod_{l'\geq l} S_{l'}$, $L_{i,l} := \alpha_l^* H^{l}({}^pH^i(f_*\mathbb{Q}_X[n])) \in Loc(S_l)$ is semisimple.

<u>Assumption.</u> Let $R\geq 0$, $m\geq 0$. Assume the statements in "the decomposition theorem package" all hold for $f: X\to Y$ with either $r(f)<R$, or $r(g)\leq R$ and $\dim f(X)<m$.

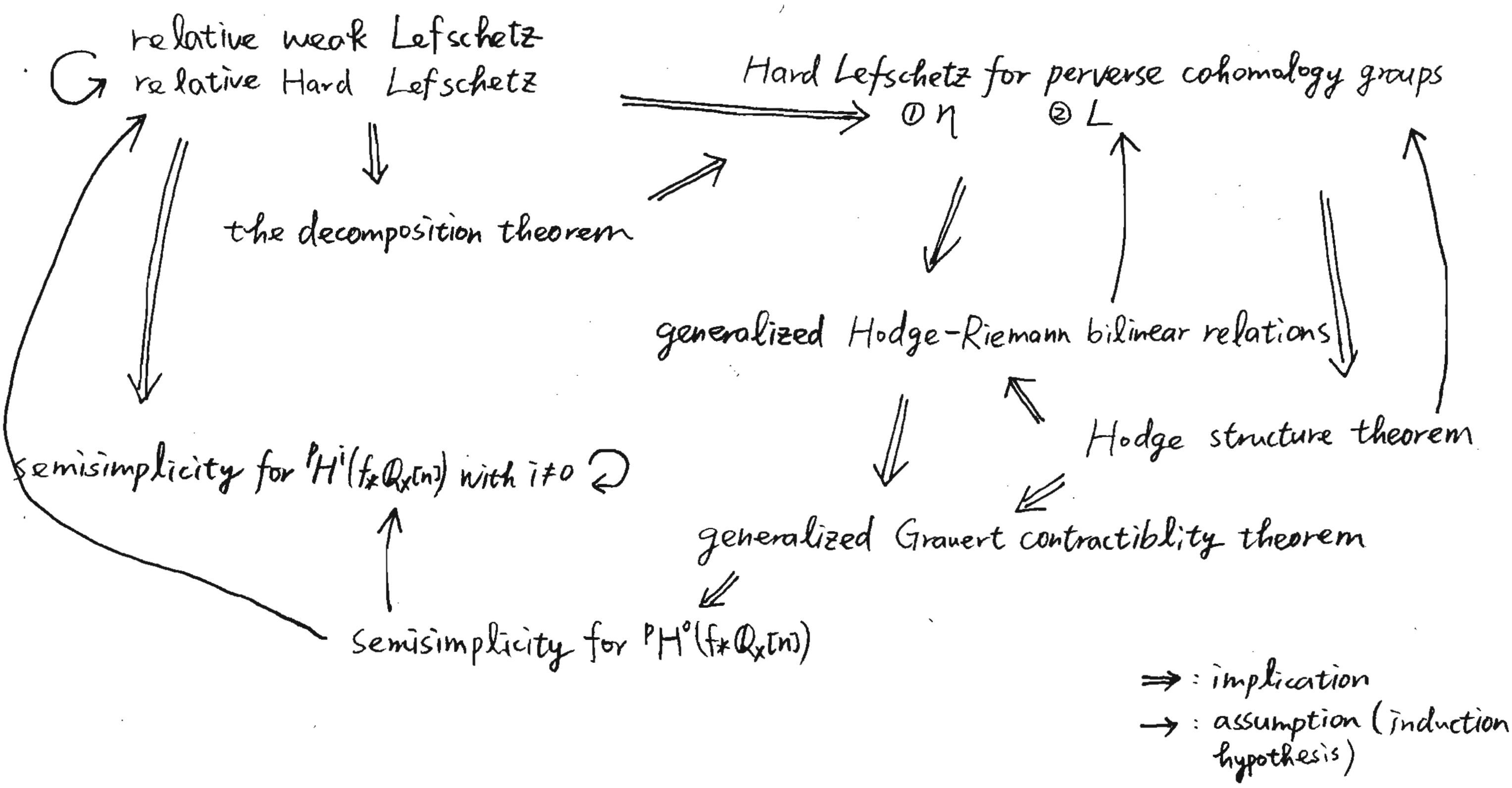


Figure: "the decomposition theorem package"

§ relative Hard Lefshetz

In this section we prove the relative Hard Lefshetz thm. This thm is a member in the decomp. thm. package and will imply the decomposition thm 2.1.1 (b). The semisimplicity for $i \neq 0$ also follows. from the "relative weak Lefschetz thm"; We then know from [14], prop 1.4.26 that, they can be expressed as a direct sum of minimal extensions from an openset and pushforward of simple objects from a closed set.

