AN ILLUSTRATIVE SURVEY FOR THE PAPER

"RIEMANN-HILBERT CORRESPONDENCE FOR HOLONOMIC D-MODULES"

BY A. D'AGNOLO AND M. KASHIWARA

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Student ID: B02201025 Name: 李自然

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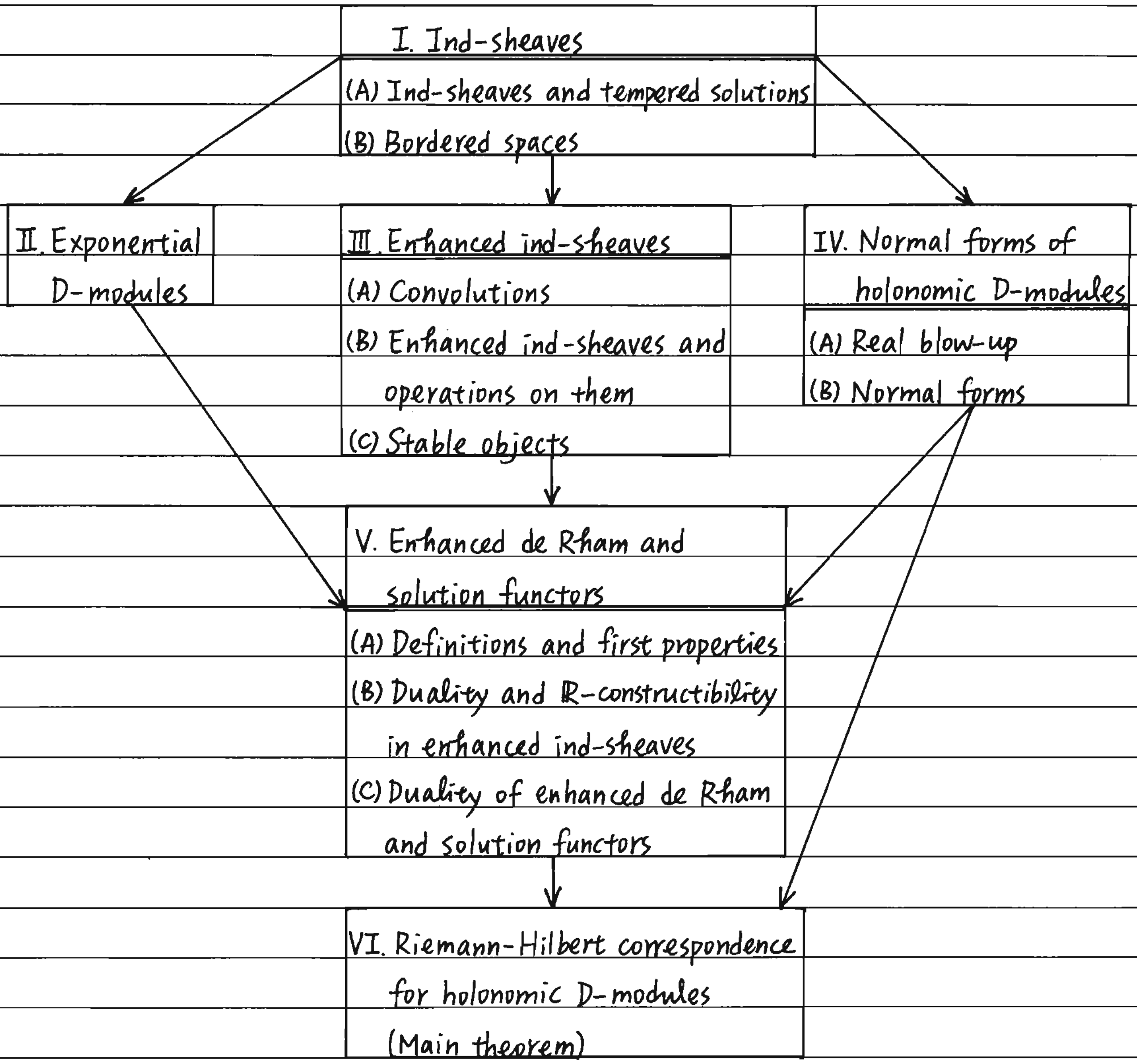


