D-module Final Report I

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0.1 Borel-Weil-Bott Theorem

Let G/k be a connected semi-simple algebraic group, T be a maximal torus of G, B be a Borel subgroup of G containing T, N be the unipotent part of B, and X be the flag variety G/B. We thus has a choice of positive roots Δ^+, simple roots $\Pi = \{\alpha_1, \cdots, \alpha_l\}$, and Weyl vector $\rho = \dfrac{\alpha_1 + \cdots + \alpha_l}{2}$. Let P be the weight lattice. For each $\lambda \in L = \mathrm{Hom}_k(B/N, k^*)$, we have a equivariant G-line bundle $\mathcal{L}(\lambda)$ on X ([HTT] p.255). Set

$$P_{sing} = \{\lambda \in P \mid \exists \alpha \in \Delta, \langle \lambda - \rho, \alpha^\vee \rangle = 0\},$$

$$P_{reg} = P - P_{sing}.$$

Define a shifted action of W on P by

$$w \bigstar \lambda = w(\lambda - \rho) + \rho.$$

Theorem 1 (Borel-Weil-Bott,[HTT] 9.11.2.)**.** Assume $\lambda \in L \subset P$.

(i) If $\langle \lambda, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^+$, then $\mathcal{L}(\lambda)$ is generated by global sections. That is, the natural morphism

$$\mathcal{O}_X \otimes_k \Gamma(X, \mathcal{L}(\lambda)) \to \mathcal{L}(\lambda)$$

is surjective.

(ii) $\mathcal{L}(\lambda)$ is ample if and only if $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta^+$.

(iii) Assume $\mathrm{char}(k) = 0$.

(a) If $\lambda \in P_{sing}$, then $H^i(X, \mathcal{L}(\lambda)) = 0$ for all $i \geq 0$.

(b) Let $\lambda \in P_{reg}$ and take $w \in W$ such that $w \bigstar \lambda \in -P^+$. Then

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} L^-(w \bigstar \lambda) & \text{if } i = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

0.2 Berlinson-Bernstein Theorems

From now on we assume $k = \mathbb{C}$. For every smooth variety Y and locally free $\mathcal{O}_Y$-module of finite rank $\mathcal{V}$, we consider the sheaf of diffenertial operators on $\mathcal{V}$, $D_Y^{\mathcal{V}} \subset \mathscr{E}nd_{\mathbb{C}_Y}(\mathcal{V})$. $D_Y^{\mathcal{V}}$ is isomorphic to $\mathcal{V} \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^*$. There's a natural filtration

$$F_p(D_Y^{\mathcal{V}}) = 0 \text{ for all } p < 0,$$

$$F_p(D_Y^{\mathcal{V}}) = \{P \mid fP - Pf \in F_{p-1}(D_Y^{\mathcal{V}}) \forall f \in \mathcal{O}_Y\} \text{ for all } p \geq 0.$$

Assume K is alinear algebraic group acting on Y and $\mathcal{V}$ is a K-equivlent vector bundle. There is a natural morphism $\partial : U(\mathfrak{k})$ to $\Gamma(Y, D_Y^{\mathcal{V}})$. Let $a \in \mathfrak{k}$, then ∂_a is defined by

$$(\partial_a s)(y) = \frac{d}{dt}(\exp(ta)s(\exp(-ta)y))|_{t=0} \ (s \in \mathcal{V}, y \in Y).$$

Here exp is the exponential map w.r.t right invariant vector fields. Algebraically, let $\varphi : p_2^*\mathcal{V} \cong \sigma^*\mathcal{V}$, then ∂_a is determined by

$$\phi((a \otimes 1) \cdot \varphi^{-1}(\sigma^* s)) = \sigma^*(\partial_a s)$$

Here a is regarded as right invariant vector fields on K acting on $k[K]$([HTT] Equation 11.1.7).

Consider X. Let $D_\lambda := D_X^{\mathcal{L}(\lambda+\rho)}$. We have $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$.

Definition 1. Let $\mathfrak{z}$ be the center of $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$([HTT] Equation 9.4.7). Let p be the projection from $U(\mathfrak{g})$ to $U(\mathfrak{h})$. f be the automorphism of $U(\mathfrak{h})$ defined by $f(h) = h - \rho(h)1$ for $h \in \mathfrak{h}$. For each $\lambda \in \mathfrak{h}^*$, define the central character

$$\chi_\lambda(z) = (f \circ p(z))(\lambda) \text{ for all } z \in \mathfrak{z}.$$

Proposition 1 ([HTT] Theorem 11.2.2)**.** Let $\lambda \in L$. Then $\Phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, D_\lambda)$ is surjective. Let $\mathfrak{z}$ be the center of $U(\mathfrak{g})$. Then $\Phi_\lambda(z) = \chi_\lambda(z)$ for all $z \in \mathfrak{z}$. Moreover, $\ker(\Phi_\lambda) = U(\mathfrak{g})(\ker(\chi_\lambda))$.

We assume the proposition.

Let $\mathrm{Mod}_{qc}(D_\lambda)$ be the abelian category of D_λ-modules which are quasi-coherent over $\mathcal{O}_X$ and $\mathrm{Mod}(\mathfrak{g})$ be the categoriy of $U(\mathfrak{g})$-modules. We have additive functors

$$\Gamma(X, \cdot) : \mathrm{Mod}_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}),$$

$$D_\lambda \otimes_{U(\mathfrak{g})} (\cdot) : \mathrm{Mod}(\mathfrak{g}) \to \mathrm{Mod}_{qc}(D_\lambda).$$

We have adjointness

$$\mathrm{Hom}_{D_\lambda}(D_\lambda \otimes_{U(\mathfrak{g})} M, \mathcal{N}) \cong \mathrm{Hom}_{U(\mathfrak{g})}(M, \Gamma(X, \mathcal{N})).$$

Let $\mathrm{Mod}(\mathfrak{g}, \chi)$ be the category of $U(\mathfrak{g})$-modules with central character χ and $\mathrm{Mod}_f(\mathfrak{g}, \chi)$ be the full subcategory of $\mathrm{Mod}(\mathfrak{g}, \chi)$ of finitely generated $U(\mathfrak{g})$-modules. The proposition shows that $\mathrm{Mod}(\mathfrak{g}, \chi_\lambda) \cong \mathrm{Mod}(\Gamma(X, D_\lambda))$.

Theorem 2 ([HTT] Theorem 11.2.3 & 11.2.4)**.** Let $\lambda \in L$.

1. Suppose
$$\langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for all } \alpha \in \Delta^+. \tag{1}$$
That is, $\lambda \in -P^+$. Then for all $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$ we have $H^k(X, \mathcal{M}) = 0$ for all $k > 0$.

2. Suppose
$$\langle \lambda, \alpha^\vee \rangle < 0 \text{ for all } \alpha \in \Delta^+. \tag{2}$$
Then for all $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$, the natural morphism
$$D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$$
is surjective.

Proof. For $v \in -P^+$, Borel-Weil-Bott theorem says $\Gamma(X, \mathcal{L}(v)) = H^0(X, \mathcal{L}(v)) = L^-(v)$ and $p_v : \mathcal{O}_X \otimes_{\mathbb{C}} L^-(v) \to \mathcal{L}(v)$ is surjective. Since $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}(v), \mathcal{O}_X) = \mathcal{L}(-v)$ and $\mathrm{Hom}_{\mathbb{C}}(L^-(v), \mathbb{C}) = L^+(-v)$, we have $\mathcal{L}(-v) \hookrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} L^+(-v)$. Apply $\mathcal{L}(v) \otimes_{\mathcal{O}_X} (\cdot)$, we have $i_v : \mathcal{O}_X \hookrightarrow \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$. Since $\mathcal{L}(v)$ is a line bundle, $\ker(p_v)$ is a direct summand of $\mathcal{O}_X \otimes_{\mathbb{C}} L^-(v)$ as an $\mathcal{O}_X$-module locally. Therefore, $\mathrm{im}(i_v)$ is a direct summand of $\mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$ as an $\mathcal{O}_X$-module locally.

Let $\lambda \in L$ and $\mathcal{M}$ be a D_λ-module. Apply $\mathcal{M} \otimes_{\mathcal{O}_X} (\cdot)$, we get

$$\overline{p_v} : \mathcal{M} \otimes_{\mathbb{C}} L^-(v) \twoheadrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v),$$

$$\overline{i_v} : \mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v).$$

Proposition 2 ([HTT] Proposition 11.4.1)**.** (i) If λ satisfies (2), then $\ker(\overline{p_v})$ is a direct summand of $\mathcal{M} \otimes_{\mathbb{C}} L^-(v)$ as a sheaf of abelian groups.

(ii) If λ satisfies (1), then $\mathrm{im}(\overline{i_v})$ is a direct summand of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$ as a sheaf of abelian groups.