__Lemma.__ ([2], lemma 4.7.6) Consider a comm. diagram of alg. var.

$$\begin{tikzcd} X' \arrow[r, hook, "i", "\text{closed emb.}"'] \arrow[dr, "g"', "\text{proper}"] & X \arrow[d, "f"] & X-X' \arrow[l, hook', "j"'] \arrow[dl, "u\text{: affine}"] \\ & Y & \end{tikzcd}$$

, and $P \in Perv(X)$. Then

① ${}^pH^l(f_*P) \to {}^pH^l(g_*i^*P)$ is $\begin{cases} \text{iso}, & l \leq -2 \\ \text{inj}, & l = -1 \end{cases}$; ② ${}^pH^l(g_*i^!P) \to {}^pH^l(f_*P)$ is $\begin{cases} \text{iso}, & l \geq 2 \\ \text{surj}, & l = 1 \end{cases}$.

__Prop.__ f is affine $\Rightarrow f_*$ is right t-exact and $f_!$ is left t-exact.

__Pf of prop.__ [14], cor 4.1.2.

__Pf of lemma.__

① We have dist. Δ $j_!j^{-1}P \to P \to i_*i^{-1}P \xrightarrow{+1}$. Apply f_* ($= f_!$ since f is proper) $\rightsquigarrow$

$$u_!\underbrace{j^{-1}P}_{\in\, Perv(X-X')} \to f_*P \to g_*i^{-1}P \xrightarrow{+1}.$$

([5], cor 8.1.41.) (j^{-1} is right t-exact on loc. closed subvar.; $j^!$ left)

want: ${}^pH^{-1}(u_!j^{-1}P) = {}^p\tau_{\leq 0}\,{}^p\tau_{\geq 0}\, u_!j^{-1}P[-1] \in {}^pD_c^{\geq 0}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{?}{0}$

$\downarrow$

$u_!j^{-1}P[-1] \in {}^pD_c^{\geq 1}$, [5], cor 8.1.23 $\Rightarrow H^s(u_!j^{-1}P[-1]) = 0$ $\forall s < -\dim Y + 1$. On the other hand, ${}^pH^{-1}(u_!j^{-1}P) \in {}^pD_c^{\leq 0}$, so cor 8.1.23 $\Rightarrow H^s({}^pH^{-1}(u_!j^{-1}P)) = 0$ $\forall s > -\dim Y$ $\therefore {}^pH^{-1}(u_!j^{-1}P) = 0$. Similarly, ${}^pH^l(u_!j^{-1}P) = 0$ $\forall l \leq -2$.

② just use the Verdier duality, noting that $D \circ {}^pH^l \simeq {}^pH^{-l} \circ D$ ([15], remark 10.1.16).

Prop. ([2], prop 4.7.7) Y: proj var, $i: Y_1 \hookrightarrow Y$ hyperplane section, $P \in Perv(Y)$. Then
① $IH^l(Y,P) \to IH^l(Y_1, i^*P)$ is $\begin{cases} \text{iso}, & l \leq -2 \\ \text{inj}, & l = -1 \end{cases}$; ② $IH^l(Y_1, P) \to IH^l(Y, P)$ is $\begin{cases} \text{iso}, & l \geq 2 \\ \text{surj}, & l = 1 \end{cases}$

Pf apply the above lemma to

$$\begin{tikzcd} Y_1 \arrow[r, hook, "i"] \arrow[dr, "g"'] & Y \arrow[d, "f"] & Y - Y_1 \arrow[l, hook', "j"'] \arrow[dl, "u"] \\ & pt & \end{tikzcd}$$

(this prop. is used in proving HL for perverse cohomology gps; we won't use it in this section.)