§I. IND-SHEAVES

(A) IND-SHEAVES AND TEMPERED SOLUTIONS

Definition (Indization). [KS01, §1.1]

$\mathcal{C}$: category $\rightsquigarrow$ $\mathcal{C} \hookrightarrow \mathcal{C}^\wedge := \mathrm{Functor}(\mathcal{C}, (\mathrm{Set}))$ by Yoneda's lemma
$X \longmapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$

Inductive limits: $\varinjlim$ in $\mathcal{C}$, "$\varinjlim$" in $\mathcal{C}^\wedge$

The category $\mathrm{Ind}(\mathcal{C})$: objects: those $A \in \mathcal{C}^\wedge$ such that $A \simeq \left[\underset{I}{\text{"}\varinjlim\text{"}}\alpha : X \in \mathcal{C} \mapsto \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i))\right]$ $\alpha: I \to \mathcal{C}$, I filtrant small
("indization of $\mathcal{C}$") morphisms: induced from $\mathcal{C}^\wedge$

$\rightsquigarrow \mathcal{C} \overset{\text{full}}{\hookrightarrow} \mathrm{Ind}(\mathcal{C}) \overset{\text{full}}{\subset} \mathcal{C}^\wedge$.

Definition (Ind-sheaves). [KS01, §4.1] k = field, X = good topological space (Hausdorff, locally compact, countable at infinity, having finite flabby dimension)

$\mathrm{Mod}^c(k_X) := \{ F \in \mathrm{Mod}(k_X) \mid \mathrm{Supp}\, F \text{ is compact} \}$

$\rightsquigarrow \mathrm{I}k_X := \mathrm{Ind}(\mathrm{Mod}^c(k_X))$ the category of ind-sheaves of k_X-modules.

Remark: $[k : \underset{\text{open}}{\mathcal{U} \subset} X \mapsto \mathrm{Mod}(k_{\mathcal{U}})]$ and $[\mathrm{I}k : \underset{\text{open}}{\mathcal{U} \subset} X \mapsto \mathrm{I}k_{\mathcal{U}}]$ are both proper stacks and stacks.
(cf. [KS01, Thm.3.2.17 & 3.3.14])

Definition/Proposition (Operations on ind-sheaves). [KS01, §4.2 & §4.3] $f: X \to Y$ continuous.

1. Internal $\otimes$ and internal hom:

$\otimes : \mathrm{I}(k_X^{op}) \times \mathrm{I}k_X \longrightarrow \mathrm{I}k_X$, $(\text{"}\varinjlim_i\text{"} F_i) \otimes (\text{"}\varinjlim_j\text{"} G_j) := \text{"}\varinjlim_{i,j}\text{"} F_i \otimes G_j$

$\mathcal{I}hom : (\mathrm{I}k_X)^{op} \times \mathrm{I}k_X \to \mathrm{I}k_X$, $\mathcal{I}hom(\text{"}\varinjlim_i\text{"} F_i, \text{"}\varinjlim_j\text{"} G_j) := \varprojlim_i \text{"}\varinjlim_j\text{"} \mathcal{H}om(F_i, G_j)$.

There is a $\otimes$-$\mathcal{I}hom$ adjunction: $\mathrm{Hom}_{\mathrm{I}k_X}(K \otimes F, G) \simeq \mathrm{Hom}_{\mathrm{I}k_X}(F, \mathcal{I}hom(K, G))$.

2. External operations:

$f^{-1} : \mathrm{I}k_Y \to \mathrm{I}k_X$, $f^{-1}(\text{"}\varinjlim_i\text{"} G_i) := \text{"}\varinjlim_i\text{"} f^{-1} G_i$

$f_* : \mathrm{I}k_X \to \mathrm{I}k_Y$, $f_*(\text{"}\varinjlim_i\text{"} F_i) := \varprojlim_{K \text{cpt.}} \text{"}\varinjlim_i\text{"} f_*(F_{iK})$.

$f_{!!} : \mathrm{I}k_X \to \mathrm{I}k_Y$, $f_{!!}(\text{"}\varinjlim_i\text{"} F_i) := \text{"}\varinjlim_i\text{"} f_! F_i$ (proper direct image).

Note that (f^{-1}, f_*) is an adjoint pair (w.r.t. $\mathrm{Hom}_{\mathrm{I}k}$).