Suppose λ satisfies (1). For all $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$,

$$H^k(X, \mathcal{M}) = \varinjlim H^k(X, \mathcal{N})$$

where $\mathcal{N}$ runs over all coherent $\mathcal{O}_X$-submodule of $\mathcal{M}$. It suffices to prove that the natural map $H^k(X, \mathcal{N}) \to H^k(X, \mathcal{M})$ is the zero map. Fix $\mathcal{N}$. Borel-Weil-Bott theorem says $\mathcal{L}(v)$ is ample if and only if v satisfies (2). Hence there is a $v \in L \cap -P^+$ such that $H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v)) = 0$ for all $k > 0$. For this v, consider the commutative diagram

$$\begin{array}{ccc} H^k(X, \mathcal{N}) & \longrightarrow & H^k(X, \mathcal{M}) \\ \downarrow & & \downarrow \overline{i_v}_* \\ H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)) & \longrightarrow & H^k(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)). \end{array}$$

$\overline{i_v}_*$ is injective. On the other hand, $H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)) = H^k(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v)) \otimes_{\mathbb{C}} L^+(-v) = 0$ for all $k > 0$. So $H^k(X, \mathcal{N}) \to H^k(X, \mathcal{M})$ is the zero map.

Suppose λ satisfies (2). For given $\mathcal{M} \in \mathrm{Mod}_{qc}(D_\lambda)$, set $\mathcal{M}'$ be the image of $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ and $\mathcal{M}''$ be the cokernel of it. If $\mathcal{M}'' \neq 0$, let $\mathcal{N} \subset \mathcal{M}''$ be a nonzero coherent $\mathcal{O}_X$-submodule. There is a $v \in L \cap -P^+$ such that $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v)$ is generated by global sections. In this case, $\Gamma(X, \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}(v)) \neq 0$, neither is $\Gamma(X, \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(v))$ On the other hand,

$$\overline{p_v}_* : \Gamma(X, \mathcal{M}'') \otimes_{\mathbb{C}} L^-(v) = \Gamma(X, \mathcal{M}'' \otimes_{\mathbb{C}} L^-(v)) \to \Gamma(X, \mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{L}(v))$$

is surjective. So $\Gamma(X, \mathcal{M}'') \neq 0$. Consider the exact sequence

$$0 \to \Gamma(X, \mathcal{M}') \to \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{M}'') \to 0.$$

By definition, $\Gamma(X, \mathcal{M}) = \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M})) \twoheadrightarrow \Gamma(X, \mathcal{M}')$. So $\Gamma(X, \mathcal{M}') = \Gamma(X, \mathcal{M})$ and hence $\Gamma(X, \mathcal{M}'') = 0$. So $\mathcal{M}''$ must be 0. The isomorphism $\Gamma(X, \mathcal{M}) = \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}))$ is proved in the proof of Corollary 1. □

0.3 Equivalences of Categories

For $\lambda \in L$ satisfying (1), we denote $\mathrm{Mod}^e_{qc}(D_\lambda)$ the full subcategory of $\mathrm{Mod}_{qc}(D_\lambda)$ consisting of objects $\mathcal{M}$ satisfying that

(a) $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective.

(b) For all nonzero subobject $\mathcal{N} \subset \mathcal{M}$ in $\mathrm{Mod}_{qc}(D_\lambda)$, we have $\Gamma(X, \mathcal{N}) \neq 0$.

Set $\mathrm{Mod}^e_c(D_\lambda) = \mathrm{Mod}^e_{qc}(D_\lambda) \cap \mathrm{Mod}_c(D_\lambda)$.

Corollary 1. $\Gamma(X, \cdot)$ induces equivlences of categories

$$\mathrm{Mod}^e_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}, \chi_\lambda), \qquad \mathrm{Mod}^e_c(D_\lambda) \cong \mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda).$$

Proof. We first prove that $M \to \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M)$ is an isomorphism for all $M \in \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$. For given $M \in \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$, consider an exact sequence

$$\Gamma(X, D_\lambda)^{\oplus I} \to \Gamma(X, D_\lambda)^{\oplus J} \to M \to 0.$$

From Theorem 2.1, $\Gamma(X, \cdot) : \mathrm{Mod}_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ is an exact functor. So $\Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} (\cdot))$ is right exact. We have an commutative diagram with exact rows

$$\begin{array}{ccccccc}
\Gamma(X, D_\lambda)^{\oplus I} & \longrightarrow & \Gamma(X, D_\lambda)^{\oplus J} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow & & \\
\Gamma(X, D_\lambda)^{\oplus I} & \longrightarrow & \Gamma(X, D_\lambda)^{\oplus J} & \longrightarrow & \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M) & \longrightarrow & 0.
\end{array}$$

So $M \cong \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M)$.

Now we show that $\Gamma(X, \cdot) : \mathrm{Mod}^e_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}, \chi_\lambda)$ is fully faithful. That is, for all $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Mod}^e_{qc}(D_\lambda)$,

$$\Gamma : \mathrm{Hom}_{D_\lambda}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}_{U(\mathfrak{g})}(\Gamma(X, \mathcal{M}_1), \Gamma(X, \mathcal{M}_2))$$

is an isomorphism. Since $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1) \to \mathcal{M}_1$ is surjective, we have

$$\begin{aligned}\mathrm{Hom}_{D_\lambda}(\mathcal{M}_1,\mathcal{M}_2) &\hookrightarrow \mathrm{Hom}_{D_\lambda}(D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1),\mathcal{M}_2)\\ &\cong \mathrm{Hom}_{U(\mathfrak{g})}(\Gamma(X,\mathcal{M}_1),\Gamma(X,\mathcal{M}_2)).\end{aligned}$$

Assume $\phi \in \mathrm{Hom}_{U(\mathfrak{g})}(\Gamma(X,\mathcal{M}_1),\Gamma(X,\mathcal{M}_2))$. Let $\mathcal{K}_1$ be the kernel of $D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1) \to \mathcal{M}_1$. Apply the exact functor $\Gamma(X,\cdot)$ on the exact sequenct

$$0 \to \mathcal{K}_1 \to D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1) \to \mathcal{M}_1 \to 0$$

and we get the exact sequence

$$0 \to \Gamma(X,\mathcal{K}_1) \to \Gamma(X,\mathcal{M}_1) \to \Gamma(X,\mathcal{M}_1) \to 0.$$

So $\Gamma(X,\mathcal{K}_1)=0$. Let $\mathcal{K}_2$ be the image of

$$\mathcal{K}_1 \longrightarrow D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1) \xrightarrow{1\otimes\phi} D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_2) \longrightarrow \mathcal{M}_2.$$

Since $\Gamma(X,\cdot)$ is exact and $\Gamma(X,\mathcal{K}_1)=0$, $\Gamma(X,\mathcal{K}_2)=0$. So $\mathcal{K}_2=0$. hence we obtain $\psi : \mathcal{M}_1 \cong D_\lambda \otimes_{U(\mathfrak{g})} \Gamma(X,\mathcal{M}_1)/\mathcal{K}_1 \to \mathcal{M}_2$ with $\Gamma(\psi)=\phi$.

Next, we prove that $\Gamma(X,\cdot) : \mathrm{Mod}^e_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g},\chi_\lambda)$ is essentially surjective. Given $M \in \mathrm{Mod}(\mathfrak{g},\chi_\lambda)$. Let $\mathcal{L}$ be a maximal element of the set of subobjects $\mathcal{K}$ of $D_\lambda \otimes_{U(\mathfrak{g})} M$ in $\mathrm{Mod}_{qc}(D_\lambda)$ satisfying that $\Gamma(X,\mathcal{K})=0$. Set $\mathcal{M} = D_\lambda \otimes_{U(\mathfrak{g})} M/\mathcal{L}$. Then $\Gamma(X,\mathcal{M}) = \Gamma(X, D_\lambda \otimes_{U(\mathfrak{g})} M)/\Gamma(X,\mathcal{L}) = M$. $D_\lambda \otimes_{U(\mathfrak{g})} M \to \mathcal{M}$ is surjective. For all $\mathcal{N} \subset \mathcal{M}$, the maximality of $\mathcal{L}$ shows that $\Gamma(X,\mathcal{N}) \neq 0$. So $\mathcal{M} \in \mathrm{Mod}^e_{qc}(D_\lambda)$.

Finally, we have to show that $\mathrm{Mod}^e_c(D_\lambda)$ and $\mathrm{Mod}_f(\mathfrak{g},\chi_\lambda)$ correspond to each other. Let $M \in \mathrm{Mod}_f(\mathfrak{g},\chi_\lambda)$. $\Gamma(X,D_\lambda)$ is left-noetherian. There is an exact sequence

$$\Gamma(X,D_\lambda)^{\oplus I} \to \Gamma(X,D_\lambda)^{\oplus J} \to M \to 0$$

with $|I|,|J| < \infty$. Apply the right exact functor $D_\lambda \otimes_{U(\mathfrak{g})} (\cdot)$ on it and we get the exact sequence

$$D_\lambda^{\oplus I} \to D_\lambda^{\oplus J} \to D_\lambda \otimes_{U_\mathfrak{g}} M \to 0.$$

We get $D_\lambda \otimes_{U_\mathfrak{g}} M \in \mathrm{Mod}^e_c(D_\lambda)$ and hence $\mathcal{M} = D_\lambda \otimes_{U_\mathfrak{g}} M/\mathcal{L}$.