The universal hyperplane section yields the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"] & & X \times \mathbb{P}^\vee \arrow[ll, "p"'] \arrow[d, "f'"] & \\ & \mathcal{X} \arrow[ur, "i"] \arrow[dr, "g"] & & X \times \mathbb{P}^\vee - \mathcal{X} \arrow[ul, "j"'] \arrow[dl, "u"] \\ Y & & \mathcal{Y} = Y \times \mathbb{P}^\vee \arrow[ll, "p"] & \end{tikzcd}$$

Let $\dim(\mathbb{P}) = d$. Then $p^*[d]$ is t-exact (use [5], prop 8.1.40.). Let $K \in Perv(X)$, $K' := p^*K[d]$, $M := i^*K'[-1]$.
$\therefore K' \in Perv(X \times \mathbb{P}^\vee)$. lemma 4.7.6 $\Rightarrow$ $\underset{p^* {}^pH^l(f_*K)[d]}{\underset{\shortparallel}{{}^pH^l(f'_*K')}} \to {}^pH^l(g_*i^*K')$ is $\begin{cases} \text{iso}, & l \leq -2 \\ \text{inj}, & l = -1 \end{cases}$.

Thm. (relative weak Lefschetz) ([2], prop 4.7.8)
① $p^* {}^pH^l(f_*K)[d] \xrightarrow{i^*} {}^pH^{l+1}(g_*M)$ is $\begin{cases} \text{iso}, & l \leq -2 \\ \text{inj}, & l = -1 \end{cases}$
② ${}^pH^{l-1}(g_*M) \xrightarrow{i_*} p^* {}^pH^l(f_*K)[d]$ is $\begin{cases} \text{iso}; & l \geq 2 \\ \text{surj}, & l = 1 \end{cases}$.

Now let $K = \mathbb{Q}_X[n]$. $\Rightarrow M = \mathbb{Q}_{\mathcal{X}}[n+d-1]$. Let $\eta \in Pic(X)$ be ample.
$\to \eta: \mathbb{Q}_X \to \mathbb{Q}_X[2]$. $\rightsquigarrow \eta: {}^pH^i(f_*\mathbb{Q}_X[n]) \to {}^pH^{i+2}(f_*\mathbb{Q}_X[n])$. Similarly, letting $\eta' := i^*p^*\eta$
$\rightsquigarrow \eta': {}^pH^i(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \to {}^pH^{i+2}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$.

Thm. ([2], 2.1.1(a), prop 5.1.3) (relative Hard Lefshetz)
(Assume $\eta'^r: {}^pH^{-r}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \xrightarrow{\sim} {}^pH^r(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$ $\forall r \geq 0$, and ${}^pH^0(g_*M)$ is semisimple. Then)
$$\eta^r: {}^pH^{-r}(f_*\mathbb{Q}_X[n]) \xrightarrow{\sim} {}^pH^r(f_*\mathbb{Q}_X[n]) \quad \forall r \geq 0.$$

Pf We have $\eta^r = i_* \circ \eta'^{r-1} \circ i^*$. If $r \neq 1 \Rightarrow$ use the relative weak Lefschetz.
For $r=1$, need: some maximality statement ([2], prop 4.7.9) and the fact that $p^*[d]$ is fully faithful ([14, proposition 4.2.5.7]).

We next prove the semisimplicity of ${}^pH^i(f_*\mathbb{Q}_X[n])$ for $i \neq 0$.

Prop. ([14, proposition 4.2.5.7]) In our situation, $p^*[d]: Perv(Y) \to Perv(\mathcal{Y})$ is fully faithful.

Pf (sketch) Let $K, L \in Perv(Y)$. $\to p^! R\mathcal{H}om(K,L) \simeq R\mathcal{H}om(p^*K, p^!L)$. p is smoth $\Rightarrow$
$$p^*R\mathcal{H}om(K,L) \simeq R\mathcal{H}om(p^*K, p^*L) (= R\mathcal{H}om(p^*K[d], p^*L[d]))$$
Applying p_* on both sides, noting $p_*p^* \simeq id$ in this case, and taking global section gives
$$\underset{Hom}{\underset{\shortparallel}{\Gamma R^0\mathcal{H}om(K,L)}} \simeq \underset{Hom}{\underset{\shortparallel}{\Gamma R^0\mathcal{H}om(p^*K[d], p^*L[d])}}.$$

This proposition, together with the relative weak Lefschetz, implies

$\underline{\text{Thm.}}$ ([2], prop 5.1.3) (Assuming ${}^pH^i(g_*\mathbb{Q}_{\tilde{X}}[n+d-1])$ is semisimple for all i, then) ${}^pH^i(f_*\mathbb{Q}_X[n])$ is semisimple $\forall i \neq 0$.