Definition/Proposition (Operations on derived categories of ind-sheaves). [KS01, §5.1~5.3] $f: X \to Y$ continuous.

$\mathcal{P}(k_X) := \{ \text{"}\bigoplus_j\text{"} G_j \mid G_j \in \mathrm{Mod}(k_X) \}$; $\mathcal{I}_q(k_X) := \{ F \in \mathrm{I}k_X \mid F \text{ is quasi-injective} \}$.
i.e. $\mathrm{Hom}_{\mathrm{I}k_X}(\cdot, F)|_{\mathrm{Mod}^c(k_X)}$ is exact, or equivalently, $F \simeq \text{"}\varinjlim_i\text{"} F_i$, each $F_i \in \mathrm{Mod}(k_X)$ injective (cf. [KS01, Prop.4.2.19])

1. $\mathcal{P}(k_X)^{op} \times \mathcal{I}_q(k_X)$ is $\mathrm{Hom}_{\mathrm{I}k_X}$-, $\mathcal{I}hom$- and $\mathcal{H}om$-injective

$\rightsquigarrow$ get $R\mathrm{Hom}_{\mathrm{I}k_X}$ ($R\mathcal{H}om$): $D^-(\mathrm{I}k_X)^{op} \times D^+(\mathrm{I}k_X) \to D^+(k)$ ($D^+(k_X)$)

$R\mathcal{I}hom : D^-(\mathrm{I}k_X)^{op} \times D^+(\mathrm{I}k_X) \to D^+(\mathrm{I}k_X)$.

2. $\mathcal{I}_q(k_X)$ is f_*- and $f_{!!}$-injective $\rightsquigarrow$ get $Rf_*, Rf_{!!} : D^+(\mathrm{I}k_X) \to D^+(\mathrm{I}k_Y)$. Note: (f^{-1}, Rf_*) adjoint pair.

3. $f^! : D^+(\mathrm{I}k_Y) \to D^+(\mathrm{I}k_X)$ is the right adjoint of $Rf_{!!}$.

Remark. The functors introduced above may not be commutative ; cf. [KS01, §5.3].

Proposition I.1. [KS01, Thm.5.2.7 & Prop.5.3.8] $f : X \to Y$ continuous.

1. For $F \in D^+(\mathrm{I}k_X)$ and $G \in D^+(\mathrm{I}k_Y)$, $Rf_{!!}(f^{-1}G \otimes F) \simeq G \otimes Rf_{!!}F$.
2. For $K \in D^-(\mathrm{I}k_Y)$ and $G \in D^+(\mathrm{I}k_Y)$, $f^! R\mathcal{I}hom(K, G) \simeq R\mathcal{I}hom(f^{-1}K, f^!G)$.

Proposition I.2 (Base change). [KS01, Thm.5.2.9 & 5.3.10 & 5.3.11]

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

1. $Rf'_{!!} \circ g'^{-1} \simeq g^{-1} \circ Rf_{!!} : D^+(\mathrm{I}k_X) \to D^+(\mathrm{I}k_{Y'})$.
2. $Rf'_* \circ g'^! \simeq g^! \circ Rf_* : D^+(\mathrm{I}k_X) \to D^+(\mathrm{I}k_{Y'})$.
3. $Rf'_{!!} \circ g'^! \simeq g^! \circ Rf_{!!} : D^+(\mathrm{I}k_X) \to D^+(\mathrm{I}k_{Y'})$.

• • •

M : real analytic manifold;
~~X : complex analytic manifold.~~

Definition (Tempered functions). [DK15, Def.5.1.1, 5.2.1]

1. Db_M : the sheaf of Schwartz's distributions on M

   Db_M^t : the subanalytic sheaf of tempered distributions on M, defined by

   $$V \text{ subanalytic} \subset M \ \mapsto Db_M^t(V) := \mathrm{im}(Db_M(M) \to Db_M(V)) \simeq Db_M(M)/\Gamma_{M \setminus V}(M, Db_M).$$