Conversely, let $\mathcal{M} \in \mathrm{Mod}_c^e(D_\lambda)$. Since $D_\lambda \otimes_{U_\mathfrak{g}} \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective, $\mathcal{M}$ is locally generated by finitely many global sections. Since X is quasi-compact, $\mathcal{M}$ is globally generated by finitely many global sections. We have an exact sequence

$$D_\lambda^{\oplus I} \to \mathcal{M} \to 0$$

where $|I| < \infty$. Apply $\Gamma(X, \cdot)$ on it and we get the exact sequence

$$\Gamma(X, D_\lambda)^{\oplus I} \to \Gamma(X, \mathcal{M}) \to 0$$

Hence $\Gamma(X, \mathcal{M})$ is an finitely generated $U(\mathfrak{g})$-module. □

Suppose λ satisfies (2). Then $\mathrm{Mod}_{qc}(D_\lambda) = \mathrm{Mod}_{qc}^e(D_\lambda)$. In this case, we have

Corollary 2. $\Gamma(X, \cdot)$ induces equivlences of categories

$$\mathrm{Mod}_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}, \chi_\lambda), \qquad \mathrm{Mod}_c(D_\lambda) \cong \mathrm{Mod}_f(\mathfrak{g}, \chi_\lambda).$$

Let K be a closed subgroup of G. We consider K-equivariant $\mathfrak{g}$-modules. That is, a $\mathfrak{g}$-module with a K-action satisfying that

$$\mathfrak{k}\text{-actions obtained from the } \mathfrak{g}\text{-action and the } K\text{-action coincide.} \tag{3}$$

$$k \cdot (a \cdot m) = \mathrm{Ad}(k)(a) \cdot (k \cdot m) \text{ for all } k \in K,\ a \in \mathfrak{g}, \text{ and } m \in M. \tag{4}$$

We denote the full subcategory consisting of K-equivariant objects of $\mathrm{Mod}(\mathfrak{g}, \chi)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi)$ by $\mathrm{Mod}(\mathfrak{g}, \chi, K)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi, K)$, respectively.

We also introduce K-equivariant D-modules. Let K acts on Y. Consider morphisms $p_2 : K \times Y \to Y$, $\sigma : K \times Y \to Y$, $m : K \times K \to K$ defined by $p_2(k, y) = y$, $\sigma(k, y) = ky$, $m(k_1, k_2) = k_1, k_2$. A K-equivariant D_Y-module is a D_Y-module $\mathcal{M}$ with a isomorphism of $D_{K\times Y}$-modules

$$\varphi : p_2^* \mathcal{M} \cong \sigma^* \mathcal{M}$$

satisfying the cocycle condition.

We consider categories $\mathrm{Mod}_{qc}(D_Y, K)$ and $\mathrm{Mod}_c(D_Y, K)$. For $\lambda = -\rho$, we have $\mathrm{Mod}(\mathfrak{g}, \chi_{-\rho}) \cong \mathrm{Mod}_{qc}(D_X)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}) \cong \mathrm{Mod}_{qc}(D_X)$.

Theorem 3. For any closed subgroup $K \le G$, we have $\mathrm{Mod}(\mathfrak{g}, \chi_{-\rho}, K) \cong \mathrm{Mod}_{qc}(D_X, K)$ and $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, K) \cong \mathrm{Mod}_{qc}(D_X, K)$.

Proof. What we have to prove is K-equivariances defined on $\mathrm{Mod}(\mathfrak{g}, \chi_{-\rho})$ and $\mathrm{Mod}_{qc}(D_X)$ coincide.

Consider $\mathcal{M} \in \mathrm{Mod}_{qc}(D_X)$. K, X and $K \times X$ are all D-affine. So $D_{K\times X}$-modules are $\Gamma(K\times X, D_{K\times X}) = \Gamma(K, D_K) \otimes_{\mathbb{C}} \Gamma(X, D_X)$-modules. Since $\Gamma(K, D_K) \cong \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} U(\mathfrak{k})$ and $\Gamma(X, D_X) \cong U(\mathfrak{g})/U(\mathfrak{g}) \ker(\chi_{-\rho})$, $D_{K\times X}$-module structures are determined by actions of $\Gamma(K, \mathcal{O}_K) \otimes 1$, $\mathfrak{k} \otimes 1$ and $1 \otimes \overline{\mathfrak{g}}$.

$\Gamma(K \times X, p_2^*\mathcal{M}) \cong \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M})$. For $\sigma^*\mathcal{M}$, consider isomorphisms $\epsilon_1 : K \times X \to K \times X$, $\epsilon_2 : K \times X \to K \times X$ defined by $\epsilon_1(k, x) = (k, kx)$, $\epsilon_2(k, x) = (k, k^{-1}x)$. $\epsilon_1 = \epsilon_2^{-1}$ and $\sigma = p_2 \circ \epsilon_1$. So

$$\begin{aligned}\Gamma(K \times X, \sigma^*\mathcal{M}) &\cong \Gamma(K \times X, \epsilon_1^* p_2^*\mathcal{M}) \cong \Gamma(K \times X, (\epsilon_2)_* p_2^*\mathcal{M}) \\ &\cong \Gamma(K \times X, p_2^*\mathcal{M}) \cong \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M}).\end{aligned}$$

For given $h \in \Gamma(X, \mathcal{O}_X)$ and $m \in \Gamma(X, \mathcal{M})$, the element $h \otimes m \in \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M})$ corresponds to the global section $h \circ p_1 \otimes p_2^{-1}m$ of $p_2^*\mathcal{M} = \mathcal{O}_{K\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\mathcal{M}$ and the global section $h \circ p_1 \otimes \sigma^{-1}m : (k, x) \mapsto (k, h(k)k^{-1} \cdot m(kx))$ of $\sigma^*\mathcal{M} = \mathcal{O}_{K\times X} \otimes_{\sigma^{-1}\mathcal{O}_X} \sigma^{-1}\mathcal{M}$. The $\Gamma(K, D_K) \otimes_{\mathbb{C}} \Gamma(X, D_X)$-action on $p_2^*\mathcal{M}$. is

$$\begin{cases} (f \otimes 1) \cdot (h \otimes m) = fh \otimes m & \text{for all } f \in \Gamma(K, \mathcal{O}_K), \\ (a \otimes 1) \cdot (h \otimes m) = a \cdot h \otimes m & \text{for all } a \in \mathfrak{k}, \\ (1 \otimes \overline{p}) \cdot (h \otimes m) = h \otimes \overline{p} \cdot m & \text{for all } p \in \mathfrak{g}. \end{cases}$$

Consider the $\Gamma(K, D_K) \otimes_{\mathbb{C}} \Gamma(X, D_X)$-action on $\sigma^*\mathcal{M}$. $(f \otimes 1) \cdot (h \otimes m) = fh \otimes m$ for all $f \in \Gamma(K, \mathcal{O}_K)$. $(a \otimes 1) \cdot (h \otimes m) = a \cdot h \otimes m - h \otimes \overline{a} \cdot m$ for all $a \in \mathfrak{k}$. Finally, $\dfrac{d}{dt} \exp(tp)k^{-1}m(k\exp(-tp)x)|_{t=0} = \dfrac{d}{dt} k^{-1}\exp(t\,\mathrm{Ad}(k)(p))m(\exp(t\,\mathrm{Ad}(k)(p))kx)|_{t=0}$. Let $\mathrm{Ad}(k)(p) = \sum_i h_i(k)p_i$. We have $(1 \otimes \overline{p}) \cdot (h \otimes m) = \sum_i hh_i \otimes \overline{p_i} \cdot m$ for all $p \in \mathfrak{g}$.

The K-equivariance of $\mathcal{M}$ is equivlent to an $\Gamma(K, D_K) \otimes_{\mathbb{C}} \Gamma(X, D_X)$-module isomorphism from $\Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M}) \cong \Gamma(K \times X, p_2^*\mathcal{M})$ to $\Gamma(K \times X, \sigma^*\mathcal{M}) \cong \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M})$ satisfying the cocycle condition. Since $\Gamma(K, \mathcal{O}_K)$-actions on both sides are the same, the condition is a $\mathbb{C}$-module homomorphism

$$\widetilde{\varphi} : \Gamma(X, \mathcal{M}) = 1 \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M}) \to \Gamma(K, \mathcal{O}_K) \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M})$$

satisfying the cocycle condition and

$$\widetilde{\varphi}((a\otimes 1)\cdot(1\otimes m)) = (a\otimes 1)\cdot\widetilde{\varphi}(1\otimes m) \text{ for all } a\in\mathfrak{k},\ m\in\Gamma(X,\mathcal{M}). \tag{5}$$

$$\widetilde{\varphi}((1\otimes\overline{p})\cdot(1\otimes m)) = (1\otimes\overline{p})\cdot\widetilde{\varphi}(1\otimes m) \text{ for all } p\in\mathfrak{g},\ m\in\Gamma(X,\mathcal{M}). \tag{6}$$

Let $\widetilde{\varphi}(m) = \sum_j g_j\otimes m_j$. The cocycle condition is equivlent to a K-representation structure of $\Gamma(X,\mathcal{M})$. (5) is

$$0 = \sum_j a\cdot g_j\otimes m_j - \overline{a}\cdot m.$$

$a : m\mapsto \sum_j a\cdot g_j\otimes m_j$ is the $\mathfrak{k}$-action obtained from the K-action while $a : m\mapsto \overline{a}\cdot m$ is the $\mathfrak{k}$-action obtained from the $\mathfrak{g}$-action. So (5) is (3).