The decomposition thm 2.1.1(b) is a consequence of our relative Hard Lefshetz thm (rel HL). First, as in the semismall case, setting $P_\eta^{-i} := \ker \eta^{i+1} \subseteq {}^pH^{-i}(f_*\mathbb{Q}_X[n])$ we have

$\underline{\text{Thm.}}$ ([2], thm 2.1.1(a)) ${}^pH^{-i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{k\geq 0} \eta^k P_\eta^{-i-2k}$; ${}^pH^{i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{k\geq 0} \eta^{i+k} P_\eta^{-i-2k}$.

$\underline{\text{Thm.}}$ ([2], prop 5.1.3) (Assuming rel HL, then) $\exists$ iso in $D(Y)$

$$f_*\mathbb{Q}_X[n] \simeq \bigoplus_i {}^pH^i(f_*\mathbb{Q}_X[n])[-i].$$

$\underline{\text{Pf}}$: step 1: E_2-degeneration ([16], théorème (1.5).) I think this comes from [18], Chapitre III, proposition 4.4.6; not very sure.

Let $K := f_*\mathbb{Q}_X[n]$. $\forall M \in D(Y)$, $\exists$ spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(M[-p], {}^pH^q(K)) \Rightarrow \mathrm{Hom}(M[-p], K[q]).$$

Take $M = {}^pH^i(K)$ (i arbitrary). We have $\eta: E_2^{p,q} \to E_2^{p,q+2}$. Putting ${}_0E_2^{p,-j} := \ker(\eta^{j+1}: E_2^{p,-j} \to E_2^{p,j+2})$, we have $\begin{cases} E_2^{p,-j} = \bigoplus_{k\geq 0} \eta^k\, {}_0E_2^{p,-j-2k} \\ E_2^{p,j} = \bigoplus_{k\geq 0} \eta^{j+k}\, {}_0E_2^{p,-j-2k} \end{cases}$ want: $d_2^{p,q} = 0$. It suffices to check this on those ${}_0E_2^{p,-j}$'s. Observe:

$$\begin{tikzcd} {}_0E_2^{p,-j} \arrow[r, "d_2^{p,q}"] \arrow[d, "\eta^{j+1}"'] \arrow[dr, phantom, "\circlearrowright"] & E_2^{p+2,-j-1} \arrow[d, "\eta^{j+1}", "\wr"'] \\ E_2^{p,j+2} \arrow[r, "d_2^{p,q}"] & E_2^{p+2,j+1} \end{tikzcd}$$

zero map! $\quad \therefore d_2^{p,q} = 0$.

step 2: constructing an iso ([17], section 2.1.)

degeneration $\Rightarrow E_2^{0,i} = \mathrm{Hom}({}^pH^i(K), {}^pH^i(K)) \twoheadleftarrow \mathrm{Hom}({}^pH^i(K)[-i], K)$. $\rightsquigarrow \phi := \sum_i \phi_i : \bigoplus_i {}^pH^i(K)[-i] \xrightarrow{\sim} K$.

$\mathrm{id} \longmapsfrom \phi_i$

§ the local systems for $i \neq 0$

We have proved that ${}^pH^i(f_*\mathbb{Q}_X[n])$ is semisimple $\forall i \neq 0$; by [14], prop 1.4.26 it follows that they can be expressed as a direct sum of local systems from an open set and pushforward of simple objects from a closed set. As in the semismall case, for a fixed s, denote $S := S_s$. $U := \coprod_{l>s} S_l$, $Y := \coprod_{l\geq s} S_l$. $\rightsquigarrow S \xrightarrow{\alpha} Y \xleftarrow{\beta} U$. Then the following condition and assumption is met for these ${}^pH^i(f_*\mathbb{Q}_X[n])$: beware of the different notation from the semismall case (P.7)!

$\underline{\text{Lemma.}}$ ([2], lemma 4.1.3) (splitting criterion) Let $P \in \mathrm{Perv}(Y)$; assume $\forall s \in S$, we have

$$\dim_{\mathbb{Q}}(H^{-s}(\alpha_!\alpha^! P))_y = \dim_{\mathbb{Q}}(H^{-s}(\alpha_*\alpha^* P))_y.$$

Then TFAE: ① $P \simeq \beta_{!*}\beta^*P \oplus H^{-s}(P)[s]$

② $\iota: H^{-s}(\alpha_!\alpha^! P) \to H^{-s}(P)$

③ $l: P \to \tau_{\leq -s}\beta_*\beta^*P$ has a lifting $\tilde{l}: P \to \beta_{!*}\beta^*P$.

the condition ② is met: use the remark at the beginning of this section, [5] cor 8.1.44 (ii) and the fact that $\alpha^! \circ \alpha_* = \mathrm{id}$ since S is closed.