2. $\mathcal{O}_X^t := R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, Db^t_{X_{\mathbb{R}}}) \underset{\uparrow}{=} [\underset{(0)}{Db^t_{X_{\mathbb{R}}}} \xrightarrow{\bar\partial} Db^{t,(0,1)}_{X_{\mathbb{R}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} Db^{t,(0,\dim_{\mathbb{C}} X)}_{X_{\mathbb{R}}}] \in D^b(\mathrm{I}\mathcal{D}_X)$

   Dolbeault resolution & Spencer resolution

   (cf. $\mathcal{O}_X \simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, Db_{X_{\mathbb{R}}}) \simeq [Db_{X_{\mathbb{R}}} \xrightarrow{\bar\partial} Db^{(0,1)}_{X_{\mathbb{R}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} Db^{(0,\dim_{\mathbb{C}} X)}_{X_{\mathbb{R}}}]$ by Dolbeault resolution)

   $\Omega_X^t := \Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X^t \in D^b(\mathrm{I}\mathcal{D}_X^{op})$.

Definition (de Rham & solution functors). [DK15, Not.5.2.2]

1. Classical : $DR_X : D^b(\mathcal{D}_X) \to D^b(\mathbb{C}_X)$, $M \mapsto \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} M$.

   $Sol_X : D^b(\mathcal{D}_X)^{op} \to D^b(\mathbb{C}_X)$, $M \mapsto R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)$.

2. Tempered : $DR_X^t : D^b(\mathcal{D}_X) \to D^b(\mathrm{I}\mathbb{C}_X)$, $M \mapsto \Omega_X^t \overset{L}{\underset{\mathcal{D}_X}{\otimes}} M$.

   $Sol_X^t : D^b(\mathcal{D}_X)^{op} \to D^b(\mathrm{I}\mathbb{C}_X)$, $M \mapsto R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X^t)$.

$d_X = \dim_{\mathbb{C}} X$, $d_Y = \dim_{\mathbb{C}} Y$

Theorem I.3. [DK15, Thm.5.2.3]; cf. also [KS01, Thm.7.4.1, 7.4.6, 7.4.12]. $f : X \to Y$ complex analytic map.

1. $f^! \mathcal{O}_Y^t [d_Y] \simeq \mathcal{D}_{Y \leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X^t [d_X]$ in $D^b(\mathrm{I}f^{-1}\mathcal{D}_Y)$.
2. For any $N \in D^b(\mathcal{D}_Y)$, $DR_X^t(Df^*N)[d_X] \simeq f^! DR_Y^t(N)[d_Y]$ in $D^b(\mathrm{I}\mathbb{C}_X)$.
3. For $M \in D^b_{good}(\mathcal{D}_X)$ such that $\mathrm{Supp}(M)$ is proper over Y, $DR_Y^t(Df_*M) \simeq Rf_{!!} DR_X^t(M)$ in $D^b(\mathrm{I}\mathbb{C}_Y)$.

   (A $\mathcal{D}_X$-module M is good if it is coherent and quasi-good; "quasi-good" means that $\forall\, \mathcal{U}$ open $\subset\subset X$, $M|_{\mathcal{U}}$ is the sum of a filtrant family of coherent $\mathcal{O}_X|_{\mathcal{U}}$-submodules.)

4. For $L \in D^b_{rh}(\mathcal{D}_X)$, $\mathcal{O}_X^t \overset{L}{\underset{\mathcal{O}_X}{\otimes}} L \simeq R\mathcal{I}hom(Sol_X(L), \mathcal{O}_X^t)$ in $D^b(\mathrm{I}\mathcal{D}_X)$;

   in particular, for $Y \subset X$ closed hypersurface, $\mathcal{O}_X^t \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{O}_X(*Y) \simeq R\mathcal{I}hom(\mathbb{C}_{X \setminus Y}, \mathcal{O}_X^t)$.

(B) BORDERED SPACES

Definition (Quotient categories). [DK15, §3.1]

$\left.\begin{array}{l} D : \text{triangulated category} \\ \cup \\ N : \text{full triangulated subcategory} \end{array}\right\} \Rightarrow$ the quotient category $D/N := D_\Sigma$ (localization) where Σ is the multiplicative system in D defined by $\Sigma := \{(X \xrightarrow{u} Y) \in \mathrm{Hom}_D(X,Y) \mid \exists \text{ distinguished } \Delta : X \xrightarrow{u} Y \to \overset{N}{\textcircled{Z}} \xrightarrow{+1}\}$.