(6) is

$$\sum_j g_j\otimes\overline{p}\cdot m_j = \sum_{i,j} h_i g_j\otimes\overline{p_i}\cdot m_j,$$

which is (4). □

Theorem 4. Let Y be a smooth variety and K be a linear algebraic group action on Y. Suppose there are only finitely many K-orbits in Y. Then $\mathrm{Mod}_c(D_Y,K)\cong\mathrm{Mod}_{rh}(D_Y,K)$. Moreover, the simple objects in $\mathrm{Mod}_c(D_Y,K)$ is parametrized by $\Upsilon(Y,K)$, the set of pairs (O,L), where $O\subset Y$ is an irreducible K-orbit and L is a K-equivariant local system on O^{an}.

Proof. We use induction on the number of K-orbits of Y. Suppose Y is a homogeneous K-space. Then $Y\cong K/K'$ for some $K'\leq K$. Consider morphisms $\sigma : K\times Y\to Y$ the natural action, $p_2 : K\times Y\to Y$ the second projection, $l : K\to\operatorname{Spec}\mathbb{C}$, $\pi : K\to Y$ the quotient map, $j : \operatorname{Spec}(\mathbb{C})\to K$, $j(x) = K'$ and $i : K\to K\times Y$, $i(k) := (k^{-1}, kK')$. Then for any $\mathcal{M}\in\mathrm{Mod}_c(D_Y,K)$, we have

$$\begin{aligned}\pi^*\mathcal{M} &= (p_2\circ i)^*\mathcal{M} = i^*p_2^*\mathcal{M}\cong i^*\sigma^*\mathcal{M} = (\sigma\circ i)^*\mathcal{M}\\ &= (j\circ l)^*\mathcal{M} = l^*j^*\mathcal{M} = \mathcal{O}_X\otimes_{\mathbb{C}}(j^*\mathcal{M}).\end{aligned}$$

$j^*\mathcal{M}$ is a finite dimensional $\mathbb{C}$-vector space, so $\pi^*\mathcal{M}\in\mathrm{Mod}_{rh}(D_K)$. Since π is smooth, $\mathcal{M}\in\mathrm{Mod}_{rh}(D_Y)$.

Now consider the general case. Let O be a closed K-orbit of Y and $Y' = Y - O$. Suppose $i : O \hookrightarrow Y$ and $j :\hookrightarrow Y$. Then we have the distinguish triagle

$$\int_i i^\dagger \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \int_j j^\dagger \mathcal{M} \xrightarrow{+1} .$$

We have $i^\dagger \mathcal{M} \in D^b_c(D_O)$ and $j^\dagger \mathcal{M} \in D^b_c(D_{Y'})$. By induction hypothesis, $i^\dagger \mathcal{M} \in D^b_{rh}(D_O)$ and $j^\dagger \mathcal{M} \in D^b_{rh}(D_{Y'})$. We conclude that $\int_i i^\dagger \mathcal{M}, \int_j j^\dagger \mathcal{M} \in D^b_{rh}(D_Y)$ and hence $\mathcal{M} \in \mathrm{Mod}_{rh}(D_Y)$.

By Riemann-Hilbert correspondence, $\mathrm{Mod}_{rh}(D_Y, K) \cong \mathrm{Perv}(\mathbb{C}_Y, K)$, which is parametrized by $\Upsilon(Y, K)$. $\square$

In particular, B has only finite orbits in X. We conclude that simple objects in $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B)$ are parametrized by $\Upsilon(X, B)$.

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D-module Final Report II

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Proposition 1 ([HTT] Proposition 9.4.5, proved in [HC])**.** $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same W-orbit.

$$\langle \lambda, \alpha^\vee \rangle \leq 0 \text{ for all } \alpha \in \Delta^+. \tag{1}$$

$$\langle \lambda, \alpha^\vee \rangle < 0 \text{ for all } \alpha \in \Delta^+. \tag{2}$$

For $v \in -P^+$, Borel-Weil-Bott theorem says $\Gamma(X, \mathcal{L}(v)) = H^0(X, \mathcal{L}(v)) = L^-(v)$ and $p_v : \mathcal{O}_X \otimes_{\mathbb{C}} L^-(v) \to \mathcal{L}(v)$ is surjective. Since $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}(v), \mathcal{O}_X) = \mathcal{L}(-v)$ and $\mathrm{Hom}_{\mathbb{C}}(L^-(v), \mathbb{C}) = L^+(-v)$, we have $\mathcal{L}(-v) \hookrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} L^+(-v)$. Apply $\mathcal{L}(v) \otimes_{\mathcal{O}_X} (\cdot)$, we have $i_v : \mathcal{O}_X \hookrightarrow \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$. Since $\mathcal{L}(v)$ is a line bundle, $\ker(p_v)$ is a direct summand of $\mathcal{O}_X \otimes_{\mathbb{C}} L^-(v)$ as an $\mathcal{O}_X$-module locally. Therefore, $\mathrm{im}(i_v)$ is a direct summand of $\mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$ as an $\mathcal{O}_X$-module locally.

Let $\lambda \in L$ and $\mathcal{M}$ be a D_λ-module. Apply $\mathcal{M} \otimes_{\mathcal{O}_X} (\cdot)$, we get

$$\overline{p_v} : \mathcal{M} \otimes_{\mathbb{C}} L^-(v) \twoheadrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v),$$

$$\overline{i_v} : \mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v).$$

Proposition 2 ([HTT] Proposition 11.4.1)**.** (i) If λ satisfies (2), then $\ker(\overline{p_v})$ is a direct summand of $\mathcal{M} \otimes_{\mathbb{C}} L^-(v)$ as a sheaf of abelian groups.

(ii) If λ satisfies (1), then $\mathrm{im}(\overline{i_v})$ is a direct summand of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$ as a sheaf of abelian groups.

Proof. Let

$$L^-(v) = L^1 \supset L^2 \supset \cdots \supset L^r = 0$$

be a filtration of B-modules of $L^-(v)$ satisfying that L^i/L^{i+1} is the character μ_i of B, $\mu_1 = v$, and $\mu_i < \mu_j$ only if $i < j$. Then we obtain corresponding filtrations

$$\mathcal{O}_X \otimes_{\mathbb{C}} L^-(v) = \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots \supset \mathcal{V}^r = 0,$$

$$\mathcal{M} \otimes_{\mathbb{C}} L^-(v) = \overline{\mathcal{V}}^1 \supset \overline{\mathcal{V}}^2 \supset \cdots \supset \overline{\mathcal{V}}^r = 0,$$

and

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v) = \overline{\mathcal{W}}^r \supset \overline{\mathcal{W}}^{r-1} \supset \cdots \supset \overline{\mathcal{W}}^1 = 0.$$

The corresponding composition factors are

$$\overline{\mathcal{V}}^i/\overline{\mathcal{V}}^{i+1} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu_i), \qquad \overline{\mathcal{W}}^{i+1}/\overline{\mathcal{W}}^i \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v - \mu_i).$$

Since $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ is a $D_{\lambda+\mu}$-module, the action of $\mathfrak{z}$ on it is $\chi_{\lambda+\mu}$. So we have

$$\prod_{i=1}^{r-1}(z - \chi_{\lambda+\mu_i})(\mathcal{M} \otimes_{\mathbb{C}} L^-(v)) = 0$$

and

$$\prod_{i=1}^{r-1}(z - \chi_{\lambda+v-\mu_i})(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)) = 0.$$

Seen as sheaves of abelian groups, $\mathcal{M} \otimes_{\mathbb{C}} L^-(v)$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v)$ are equipped with locally finite $\mathfrak{z}$-actions and thus have decompositions into χ-primary parts:

$$\mathcal{M} \otimes_{\mathbb{C}} L^-(v) = \bigoplus_{\chi}(\mathcal{M} \otimes_{\mathbb{C}} L^-(v))_{\chi},$$

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v) = \bigoplus_{\chi}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v))_{\chi}.$$

The morphisms $\overline{p_v}$ and $\overline{i_v}$ are $\overline{\mathcal{V}}^1 \to \overline{\mathcal{V}}^1/\overline{\mathcal{V}}^2$ and $\overline{\mathcal{W}}^2 \to \overline{\mathcal{W}}^r$, respectively. It suffices to prove that

(i) If λ satisfies (2), then $\ker(\overline{p_v}) = (\mathcal{M} \otimes_{\mathbb{C}} L^-(v))_{\chi_{\lambda+v}}$. That is, $\chi_{\lambda+\mu_i} = \chi_{\lambda+v} \Leftrightarrow i = 1$.