The above lemma allows us to write down the summands of ${}^pH^i(f_*\mathbb{Q}_X[n])$ explicitely ($i \neq 0$). We have:

$\underline{\text{Lemma.}}$ ([2], lemma 6.1.3) ${}^pH^i(f_*\mathbb{Q}_X[n])|_{U_s} \simeq \beta_{s,!*}({}^pH^i(f_*\mathbb{Q}_X[n])|_{U_{s+1}}) \oplus H^{-s}({}^pH^i(f_*\mathbb{Q}_X[n])|_{U_s})[s]$.

Therefore, remembering minimal extensions can be composited ([5], lemma 8.2.4) and using induction on s, we conclude:

$\underline{\text{Thm.}}$ ([2], prop 6.3.2) $\forall i \neq 0$, ${}^pH^i(f_*\mathbb{Q}_X[n]) \simeq \bigoplus_{l=0}^{\dim Y} IC_{\overline{S_l}}(\alpha_l^* H^{-l}({}^pH^i(f_*\mathbb{Q}_X[n])))$.

§ the way to the semisimplicity of ${}^pH^0(f_*\mathbb{Q}_X[n])$

It remains to discuss the semisimplicity for ${}^pH^0(f_*\mathbb{Q}_X[n])$. First, as in the mid-term report, let me present a heuristic argument. See [3], p.26-p.27.

<u>Def.</u> ([3], p.18; [2], p.31) The perverse filtration on $f_*\mathbb{Q}_X[n]$ is $\cdots \to {}^p\tau_{\le i} f_*\mathbb{Q}_X[n] \to {}^p\tau_{\le i+1} f_*\mathbb{Q}_X[n] \to \cdots$

This induces the perverse filtration on $\mathbb{H}^j(Y, f_*\mathbb{Q}_X[n])$:

$$\mathbb{H}^j_{\le i}(Y, f_*\mathbb{Q}_X[n]) := \mathrm{im}\left(\mathbb{H}^j(Y, {}^p\tau_{\le i} f_*\mathbb{Q}_X[n]) \to \mathbb{H}^j(Y, f_*\mathbb{Q}_X[n])\right).$$

The graded pieces are $\mathbb{H}^j_i(Y, f_*\mathbb{Q}_X[n]) := \mathbb{H}^j_{\le i}(Y, f_*\mathbb{Q}_X[n]) / \mathbb{H}^j_{\le i-1}(Y, f_*\mathbb{Q}_X[n])$. $\forall y \in Y$, $\alpha: y \hookrightarrow Y$, there is also an induced perverse filtration on $\alpha^! f_*\mathbb{Q}_X[n] \rightsquigarrow$ perverse filtration on

$$H^0(\alpha^! f_*\mathbb{Q}_X[n]) = \mathrm{Hom}(\alpha_*\mathbb{Q}_y, f_*\mathbb{Q}_X[n]) = H_n^{BM}(f^{-1}(y)),$$

Here we use the lemma proved in the mid-term report:

<u>Lemma.</u> $\mathrm{Hom}(\alpha_*\mathbb{Q}_y, f_*\mathbb{Q}_X[n]) = H_n^{BM}(f^{-1}(y)) = \mathrm{Hom}(f_*\mathbb{Q}_X[n], \alpha_*\mathbb{Q}_y)$.

<u>Pf of lemma.</u> $LHS = \mathrm{Hom}(\alpha_!\mathbb{Q}_y, f_*\mathbb{Q}_X[n]) = \mathrm{Hom}(\mathbb{Q}_y, \alpha^! f_*\mathbb{Q}_X[n]) \overset{[2], p.24}{=} \mathrm{Hom}(\mathbb{Q}_y, f_*\alpha^!\mathbb{Q}_X[n])$

$= \mathrm{Hom}(\mathbb{Q}_y, f_*\omega_{f^{-1}(y)}[-n]) \underset{[5], \text{prop C.2.4}}{=} \mathrm{Hom}(\mathbb{Q}_{f^{-1}(y)}, \omega_{f^{-1}(y)}[-n]) = H_n^{BM}(f^{-1}(y))$

ex C.2.16 $\Rightarrow H^{n-i}(f^{-1}(y), \mathbb{D}_{f^{-1}(y)}(M^\cdot)) = H_i(f^{-1}(y), M^\cdot)$ for $M^\cdot \in D^b(\mathbb{Q}_{f^{-1}(y)})$. Take $M^\cdot = \mathbb{Q}_{f^{-1}(y)}$ and $i = n$.