Basic properties. [KS06, §10.] Let $Q : D \to D/N \equiv D_\Sigma$ be the localization functor.

1. For $X \in N$, $Q(X) = 0$ in D/N.
2. For $D \xrightarrow{F} D'$ triangulated functor such that $F(X) \simeq 0$ for all $X \in N$,

$$\begin{tikzcd} D \arrow[r, "F"] \arrow[d, "Q"'] & D' \\ D/N \arrow[ur, dashed, "G", "\exists!\,\tilde{F}"'] & \end{tikzcd}$$

Definition (Bordered spaces). [DK15, Def. 3.2.1]

The category of bordered spaces is defined as follows:

objects: $(M, \check{M})$ with $M \hookrightarrow \check{M}$ open embedding of good topological spaces

morphisms: $f : (M, \check{M}) \to (N, \check{N})$ which is a continuous map $f : M \to N$ such that $\check{M} \times \check{N} \supset \overline{\mathrm{graph}(f)} \longrightarrow \check{M}$ (canonical projection) is proper.

We regard (good topological spaces) $\underset{\text{full subcategory}}{\subset}$ (bordered spaces) by $M \mapsto (M, M)$.

Definition (Derived categories of ind-sheaves on bordered spaces). [DK15, Def. 3.2.6]

$$D^b(\mathrm{I}k_{(M,\check{M})}) := D^b(\mathrm{I}k_{\check{M}}) / D^b(\mathrm{I}k_{\check{M}\setminus M}).$$

Definition (Operations on bordered spaces). [DK15, §3.3] $f : (M, \check{M}) \to (N, \check{N})$.

The functors $\otimes$ and $R\mathcal{H}om$ on $D^b(\mathrm{I}k_{\check{M}})$ induce

$$\begin{cases} \otimes : D^b(\mathrm{I}k_{(M,\check{M})}) \times D^b(\mathrm{I}k_{(M,\check{M})}) \longrightarrow D^b(\mathrm{I}k_{(M,\check{M})}) \\ R\mathcal{H}om : D^b(\mathrm{I}k_{(M,\check{M})})^{op} \times D^b(\mathrm{I}k_{(M,\check{M})}) \to D^b(\mathrm{I}k_{(M,\check{M})}). \end{cases}$$

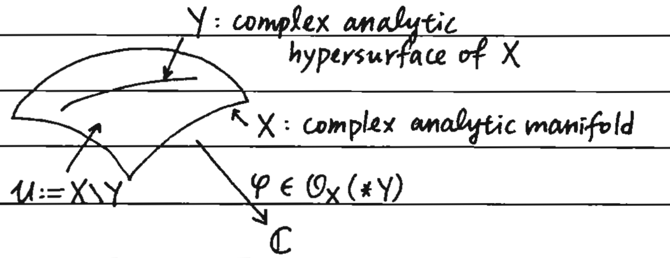
We also define the following functors: $\left(\begin{tikzcd}[column sep=small] & \check{M} \times \check{N} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \check{M} & & \check{N} \end{tikzcd} \text{ projections} \right)$

$$\left.\begin{array}{l} Rf_{!!} : D^b(\mathrm{I}k_{(M,\check{M})}) \to D^b(\mathrm{I}k_{(N,\check{N})}), \quad Rf_{!!}(F) := Rq_{2!!}(k_{\mathrm{graph}(f)} \otimes q_1^{-1}F). \\ Rf_* : D^b(\mathrm{I}k_{(M,\check{M})}) \to D^b(\mathrm{I}k_{(N,\check{N})}), \quad Rf_*(F) := Rq_{2*}R\mathcal{H}om(k_{\mathrm{graph}(f)}, q_1^!F). \\ f^{-1} : D^b(\mathrm{I}k_{(N,\check{N})}) \to D^b(\mathrm{I}k_{(M,\check{M})}), \quad f^{-1}(G) := Rq_{1!!}(k_{\mathrm{graph}(f)} \otimes q_2^{-1}G). \\ f^! : D^b(\mathrm{I}k_{(N,\check{N})}) \to D^b(\mathrm{I}k_{(M,\check{M})}), \quad f^!(G) := Rq_{1*}R\mathcal{H}om(k_{\mathrm{graph}(f)}, q_2^!G). \end{array}\right\} \begin{array}{l} \text{generalizations} \\ \text{of their counterpart} \\ \text{on } D^b(\mathrm{I}k_{\check{M}}). \end{array}$$