(ii) If λ satisfies (1), then $\mathrm{im}(\overline{i_v}) = (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(v) \otimes_{\mathbb{C}} L^+(-v))_{\chi_\lambda}$. That is, $\chi_{\lambda+v-\mu_i} = \chi_\lambda \Leftrightarrow i = 1$.

Suppose λ satisfies (2). If $\chi_{\lambda+\mu_i} = \chi_{\lambda+v}$, then there is a $w \in W$ such that $w(\lambda+\mu_i) = \lambda + v$. That is, $(w(\lambda) - \lambda) + (w(\mu_i) - v) = 0$. Since $\langle \lambda, \alpha \rangle < 0$ for all $\alpha \in \Delta^+$, $w(\lambda) - \lambda \geq 0$ and the equality holds if and only if $w = \mathrm{id}$. Since $w(\mu_i)$ is a weight of $L^-(v)$, $w(\mu_i) \geq v$. So $w = \mathrm{id}$ and thus $\mu_i = v$. That is, $i = 1$.

Suppose λ satisfies (1). If $\chi_{\lambda+v-\mu_i} = \chi_\lambda$, then there is a $w \in W$ such that $w(\lambda) = \lambda + v - \mu_i$. That is, $(w(\lambda) - \lambda) + (\mu_i - v) = 0$. Since $\langle \lambda, \alpha \rangle \leq 0$ for all $\alpha \in \Delta^+$, $w(\lambda) \geq \lambda$. Also, $\mu_i \geq v$. So $\mu_i = v$ and thus $i = 1$. □

Theorem 1 ([HTT] Theorem 11.6.1)**.** Let Y be a smooth variety and K be a linear algebraic group action on Y. Suppose there are only finitely many K-orbits in Y. Then $\mathrm{Mod}_c(D_Y, K) \cong \mathrm{Mod}_{rh}(D_Y, K)$. Moreover, the simple objects in $\mathrm{Mod}_c(D_Y, K)$ is parametrized by $\Upsilon(Y, K)$, the set of pairs (O, L), where $O \subset Y$ is an irreducible K-orbit and L is a K-equivariant local system on O^{an}.

Proof. We use induction on the number of K-orbits of Y. Suppose Y is a homogeneous K-space. Then $Y \cong K/K'$ for some $K' \leq K$. Consider morphisms $\sigma : K \times Y \to Y$ the natural action, $p_2 : K \times Y \to Y$ the second projection, $l : K \to \mathrm{Spec}\,\mathbb{C}$, $\pi : K \to Y$ the quotient map, $j : \mathrm{Spec}(\mathbb{C}) \to K$, $j(x) = K'$ and $i : K \to K \times Y$, $i(k) := (k^{-1}, kK')$. Then for any $\mathcal{M} \in \mathrm{Mod}_c(D_Y, K)$, we have

$$\begin{aligned}\pi^*\mathcal{M} &= (p_2 \circ i)^*\mathcal{M} = i^* p_2^* \mathcal{M} \cong i^*\sigma^*\mathcal{M} = (\sigma \circ i)^*\mathcal{M} \\ &= (j \circ l)^*\mathcal{M} = l^* j^* \mathcal{M} = \mathcal{O}_X \otimes_{\mathbb{C}} (j^*\mathcal{M}).\end{aligned}$$

$j^*\mathcal{M}$ is a finite dimensional $\mathbb{C}$-vector space, so $\pi^*\mathcal{M} \in \mathrm{Mod}_{rh}(D_K)$. Since π is smooth, $\mathcal{M} \in \mathrm{Mod}_{rh}(D_Y)$.

Now consider the general case. Let O be a closed K-orbit of Y and $Y' = Y - O$. Suppose $i : O \hookrightarrow Y$ and $j : \hookrightarrow Y$. Then we have the distinguish triagle

$$\int_i i^\dagger \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \int_j j^\dagger \mathcal{M} \xrightarrow{+1} .$$

We have $i^\dagger\mathcal{M} \in D_c^b(D_O)$ and $j^\dagger\mathcal{M} \in D_c^b(D_{Y'})$. By induction hypothesis, $i^\dagger\mathcal{M} \in D_{rh}^b(D_O)$ and $j^\dagger\mathcal{M} \in D_{rh}^b(D_{Y'})$. We conclude that $\int_i i^\dagger\mathcal{M}, \int_j j^\dagger\mathcal{M} \in D_{rh}^b(D_Y)$ and hence $\mathcal{M} \in \mathrm{Mod}_{rh}(D_Y)$.

By Riemann-Hilbert correspondence, $\mathrm{Mod}_{rh}(D_Y, K) \cong \mathrm{Perv}(\mathbb{C}_Y, K)$, which is parametrized by $\Upsilon(Y, K)$. □

In particular, B has only finite orbits in X. We conclude that simple objects in $\mathrm{Mod}_f(\mathfrak{g}, \chi_{-\rho}, B)$ are parametrized by $\Upsilon(X, B)$.

0.1 Highest Weight Module

Definition 1. Let $\lambda \in \mathfrak{h}^*$ and M be a $\mathfrak{g}$-module. If there exists $0 \neq m \in M$ such that $m \in M_\lambda$, $\mathfrak{n}m = 0$, and $M = U(\mathfrak{g})m$, then M is called a highest weight module with highest weight λ. m is called a highest weight vector.

In this case, $M = U(\mathfrak{n}^-)m$ and $M = \bigoplus_{\mu \leq \lambda} M_\mu$. $M_\lambda = \mathbb{C}m$. Since M is generated by m, M is the quotiend $U(\mathfrak{g})/N$ as a $U(\mathfrak{g})$-module. The relation contains at least $\mathfrak{n}$ and $h - \lambda(h)$ for all $h \in \mathfrak{h}$.

Definition 2. The Verma module is defined as

$$M(\lambda) := U(\mathfrak{g})/(U(\mathfrak{g})\mathfrak{n} + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1)).$$

$M(\lambda)$ is the unique maximal highest weight module. If M be a highest weight module, there is a unique surjective homomorphism $f : M(\lambda) \to M$ such that $f(\bar{1}) = m$.

Lemma 1 ([HTT] Lemma 12.1.3)**.** $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module. In particular, we compute that

$$\begin{aligned} \mathrm{ch}(M(\lambda)) =& \sum_\mu \dim(M(\lambda)_\mu) e^\mu = \sum_{\beta \leq 0} \dim(U(\mathfrak{n}^-)_\beta) e^{\lambda+\beta} \\ =& e^\lambda \prod_{\beta \in \Delta^+} (1 + e^{-\beta} + e^{-2\beta} + \cdots) \\ =& \frac{e^\lambda}{\prod_{\beta \in \Delta^+} (1 - e^{-\beta})}. \end{aligned}$$

Proof. Let $I = (U(\mathfrak{g})\mathfrak{n} + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1))$. We want to prove that $U(\mathfrak{g}) = U(\mathfrak{n}^-) \oplus I$. By PBW theorem, we have a canonical isomorphism

$$U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \cong U(\mathfrak{g}).$$

So we have

$$\begin{aligned} \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1)) =& \sum_{h \in \mathfrak{h}} U(\mathfrak{n}^-) U(\mathfrak{h}) U(\mathfrak{n})(h - \lambda(h)1)) \\ =& \sum_{h \in \mathfrak{h}} U(\mathfrak{n}^-) U(\mathfrak{h})(\mathbb{C} + U(\mathfrak{n})\mathfrak{n})(h - \lambda(h)1)) \\ \subset& U(\mathfrak{n}^-) \left(\sum_{h \in \mathfrak{h}} U(\mathfrak{h})(h - \lambda(h)1)) \right) + U(\mathfrak{g})\mathfrak{n}. \end{aligned}$$

So $I = U(\mathfrak{n}^-)\left(\sum_{h\in\mathfrak{h}} U(\mathfrak{h})(h-\lambda(h)1))\right) + U(\mathfrak{g})\mathfrak{n}$. Finally we have the isomorphism

$$\begin{aligned}
U(\mathfrak{g}) =& U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}) \\
=& U(\mathfrak{n}^-)U(\mathfrak{h})(\mathbb{C}\oplus U(\mathfrak{n})\mathfrak{n}) \\
=& U(\mathfrak{n}^-)U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n} \\
=& U(\mathfrak{n}^-)\left(\mathbb{C}\oplus \sum_{h\in\mathfrak{h}} U(\mathfrak{h})(h-\lambda(h)1))\right) \oplus U(\mathfrak{g})\mathfrak{n} \\
=& U(\mathfrak{n}^-) \oplus I.
\end{aligned}$$

□

Lemma 2 ([HTT] Lemma 12.1.4)**.** There is a unique maximal proper $U(\mathfrak{g})$-submodule $N \subset M(\lambda)$.

Proof. Any proper $U(\mathfrak{g})$-submodule of $M(\lambda)$ is a weight module whose weights $< \lambda$. So the sum of them is also a proper $U(\mathfrak{g})$-submodule. □

Define $L(\lambda) = M(\lambda)/N$. $L(\lambda)$ is the minimal highest weight module.