Similarly, $RHS = H_n^{BM}(f^{-1}(y))$.

i.e. $H^{BM}_{n,\le i}(f^{-1}(y)) := \mathrm{im}\left(H^0(\alpha^!\,{}^p\tau_{\le i} f_*\mathbb{Q}_X[n]) \to H^0(\alpha^! f_*\mathbb{Q}_X[n])\right)$. Now we say ${}^pH^0(f_*\mathbb{Q}_X[n])$ is semisimple at y if

$${}^pH^0(f_*\mathbb{Q}_X[n]) = \alpha_* V \oplus F,$$

where F is some perverse sheaf, $V = \alpha^* H^0({}^pH^0(f_*\mathbb{Q}_X[n]))$. Again this holds iff the following form is nondegenerate:

$$\mathrm{Hom}(\alpha_*\mathbb{Q}_y, {}^pH^0(f_*\mathbb{Q}_X[n])) \times \mathrm{Hom}({}^pH^0(f_*\mathbb{Q}_X[n]), \alpha_*\mathbb{Q}_y) \to \mathrm{End}(\alpha_*\mathbb{Q}_y) = \mathbb{Q}$$

We seek for other equivalent description for this. Conider the familiar form in the mid-term:

$$Q: \underset{H_n^{BM}(f^{-1}(y))}{\underset{\parallel}{\mathrm{Hom}(\alpha_*\mathbb{Q}_y, f_*\mathbb{Q}_X[n])}} \times \underset{H_n^{BM}(f^{-1}(y))}{\underset{\parallel}{\mathrm{Hom}(f_*\mathbb{Q}_X[n], \alpha_*\mathbb{Q}_y)}} \to \mathrm{End}(\alpha_*\mathbb{Q}_y) = \mathbb{Q}$$

Observe: ① By the decomp. thm we have proved, we may write $f_*\mathbb{Q}_X[n] = \tau_{\le 0} f_*\mathbb{Q}_X[n] \oplus \tau_{\ge 1} f_*\mathbb{Q}_X[n]$. Recalling y is assumed to be a point stratum now, [5] prop 8.1.40 (ii) gives $H^0(\alpha^!\,{}^p\tau_{\ge 1} f_*\mathbb{Q}_X[n]) = 0$.
$\therefore H^{BM}_{n,\le 0}(f^{-1}(y)) = H_n^{BM}(f^{-1}(y))$.

② ${}^p\tau_{\le -1} f_*\mathbb{Q}_X[n]$ doesn't contain any summand iso. to $\alpha_*\mathbb{Q}_y$; if so, $\alpha^{-1}\,{}^p\tau_{\le -1} f_*\mathbb{Q}_X[n]$ would have nontrivial H^0, while [5] prop 8.40 (i) gives a contradiction.

From these observations we can conclude:

<u>Prop.</u> ([3], prop 3.27) $\left({}^pH^0(f_*\mathbb{Q}_X[n])\text{ is semisimple at } y\right) \Leftrightarrow \left(Q \text{ induces a nondegenerate form on } H^{BM}_{n,0}(f^{-1}(y)) = H^{BM}_{n,\le 0}(f^{-1}(y)) / H^{BM}_{n,\le -1}(f^{-1}(y))\right)$.

Again, as in the semismall case, a result stronger than nondegeneracy can be proven. However, let's not pursue the precise result at this moment.

§ after the generalized Grauert contractility criterion

<u>Thm.</u> ([2], thm 2.1.8) (the generalized Grauert contractibiliton criterion) (Assuming the decomposition thm and the "generalized Hodge-Riemann bilinear relations", then) the intersection form induces an iso. $H^{BM}_{n,0}(f^{-1}(y)) \xrightarrow{\sim} {}^{\prime\prime}H^n_0(f^{-1}(y))^{\prime\prime}$.

Now, consider the natural map $A: \alpha_!\alpha^{!}\,{}^pH^0(f_*\mathbb{Q}_X[n]) \to {}^pH^0(f_*\mathbb{Q}_X[n])$.