Remark. A typical example of bordered spaces which we shall encounter often later is the "extended real line" $\mathbb{R}_\infty := (\mathbb{R}, \mathbb{R} \sqcup \{\pm\infty\})$. In this case $D^b(\mathrm{I}k_{\mathbb{R}_\infty}) = \dfrac{D^b(\mathrm{I}k_{\mathbb{R} \sqcup \{\pm\infty\}})}{D^b(\mathrm{I}k_{\{\pm\infty\}})}$, so it seems that this is devised to deal with the infinity points. But in my opinion one may just regard "$D^b(\mathrm{I}k_{\mathbb{R}_\infty}) \approx D^b(\mathrm{I}k_{\mathbb{R} \sqcup \{\pm\infty\}})$" first to get a whole picture.

§II. EXPONENTIAL D-MODULES

Definition (Exponential D-modules). [DK15, Def.6.1.1]



The exponential D-module
$\mathcal{E}^{\varphi}_{\mathcal{U}|X}:=\mathcal{D}_X e^{\varphi}(*Y)\in\mathrm{Mod}(\mathcal{D}_X)$
where $\mathcal{D}_X e^{\varphi}=\mathcal{D}_X/\{P\in\mathcal{D}_X\mid Pe^{\varphi}|_{\mathcal{U}}=0\}$.

Basic properties.

1. $\mathcal{E}^{\varphi}_{\mathcal{U}|X}$ is a holonomic $\mathcal{D}_X$-module which satisfies $\mathcal{E}^{\varphi}_{\mathcal{U}|X}\simeq\mathcal{E}^{\varphi}_{\mathcal{U}|X}(*Y)$ and $\mathrm{sing.supp}(\mathcal{E}^{\varphi}_{\mathcal{U}|X})\subset Y$.
2. We have a canonical isomorphism of $\mathcal{O}_X$-modules: $\mathcal{O}_X(*Y)\xrightarrow[\sim]{\cdot e^{\varphi}}\mathcal{E}^{\varphi}_{\mathcal{U}|X}$.

Notation: [DK15, Not.6.2.1]

- $\mathbb{C}_{\mathrm{Re}\varphi<*}:=\text{"}\varinjlim_{c\to+\infty}\text{"}\,\mathbb{C}_{\mathrm{Re}\varphi<c}\in\mathrm{I}\mathbb{C}_X$, where $\{\mathrm{Re}\varphi<c\}=\{x\in\mathcal{U}\mid\mathrm{Re}\varphi(x)<c\}\subset X$.
- $\mathrm{E}^{\varphi}_{\mathcal{U}|X}:=R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}},\mathbb{C}_{\mathrm{Re}\varphi<*})\in D^b(\mathrm{I}\mathbb{C}_X)$.

Proposition II.1. [DK15, Prop.6.2.2]

Y closed in X, other setting as in the above definition of exponential D-modules
$\Rightarrow DR^{t}_{X}(\mathcal{E}^{-\varphi}_{\mathcal{U}|X})\simeq\mathrm{E}^{\varphi}_{\mathcal{U}|X}[d_X]$ in $D^b(\mathrm{I}\mathbb{C}_X)$.

(Proof.) The proof consists of 5 steps.

Step 1: $DR^{t}_{X}(\mathcal{E}^{-\varphi}_{\mathcal{U}|X})\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}},DR^{t}_{X}(\mathcal{E}^{-\varphi}_{\mathcal{U}|X}))$. [DK15, Lem.6.2.3]

(Proof.) $DR^{t}_{X}(\mathcal{E}^{-\varphi}_{\mathcal{U}|X})=\Omega^{t}_{X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{E}^{-\varphi}_{\mathcal{U}|X}=\Omega^{t}_{X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}(\mathcal{E}^{-\varphi}_{\mathcal{U}|X}\overset{D}{\otimes}\mathcal{O}_X(*Y))\simeq(\mathcal{O}_X(*Y)\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\Omega^{t}_{X})\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{E}^{-\varphi}_{\mathcal{U}|X}$