Problem 1. Compute $\mathrm{ch}(L(\lambda))$.

Example 1. If $\lambda \in \Delta^+$, then $L(\lambda) = L^+(\lambda)$. Weyl's character formula says

$$\begin{aligned}
\mathrm{ch}(L(\lambda)) =& \frac{\sum_{w\in W}(-1)^{l(w)} w^{w(\lambda+\rho)-\rho}}{\prod_{\beta\in\Delta^+}(1-e^{-\beta})} \\
=& \sum_{w\in W}(-1)^{l(w)}\,\mathrm{ch}(M(w(\lambda+\rho)-\rho)).
\end{aligned}$$

Lemma 3 ([HTT] Lemma 12.1.6)**.** For $z \in \mathfrak{z}$, $zm = \chi_{\lambda+\rho}(z)m$.

Proof. We decompose z into $u+v$ where $z \in U(\mathfrak{h})$ and $v \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$. Then $zm = um = \lambda(u)m = \chi_{\lambda+\rho}(z)m$. □

Proposition 3 ([HTT] Proposition 12.1.7)**.** Let M be a highest weight module with highest weight λ. The M has a decomposition series with finite length and each composition factor of it has the form $L(\mu)$ where $\mu \le \lambda$ and $\mu+\rho \in W(\lambda+\rho)$.

Proof. If M is simple then we are done. If M is not simple, then we take a nonzero proper submodule $N \subset M$. Let μ be a maximal weight of N and $0 \neq n \in N_\mu$, then $U(\mathfrak{g})m \subset N$ is a highest weight module with highest weight μ. $\chi_{\mu+\rho}(z)n = zn = \chi_{\lambda+\rho}(z)n$ for all $z \in \mathfrak{z}$. So $\chi_{\mu+\rho} = \chi_{\lambda+\rho}$. We have $\mu < \lambda$ and $\mu + \rho \in W(\lambda+\rho)$. Replace M by N and repeat the process. We can repeat only finitely many times and obtain a simple $U(\mathfrak{g})$-module N_1, which is a highest weight module with highest weight μ_1. $N_1 \cong L(\mu_1)$. Replace M by M/N_1 and repeat the process. We obtain a sequence

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots,$$

the composition factors of which have the form $L(\mu)$ for some $\mu \leq \lambda$ and $\mu + \rho \in W(\lambda+\rho)$. Since $|W(\lambda+\rho)| < \infty$ and $L(\mu)$ can occur no more than $\dim(M_\mu)$ times, the sequence is finite. □

Fix a equivlence class $\Lambda = W(\lambda+\rho) - \rho$. Let $a_{\mu\lambda}$ the the multiplicity of $L(\mu)$ appearing in the deconposition series of $M(\lambda)$. We have $a_{\mu\lambda} \neq 0$ only if $\mu \sim \lambda$ and $\mu \leq \lambda$. $a_{\lambda\lambda} = 1$. Let $(b_{\mu\lambda})$ be the inverse matrix of $(a_{\mu\lambda})$. Then $b_{\mu\lambda} \in \mathbb{Z}$ and

$$\mathrm{ch}(M(\lambda)) = \sum_{\mu\in\Lambda} a_{\mu\lambda}\,\mathrm{ch}(L(\mu)).$$

$$\mathrm{ch}(L(\lambda)) = \sum_{\mu\in\Lambda} b_{\mu\lambda}\,\mathrm{ch}(M(\mu)).$$

It suffices to compute $b_{\mu\lambda}$.

0.2 Kazhdan-Lusztig Conjecture

The problem is answered when $\Lambda = W(-\rho) - \rho$. In this case, $\Lambda \subset P$. We are considering objects $M(-w(\rho)-\rho), L(-w\rho-\rho) \in \mathrm{Mod}_f(\mathfrak{g}, \chi_\rho, B) = \mathrm{Mod}_{rh}(D_X, B)$. Every object in $Mod_f(\mathfrak{g}, \chi_\rho, B)$ has a composition series of finite length, which is proved similarly as in the proof above. We consider the Grothdieck group $K(\mathrm{Mod}_f(\mathfrak{g}, \chi_\rho, B))$. We have

$$[L(-w\rho-\rho)] = \sum_{y\in W} b_{yw}[M(-y\rho-\rho)].$$

$$[M(-w\rho-\rho)] = \sum_{y\in W} a_{yw}[L(-y\rho-\rho)].$$

We want to compute b_{yw}.

Definition 3. The Hecke algebra $H(W)$ is the $\mathbb{Z}[q^1, q^{-1}]$ algebra which is freely generated by $\{T_w \mid w \in W\}$ as a $\mathbb{Z}[q^1, q^{-1}]$-module with multiplicative relations

$$\begin{aligned} T_y T_w &= T_{yw}, && \text{if } l(yw) = l(y) + l(w). \\ (T_s + 1)(T_s - q) &= 0, && \text{if } s \in W. \end{aligned}$$

Proposition 4 ([HTT] Proposition 12.2.3)**.** There exists a unique family $\{P_{y,w}(q)\}$ of polynomials in $\mathbb{Z}[q]$ satisfying the following conditions:

$$P_{y,w}(q) = 0 \text{ if } y \not\leq w,$$

$$P_{w,w}(q) = 1,$$

$$\deg(P_{y,w}(q)) \leq \frac{l(w) - l(y) - 1}{2} \text{ if } y < w,$$

$$\sum_{y \leq w} P_{y,w}(q) T_y = q^{l(w)} \sum_{y \leq w} P_{y,w}(q^{-1}) T_{y^{-1}}^{-1}.$$

Conjecture 1 (Kazhdan-Lusztig)**.**

$$b_{y,w} = (-1)^{l(w)-l(y)} P_{y,w}(1).$$

Definition 4. For each $w \in W$ we define

$$X_w = BwB/B.$$

Here w is seen as an element in $W = N_G(H)/H$. The Schubert variety is defined as $\overline{X_w}$.

Proposition 5 ([HTT] Theorem 9.9.4, 9.9.5)**.** X is the disjoint union of $\{X_w \mid w \in W\}$. Each X_w is isomorphic to $\mathbb{C}^{l(w)}$. $\overline{X_w} = \coprod_{y \leq w} X_w$.

We denote by $IC(\mathbb{C}_{X_w})$ the intersection complex on $\overline{X_w}$ and set

$$\mathbb{C}^{\pi}_{X_w} = IC(\mathbb{C}_{X_w})[-\dim(X_w)].$$

We'll show that the Kazhdan-Lusztig conjecture is reduced the theorem below.

Theorem 2 (Kazhdan-Lusztig,[HTT] Theorem 12.2.5)**.** For any $y, w \in W$, we have

$$\sum_i \dim(H^i(\mathbb{C}^\pi_{X_w})_{yB})q^{i/2} = P_{y,w}(q).$$

In particular, We have $H^i(\mathbb{C}^\pi_{X_w})_{yB} = 0$ for all odd i and

$$\sum_j (-1)^j \dim(H^j(\mathbb{C}^\pi_{X_w})_{yB}) = P_{y,w}(1).$$

Let $\mathcal{M}_w = D_X \otimes_{U(\mathfrak{g})} M(-w(\rho) - \rho)$, $\mathcal{L}_w = D_X \otimes_{U(\mathfrak{g})} L(-w(\rho) - \rho)$, and

$$\mathcal{N}_w = \int_{i_w} \mathcal{O}_{X_w} = (i_w)_*(D_{X \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}).$$

$\mathcal{N}_w \in \mathrm{Mod}_c(D_X, B) = \mathrm{Mod}_{rh}(D_X, B)$.

Lemma 4 ([HTT] Lemma 12.3.1)**.** Let $w \in W$. Then

(i) $\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \mathrm{ch}(M(-w(\rho)-\rho))$. In particular, $[\mathcal{M}_w] = [\mathcal{N}_w]$ in $K(\mathrm{Mod}_{rh}(D_X, B))$.

(ii) The only D_X submodule of $\mathcal{N}_w$ whose support is contained in $\overline{X_w} - X_w$ is 0.