<u>Prop.</u> ([2], prop 6.3.1) $H^0(A)_y: H^0(\alpha_!\alpha^!\,{}^pH^0(f_*\mathbb{Q}_X[n]))_y \to H^0({}^pH^0(f_*\mathbb{Q}_X[n]))_y$ is iso.

<u>pf.</u> Recall in the semismall case, the stalk of
$$\alpha_*\alpha^!(Rf_*\mathbb{Q}_X[n]|_{V'}) \to (Rf_*\mathbb{Q}_X[n])|_{V'} \to R\beta_*(Rf_*\mathbb{Q}_X[n]|_V) \xrightarrow{+1}$$
of deg $-i$ at y is $\to H^{n-i}(f^{-1}(U), f^{-1}(U-S')) \to H^{n-i}(f^{-1}(U)) \to H^{n-i}(f^{-1}(U-S')) \to \cdots$
$$H^{BM}_{n+i}(f^{-1}(S')) \cong H^{BM}_{n-i}(f^{-1}(s)) \qquad H^{n-i}(f^{-1}(s)) = H^{BM}_{n-i}(f^{-1}(s))^*$$

It's similar for the general case here. Noting that ${}^pH^0(f_*\mathbb{Q}_X[n]) = {}^p\tau_{\le 0} f_*\mathbb{Q}_X[n] / {}^p\tau_{\le -1} f_*\mathbb{Q}_X[n]$, the map $H^0(A)_y$ is just $H^{BM}_{n,0}(f^{-1}(y)) \to H^n_0(f^{-1}(y))$, which is an iso by the generalized Grauert contractiblity thm.

<u>Rmk.</u> In the original context of [2], the authors state that it's the composited map
$$\alpha_!\alpha^!\,{}^pH^0(f_*\mathbb{Q}_X[n]) \to {}^pH^0(f_*\mathbb{Q}_X[n]) \to \alpha_*\alpha^*\,{}^pH^0(f_*\mathbb{Q}_X[n])$$
whose stalk at y is $H^{BM}_{n,0}(f^{-1}(y)) \to H^n_0(f^{-1}(y))$. I think they made a mistake, because this seems not compatible with the semismall case argument.

The remaining argument again goes like that in semismall case. Just cut out by hyperplane sections to reduce things to point strata ([2], lemma 6.1.3), and we find

<u>Thm.</u> ([2], prop 6.3.2) ${}^pH^0(f_*\mathbb{Q}_X[n]) \simeq \bigoplus_{\ell=0}^{\dim Y} IC_{\overline{S_\ell}}(\alpha_\ell^* H^{-\ell}({}^pH^0(f_*\mathbb{Q}_X[n])))$.

§ the generalized Grauert contractibility criterion and the generalized Hodge Riemann bilinear relations

This section serves to make precise statements of the involving theorems in the last section.

Let $\eta \in Pic(X)$ and $A \in Pic(Y)$ be ample; $L := f^*A$.

Note that $H^{n+j}(X) \simeq \mathbb{H}^j(Y, f_*\mathbb{Q}_X[n])$. ↙ the perverse cohomology gp of X

<u>Def.</u> $H^{n+j}_{\le i}(X) := \mathbb{H}^j_{\le i}(Y, f_*\mathbb{Q}_X[n])$; $H^{n+j}_{i}(X) := H^{n+j}_{\le i}(X) / H^{n+j}_{\le i-1}(X)$.

Recall, $\eta: \mathbb{Q}_X \to \mathbb{Q}_X[2] \rightsquigarrow \eta: \mathbb{H}^*(Y, f_*\mathbb{Q}_X[n]) \to \mathbb{H}^{*+2}(Y, f_*\mathbb{Q}_X[n])$. Moreover we have the diagram
$$\begin{tikzcd} \tau_{\le i}\mathbb{Q}_X[n] \arrow[r, "\eta"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & (\tau_{\le i+2}\mathbb{Q}_X[n])[2] \arrow[d] \\ \mathbb{Q}_X[n] \arrow[r, "\eta"] & \mathbb{Q}_X[n][2] \end{tikzcd}$$
$\rightsquigarrow \eta: \mathrm{im}(\mathbb{H}^*({}^p\tau_{\le i} f_*\mathbb{Q}_X[n]) \to \mathbb{H}^*(f_*\mathbb{Q}_X[n])) \to \mathrm{im}(\mathbb{H}^{*+2}({}^p\tau_{\le i+2} f_*\mathbb{Q}_X[n]) \to \mathbb{H}^{*+2}(f_*\mathbb{Q}_X[n]))$,

i.e. $\eta: H^*_i(X) \to H^{*+2}_{i+2}(X)$.