$\underset{\substack{\uparrow\\ \text{Thm.I3\#4}}}{\simeq}R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}},\Omega^{t}_{X})\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{E}^{-\varphi}_{\mathcal{U}|X}\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}},\Omega^{t}_{X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{E}^{-\varphi}_{\mathcal{U}|X})$. #

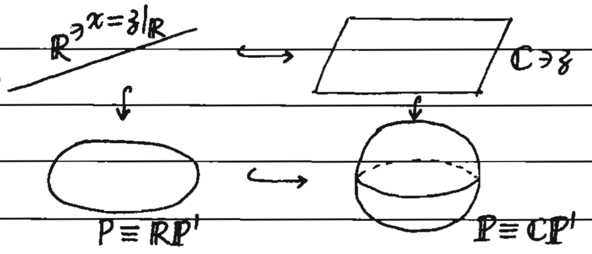
Definition. $i:\underset{\text{real manifold}}{M}\longrightarrow X$ a complexification of M

$\rightsquigarrow DR^{t}_{M}(\underset{\substack{\cap\\ D^b(\mathcal{D}_X)}}{\mathcal{M}}):=Db^{t,\vee}_{M}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}\simeq i^{!}DR^{t}_{X}(\mathcal{M})[d_X]\in D^b(\mathrm{I}\mathbb{C}_M)$.

$\hookrightarrow Db^{t,\vee}_{M}:=Db^{t}_{M}\otimes or_M\otimes_{i^{-1}\mathcal{O}_X}i^{-1}\Omega_X\simeq i^{!}\Omega^{t}_{X}[d_X]$

Step 2:

[DK15, Lem.6.2.4]



In $D^b(\mathrm{I}\mathbb{C}_P)$,
$DR^{t}_{P}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})\simeq\mathcal{I}hom(\mathbb{C}_{\mathbb{R}},\mathbb{C}_{x<*})[1]$.

(Proof of Prop. II.1, cont'd)

(Proof of Step 2.)

1. $\mathrm{DR}^{t}_{\mathrm{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) = \mathcal{D}b^{t,\vee}_{\mathrm{P}} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}} \mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \simeq \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathcal{D}b^{t,\vee}_{\mathrm{P}}) \overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}} \mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}$ (cf. Step 1.)

$\simeq (\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}} \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathcal{D}b^{t}_{\mathrm{P}})$ (Recall: $(\cdot)^{r} = \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} (\cdot) : D^b(\mathcal{D}_X) \xrightarrow{\sim} D^b(\mathcal{D}_X^{op})$)

$\simeq [\underset{(-1)}{\mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathcal{D}b^{t}_{\mathrm{P}})} \xrightarrow{\partial_x - 1} \underset{(0)}{\mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathcal{D}b^{t}_{\mathrm{P}})}] =: \mathcal{S}$.

$(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})^{r} = [\underset{(-1)}{\mathcal{D}_{\mathbb{P}}(*Y)} \xrightarrow{\cdot(\partial_z + 1)} \underset{(0)}{\mathcal{D}_{\mathbb{P}}(*Y)}]^{\vee} = [\underset{(-1)}{\mathcal{D}_{\mathbb{P}}(*Y)} \xrightarrow{\partial_z - 1} \underset{(0)}{\mathcal{D}_{\mathbb{P}}(*Y)}]$

$\mathcal{D}b^{t}_{\mathrm{P}}$ is tempered $\Rightarrow$ "$(*Y)$" is absorbed in $\mathcal{S}$

2. $H^{-1}(\mathcal{S}) \simeq \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<*})$ and $H^{-1}(\mathcal{S}) = 0$

(thus $\mathrm{DR}^{t}_{\mathrm{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq \mathcal{S} \simeq [\underset{(-1)}{\mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<*})} \to \underset{(0)}{0}] = \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<*})[1]$):

$\underline{H^{-1}(\mathcal{S})}$: $\mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathcal{D}b^{t}_{\mathrm{P}}) = \varprojlim_{\mathcal{U} \subset\subset X} \mathcal{H}om(\mathbb{C}_{\mathcal{U}}, \mathcal{D}b^{t}_{\mathrm{P}})$