Proof. Define two subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}_1 = \bigoplus_{\alpha \in \Delta^+ \cap w(\Delta^+)} \mathfrak{g}_{-\alpha}, \qquad \mathfrak{n}_2 = \bigoplus_{\alpha \in \Delta^+ \cap -w(\Delta^+)} \mathfrak{g}_{\alpha}.$$

Let the corresponding unipotent subgroup of G be N_1 and N_2 respectively. Define a morphism $\varphi : N_1 \times N_2 \to X$ by

$$\varphi(n_1, n_2) = n_1 n_2 w B/B.$$

Then φ is an open embedding. $\varphi(\{e\} \times N_2) = X_w$. Let $V = \mathrm{im}(\varphi)$, we have the commutative diagram

$$\begin{array}{ccccc} N_1 \times N_2 & \xrightarrow{\ \varphi\ } & V & \hookrightarrow & X \\ \uparrow & & \uparrow & \nearrow_{i_w} & \\ N_2 = \{e\} \times N_2 & \longrightarrow & X_w & & \end{array}$$

So

$$\begin{aligned} \Gamma(X, \mathcal{N}_w) =& \Gamma\left(X, (i_w)_*\left(D_{X \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}\right)\right) \\ =& \Gamma(X_w, D_{X \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}) \\ =& \Gamma(X_w, D_{V \leftarrow X_w} \otimes_{D_{X_w}} \mathcal{O}_{X_w}) \\ \cong& \Gamma(N_2, D_{N_1 \times N_2 \leftarrow N_2} \otimes_{D_{N_2}} \mathcal{O}_{N_2}). \end{aligned}$$

$$D_{N_1\times N_2\leftarrow N_2} \cong \left(D_{N_1\times\{e\}} \otimes_{\mathcal{O}_{N_1\times\{e\}}} \mathbb{C}\right) \otimes_{\mathbb{C}} \left(\Omega^{\otimes -1}_{N_1\times\{e\}} \otimes_{\mathcal{O}_{N_1\times\{e\}}} \mathbb{C}\right) \otimes_{\mathbb{C}} \otimes_{\mathbb{C}} \Gamma(N_2, \mathcal{N}_2).$$

First, $D_{N_1} = U(\mathfrak{n}_1) \otimes_{\mathbb{C}} \mathcal{O}_X$, so $D_{N_1 \times e} \otimes_{\mathcal{O}_{N_1\times\{e\}}} \mathbb{C} \cong U(\mathfrak{n}_1)$. Second,

$$\Omega^{\otimes -1}_{N_1\times\{e\}} \otimes_{\mathcal{O}_{N_1\times\{e\}}} \mathbb{C} \cong \wedge^{\dim(\mathfrak{n}_1)}(\mathfrak{n}_1 \otimes_{\mathbb{C}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathbb{C} = \wedge^{\dim(\mathfrak{n}_1)}\mathfrak{n}_1.$$

Finally, the exponential map gives the isomorphism $\mathfrak{n}_2 \cong N_2$. Therefore, $\Gamma(N_2, \mathcal{N}_2) \cong \Gamma(\mathfrak{n}_2, \mathcal{O}_{\mathfrak{n}_2}) = S(\mathfrak{n}_2^*)$.

$$\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \mathrm{ch}(U(\mathfrak{n}_1))\,\mathrm{ch}(\wedge^{\dim(\mathfrak{n}_1)}\mathfrak{n}_1)\,\mathrm{ch}(S(\mathfrak{n}_2^*)).$$

We compute that

$$\mathrm{ch}(U(\mathfrak{n}_1)) = \prod_{\alpha\in\Delta^+\cap w(\Delta^+)} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = \prod_{\alpha\in\Delta^+\cap w(\Delta^+)} \frac{1}{1-e^{-\alpha}},$$

$$\mathrm{ch}(\wedge^{\dim(\mathfrak{n}_1)}) = e^{\sum_{\alpha\in\Delta^+\cap w(\Delta^+)} -\alpha} = e^{-w(\rho)-\rho},$$

and

$$\mathrm{ch}(S(\mathfrak{n}_2^*)) = \prod_{\alpha\in\Delta^+\cap -w(\Delta^+)} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = \prod_{\alpha\in\Delta^+\cap -w(\Delta^+)} \frac{1}{1-e^{-\alpha}}.$$

So

$$\mathrm{ch}(\Gamma(X, \mathcal{N}_w)) = \frac{e^{-w(\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} = \mathrm{ch}(M(-w(\rho)-\rho)).$$

Set $Z = X - V$ and $j : V \to X$ be the open embedding, we have a distinguished traingle

$$R\Gamma_Z(\mathcal{N}_w) \longrightarrow \mathcal{N}_w \longrightarrow j_*(\mathcal{N}_w) \xrightarrow{+1} .$$

By definition, $\mathcal{N}_w \to j_*(\mathcal{N}_w)$ is an isomorphism, so $R\Gamma_Z(\mathcal{N}_w) = 0$. So $\Gamma_Z(\mathcal{N}_w)$. Hence the only D_X submodule of $\mathcal{N}_w$ whose support is contained in Z is 0. Since $\overline{X_w} - X_w \subset Z$, the assertion follows. $\square$

Let $\mathcal{L}(X_w, \mathcal{O}_{X_w})$ be the minimal extension of the D_{X_w}-module X_w. $\mathcal{L}(X_w, \mathcal{O}_{X_w}) \in \mathrm{Mod}_{rh}(D_X, B)$.

Proposition 6 ([HTT] Lemma 12.3.2)**.** Let $w \in W$. Then we have

(i)

$$\mathcal{L}_w = \mathcal{L}(X_w, \mathcal{O}_{X_w}).$$

(ii)

$$\mathcal{M}_w = \mathbb{D}(\mathcal{N}_w).$$

Proof. Since X_w^{an} is simply connected, simple objects in $\mathrm{Mod}_{rh}(D_X, B)$ is given by $\{\mathcal{L}(X_w, \mathcal{O}_{X_w}) \mid y \in W\}$. On the other hand, simple objects in $\mathrm{Mod}_f(\mathfrak{g}, B, \chi_{-\rho})$ is given by $\{\mathcal{L}(-w(\rho) - \rho)\}$. So for each $w \in W$, there is a $y \in W$ such that $\mathcal{L}_w = \mathcal{L}(X_y, \mathcal{O}_{X_y})$. For this y, $\mathcal{L}(X_y, \mathcal{O}_{X_y})$ is a composition factor of $\mathcal{M}_w$ and hence one of $\mathcal{N}_w$. Since $\mathcal{N}_w$ is supported on $\overline{X_w} = \coprod_{w' \leq w} X_{w'}$, $y \leq w$. The induction on Bruhat order gives the equality.

Since $\{\mathcal{L}(X_w, \mathcal{O}_{X_w}) \mid y \in W\}$ are self dual, for all $\mathcal{M} \in \mathrm{Mod}_{rh}(D_X, B)$, the composition factors of $\mathcal{M}$ and those of $\mathbb{D}(\mathcal{M})$ coincide. In particular, we get

$$\mathrm{ch}(\mathbb{D}(\mathcal{N}_w)) = \mathrm{ch}(\mathcal{N}_w) = \mathrm{ch}(M(-w(\rho) - \rho)).$$

$U(\mathfrak{g})\Gamma(X, \mathbb{D}(\mathcal{N}_w))_{-w(\rho)-\rho}$ is a highest weight module with highest weight $-w(\rho)-\rho)$. Thus we have an exact sequence

$$M(-w(\rho) - \rho) \to \Gamma(X, \mathbb{D}(\mathcal{N}_w)) \to N \to 0.$$

Tensoring D_X over $U(\mathfrak{g})$ and we get

$$\mathcal{M}_w \to \mathbb{D}(\mathcal{N}_w) \to \mathcal{N} \to 0.$$

Taking dual and we get

$$0 \to \mathbb{D}(\mathcal{N}) \to \mathcal{N}_w \to \mathbb{D}(\mathcal{M}_w)$$

$\mathcal{L}_w$ isn't in the set of composition factors of $\mathbb{D}(\mathcal{N})$, so the support of $\mathbb{D}(\mathcal{N})$ is in $\overline{X_w} - X_w$. We get $\mathbb{D}(\mathcal{N}) = 0$, $\mathcal{N} = 0$, and $N = 0$. So we have a surjective homomorphism $M(-w(\rho) - \rho) \to \Gamma(X, \mathbb{D}(\mathcal{N}_w))$. The injectivity follows from that $\mathrm{ch}(\mathcal{N}_w) = \mathrm{ch}(M(-w(\rho) - \rho))$. □

Corollary 1 ([HTT] Corollary 12.3.3)**.** The Riemann-Hilbert correspondence gives

$$DR_X(\mathcal{M}_w) = \mathbb{C}_{X_w}[\dim(X_w)]$$

and

$$DR_X(\mathcal{L}_w) = \mathbb{C}_{X_w}^{\pi}[\dim(X_w)].$$

To prove the conjecture, it suffices to prove that

$$[\mathcal{L}_w] = \sum_{y\leq w}(-1)^{l(w)-l(y)}P_{y,w}(1)[\mathcal{M}_w]$$

in $K(\mathrm{Mod}_{rh}(D_X, B)$. We define a $\mathbb{Z}$-module homomorphism $\varphi : K(\mathrm{Mod}_{rh}(D_X, B) \to \mathbb{Z}[W]$ given by

$$\varphi([\mathcal{M}]) = \sum_{y\in W}\left(\sum_i (-1)^i \dim(H^i(DR_X(\mathcal{M}))_{yB})\right) y.$$

Form the corollary above, we have $\varphi([\mathcal{M}_w]) = (-1)^{l(w)}m$, so φ is an isomorphism. Assume the Kazhdan-Lusztig theorem and we get

$$\begin{aligned}
\varphi([\mathcal{L}_w]) =& \sum_{y\in W}\left(\sum_i (-1)^i \dim(H^i(DR_X(\mathcal{L}_w))_{yB})\right) y \\
=& \sum_{y\in W}\left(\sum_i (-1)^i \dim(H^i(\mathbb{C}^{\pi}_{X_w}[\dim(X_w)]))_{yB})\right) y \\
=& (-1)^{l(w)}\sum_{y\in W} P_{y,w}(1)y \\
=& (-1)^{l(w)-l(y)}P_{y,w}(1)\varphi([\mathcal{M}_w]).
\end{aligned}$$

0.3 Sketch of the Proof of Kazhdan-Lusztig Theorem

Let ΔG be the diagonal group of $G \times G$ and act diagonally on $X \times X$. ΔG- orbits of $X \times X$ has a natural bijection to $\{X_w\}$ given by

$$Z_w := \Delta G(eB, wB) \leftrightarrow X_w$$

Let $p_k : X\times X \to X$, $i_k : X \to X\times X (k = 1, 2)$ be given by $p_1(a, b) = a$, $p_2(a, b) = b$, $i_1(b) = (eB, b)$, $i_2(a) = (a, eB)$.