For L, things are a bit different. We have $f_*\mathbb{Q}_X[n] \simeq \bigoplus_i {}^pH^i(f_*\mathbb{Q}_X[n])[-i]$, and $A: {}^pH^i(f_*\mathbb{Q}_X[n]) \to {}^pH^i(f_*\mathbb{Q}_X[n])[2]$, which means the ${}^p\tau_{\le i}$ part of $\mathrm{im}(f_*\mathbb{Q}_X[n] \xrightarrow{A} (f_*\mathbb{Q}_X[n])[2])$ comes from $\mathrm{im}(\bigoplus_{\le i} {}^pH^i(f_*\mathbb{Q}_X[n])[-i] \to \bigoplus_{\le i} {}^pH^i(f_*\mathbb{Q}_X[n])[-i+2])$.
Moreover, $f_*L = A$ ([2], rmk 4.4.3), ∴ $L: H^*_i(X) \to H^*_{i+2}(X)$.

<u>Thm.</u> ([2], thm 2.1.4) (Hard Lefshetz for perverse cohomology groups) (Assuming the relative Hard Lefschetz and the "generalized Hodge-Riemann bilinear relations," then) $\forall k \in \mathbb{Z}_{\ge 0}$, $b, j \in \mathbb{Z}$, we have
$$\eta^k: H^j_{-k}(X) \xrightarrow{\sim} H^{j+2k}_{k}(X) \ ; \ L^k: H^{n+b-k}_b(X) \to H^{n+b+k}_b(X).$$

<u>Thm.</u> ([2], thm 2.1.5) (the Hodge structure theorem) (Assuming the Hard Lefschetz for perverse cohomology groups, then) $\forall k \in \mathbb{Z}_{\ge 0}$, $b \in \mathbb{Z}$, $H^k_b(X)$ inherits a pure Hodge structure of weight k.

The rel HL for perverse cohomology groups allows us to perform a direct sum decomposition, just like we had in P.1. Define $P^{-j}_{-i} := \begin{cases} \ker \eta^{i+1} \cap \ker L^{j+1} \subseteq H^{n-i-j}_{-i}(X), & i, j \ge 0 \\ 0, & \text{o.w} \end{cases}$

<u>Cor.</u> ([2], cor 2.1.6) ((η, L)-decomposition) $\forall i, j \in \mathbb{Z}$, we have a direct sum decomposition into pure Hodge substructures of weight $(n-i-j)$:
$$H^{n-i-j}_{-i}(X) = \bigoplus_{\ell, m \in \mathbb{Z}} \eta^{-i+\ell} L^{-j+m} P^{j-2m}_{i-2\ell}.$$

The bilinear form in the general case is: $\forall i,j \geq 0$,

$$S_{ij}^{\eta L}: H_{-i}^{n-i-j}(X) \times H_{-i}^{n-i-j}(X) \to \mathbb{R}$$
$$([\alpha], [\beta]) \mapsto \int_X \eta^i \wedge L^j \wedge \alpha \wedge \beta .$$

(<u>Rmk.</u> ([2], p.9) Using thm 2.1.4, $S_{ij}^{\eta L}$ can be defined $\forall i,j \in \mathbb{Z}$.)

<u>Thm.</u> ([2], thm 2.1.7) (the generalized Hodge-Riemann bilinear relations) (Assuming rel HL for perverse cohomology groups, then) The (η, L)-decomposition is orthogonal with respect to $S_{ij}^{\eta L}$. Up to a sign ([2], rmk 4.5.2), $S_{ij}^{\eta L}$ is a polarization on each (η, L)-direct summand.

<u>Thm.</u> ([2], thm 2.1.8) (the generalized Grauert contractibility criterion) (Assuming the generalized Hodge-Riemann bilinear relations, then) $\forall b \in \mathbb{Z}$, the filtered class map

$$cl_b: H_{n-b,b}^{BM}(f^{-1}(y)) \to H_b^{n+b}(X)$$

is injective and gives $H_{n-b,b}^{BM}(X)$ a pure Hodge substructure, compatible with the (η, L)-decomposition. Each direct summand is up to sign polarized by $S_{-b,0}^{\eta L}$.

The last statement of thm 2.1.8 implies $S_{-b,0}^{\eta L}|_{H_{n-b,b}^{BM}(f^{-1}(y))}$ is nondegenerate, whose particular case $b=0$ is our original statement in p.12.

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