Proposition 7 ([HTT] Proposition 13.1.2)**.** $i_k(k = 1, 2)$ induce equivlences of categories:

$$i_k^* : \mathrm{Mod}_c(D_{X\times X}, \Delta G) \cong \mathrm{Mod}_c(D_X, B).$$

Since $X \times X$ has only finite ΔG-orbits, $\mathrm{Mod}_c(D_{X\times X}, \Delta G) = \mathrm{Mod}_{rh}(D_{X\times X}, \Delta G)$. For $w \in W$, consider the embedding $j_w : Z_w \to X \times X$ and set

$$\widetilde{\mathcal{N}}_w = \int_{j_w} \mathcal{O}_{Z_w}, \qquad \widetilde{\mathcal{M}}_w = \mathbb{D}(\widetilde{\mathcal{N}}_w), \qquad \widetilde{\mathcal{L}}_w = \mathcal{L}(Z_w, \mathcal{O}_{Z_w}).$$

They are in $\mathrm{Mod}_c(D_{X\times X}, \Delta G)$. Moreover,

$$i_1^*(\widetilde{\mathcal{N}}_w) = \mathcal{N}_w, \qquad i_1^*(\widetilde{\mathcal{M}}_w) = \mathcal{M}_w, , \qquad i_1^*(\widetilde{\mathcal{L}}_w) = \mathcal{L}_w,$$

$$i_2^*(\widetilde{\mathcal{N}}_w) = \mathcal{N}_{w^{-1}}, \qquad i_2^*(\widetilde{\mathcal{M}}_w) = \mathcal{M}_{w^{-1}}, \qquad i_2^*(\widetilde{\mathcal{L}}_w) = \mathcal{L}_{w^{-1}}.$$

Proposition 8 ([HTT] Proposition 13.1.5)**.** Let $p_{13} : X \times X \times X \to X \times X$ and $r : X \times X \times X \to X \times X \times X \times X$ be given by $p_{13}(a, b, c) = (a, c)$ and $r(a, b, c) = (a, b, b, c)$. Then $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ has a ring structure given by

$$[\widetilde{\mathcal{M}}] \cdot [\widetilde{\mathcal{N}}] = \sum_k (-1)^k H^k \left(\int_{p_{13}} r^*(\widetilde{\mathcal{M}} \boxtimes \widetilde{\mathcal{N}}) \right)$$

and $K(\mathrm{Mod}_c(D_{X\times X}, \Delta G))$ is isomorphic to $\mathbb{Z}[W]$ by the correspondence $\widetilde{\mathcal{M}}_w \leftrightarrow (-1)^{l(w)} w$.

We should consider the categories of Hodge modules ([HTT] 8.3) to relate the objects and the Hecke algebra $H(W)$. We need the categories $SH(n)$, $SH(n)^p$, and $MHM(Y)$. An object in $MHM(Y)$ is a tuple $(\mathcal{M}, F, K, W)$, where $\mathcal{M} \in \mathrm{Mod}_{rh}(D_Y)$, F is a good filtration of $\mathcal{M}$, $K \in \mathrm{Perv}(Y)/\mathbb{Q}$ such that $DR_Y(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} K$, and W is an increasing filtration of the tuple $(\mathcal{M}, F, K)$.

Consider $R = K(MHM(pt)) = K(SHM^p)$ ([HTT] (m12), p.224). $R = \bigoplus_{n\in\mathbb{Z}} R_n$ where $R_n = K(SH(n)^p)$. The unit is $\mathbb{Q}^H$. The morphism $q^n \mapsto [\mathbb{Q}^H[-n]]$ gives R a $\mathbb{Z}[q, q^{-1}]$-algebra structure $q \in R_2$.

Consider the category $K(MHM(X \times X, \Delta G))$. It has a ring structure

$$[\mathcal{V}_1] \cdot [\mathcal{V}_1] = (-1)^{\dim(X)} \sum_j (-1)^j [H^j(p_{13!} r^{\bigstar}(\mathcal{V}_1 \boxtimes \mathcal{V}_2))]$$

([HTT] Equation 13.2.7.)

The tensor product

$$MHM(pt) \times MHM(X \times X, \Delta G) \to MHM(X \times X, \Delta G)$$

gives $K(MHM(X\times X,\Delta G))$ a R-algebra structure.

For $w\in W$ we set

$$\begin{aligned}\overline{\mathcal{N}}_w^H =&(j_w)_\bigstar(\mathbb{Q}^H_{Z_w}[\dim(Z_w)]), \quad \overline{\mathcal{M}}_w^H = (j_w)_!(\mathbb{Q}^H_{Z_w}[\dim(Z_w)]),\\ \overline{\mathcal{L}}_w^H =&(j_w)_{!\bigstar}(\mathbb{Q}^H_{Z_w}[\dim(Z_w)]) = IC^H_{\overline{\mathbb{Z}_w}} \in MHM(X\times X,\Delta G).\end{aligned}$$

The underlying D-modules are $\overline{\mathcal{N}}_w, \overline{\mathcal{M}}_w, \overline{\mathcal{L}}_w$, respectively. In[HTT] they defined a R-algebra isomorphism

$$F: K(MHM(X\times X,\Delta G))\to R\otimes_{\mathbb{Z}[g,g^{-1}]} H(W)$$

by $F([\overline{\mathcal{M}}_w^H]) = (-1)^{l(w)}T_w$ ([HTT] Theorem 13.2.8) and for each $w\in W$ a R-module homomorphism $F_w: K(MHM(X\times X,\Delta G))\to R$ given by $F(m) = \sum_{w\in W}(-1)^{l(w)}F_w(m)T_w$ ([HTT] Equation 13.2.26). The morphisms $\{F_w\}$ satisfying that

$$\sum_k(-1)^k[H^k(j_w^\bigstar(\mathcal{V}))] = F_w([V])[Q^H_{Z_w}[\dim(Z_w)]]$$

([HTT] Equation 13.2.25). Next, they defined

$$C'_w = (-1)^{l(w)}F([\overline{\mathcal{L}}_w^H]) = \sum_{y\le w}P'_{y,w}T_y \qquad (P'_{y,w}\in R)$$

([HTT] Equation 13.2.34). Put $m = [[\overline{\mathcal{L}}_w^H]]$ in[HTT] Equation 13.2.26 and get

$$(-1)^{l(w)-l(y)}\sum_k(-1)^k[H^k(j_w^\bigstar(\overline{\mathcal{L}}_w^H))] = P'_{y,w}[Q^H_{Z_w}[\dim(Z_w)]]$$

([HTT] Equation 13.2.38). Comparing the weight and relations[HTT] Equation 13.2.35, 13.2.36, 13.2.37 of $\{P'_{y,w}\}$, they proved $P'_{y,w} = P_{y,w}(q)$ and thus $C'_w = C_w := \sum_{y\le w}P_{y,w}(q)T_y$.

Proposition 9 ([HTT] Proposition 13.2.9)**.** If $y\le w$, $H^k(j_w^\bigstar(\overline{\mathcal{L}}_w^H))$ has pure weight $\dim(Z_w)+k$.

With this proposition, they can write $H^k(j_w^\bigstar(\overline{\mathcal{L}}_w^H)) = N_k\otimes\mathbb{Q}^H_{Z_y}[\dim(Z_y)]$ where $Z_y\in SH(k+l(w)-l(y))^p$.[HTT] Equation 13.2.38 gives

$$\sum_k(-1)^{l(w)+l(y)-k}[N_k] = P_{y,w} := c_{y,w,j}q^j.$$

Since $q \in R_2$, we have $[N_k] = 0$ if $k + l(w) - l(y)$ is odd and $[N_k] = c_{y,w,j}q^j$ if $k + l(w) - l(y) = 2j$. Thus $\dim(H^k(j_w^{\bigstar}(\overline{\mathcal{L}}_w^H))) = c_{y,w,j}$ if $k + l(w) - l(y) = 2j$ and 0 if $k + l(w) - l(y) = 2j$ is odd. Kazhdan-Lusztig theorem follows.

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