$\rightsquigarrow H^{-1}(\mathcal{S}) = \varprojlim_{\mathcal{U} \subset\subset X} \langle e^{x} \rangle \cap \mathcal{H}om(\mathbb{C}_{\mathcal{U}}, \mathcal{D}b^{t}_{\mathrm{P}}) = \varprojlim_{\mathcal{U} \subset\subset X} \text{“}\varinjlim_{c \to +\infty}\text{”} \mathcal{H}om(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<c})$

subanalytic $\ker(\partial_x - 1) = \langle e^{x} \rangle$ $e^{x} \in \mathcal{D}b^{t}_{\mathrm{P}}(\mathcal{U} \cap \mathbb{R}) \Leftrightarrow \mathcal{U} \cap \mathbb{R}$ is bounded above

(i.e. $e^{x}|_{\mathcal{U} \cap \mathbb{R}}$ tempered)

$= \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<*})$.

$\underline{H^{0}(\mathcal{S})}$:

$$\begin{tikzcd}
\mathcal{D}b^{t}_{\mathrm{P}}(\mathbb{R}) \arrow[r, two heads, "\partial_x - 1"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{D}b^{t}_{\mathrm{P}}(\mathbb{R}) \arrow[d, two heads] \\
\mathcal{D}b^{t}_{\mathrm{P}}(\mathcal{U} \cap \mathbb{R}) \arrow[r, "\partial_x - 1"'] & \mathcal{D}b^{t}_{\mathrm{P}}(\mathcal{U} \cap \mathbb{R})
\end{tikzcd}$$

$\leftarrow$ by definition of $\mathcal{D}b^{t}_{\mathrm{P}}$ ($\mathcal{U}$: subanalytic)

$\Rightarrow$ thus surjective $\therefore H^{0}(\mathcal{S}) = \mathrm{coker}\left(\varprojlim_{\mathcal{U} \subset\subset \mathbb{R}} \mathcal{D}b^{t}_{\mathrm{P}}(\mathcal{U} \cap \mathbb{R}) \xrightarrow{\partial_x - 1} \varprojlim_{\mathcal{U} \subset\subset \mathbb{R}} \mathcal{D}b^{t}_{\mathrm{P}}(\mathcal{U} \cap \mathbb{R})\right)$

$= 0$.

3. By 1. and 2., $\mathrm{DR}^{t}_{\mathrm{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq \mathcal{S} \simeq [\overset{(-1)}{H^{-1}(\mathcal{S})} \to \overset{(0)}{0}] \simeq H^{-1}(\mathcal{S})[1] \simeq \mathcal{I}hom(\mathbb{C}_{\mathbb{R}}, \mathbb{C}_{x<*})[1]$. #

Step 3: $\mathbb{P} \supset \mathbb{C} \ni z$: coordinate $(\mathbb{C} = \mathbb{P} \setminus \{\infty\}) \Rightarrow$ In $D^b(\mathrm{I}\mathbb{C}_{\mathbb{P}})$, $\mathrm{DR}^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq E^{z}_{\mathbb{C}|\mathbb{P}}[1]$. [DK15, Lem. 6.2.5]

(Proof.) Consider the morphisms

$$\begin{tikzcd}
 & (\mathbb{R}^2, P^2) \arrow[dl, "k"'] \arrow[r, "f", "\sim"'] & (\mathbb{C}_{\mathbb{R}}, \mathbb{P}_{\mathbb{R}}) \arrow[dr, "j"] & \\
P^2 & (x,y) \arrow[r, mapsto] & z = x + \sqrt{-1}y & \mathbb{P}_{\mathbb{R}}.
\end{tikzcd}$$

"realizations" (k, j canonical)

1. $\mathrm{DR}^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq R\mathcal{I}hom(\mathbb{C}_{\mathbb{C}}, \mathrm{DR}^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}))$ (Step 1)

$\simeq Rj_{*}j^{-1}\mathrm{DR}^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})$ ([DK15, Lem. 3.3.7])

$\simeq Rj_{*}j^{-1}\mathrm{DR}^{t}_{\mathbb{P}_{\mathbb{R}}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}})[-\dim_{\mathbb{C}}\mathbb{P}] \simeq Rj_{*}j^{-1}((\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P} \times \overline{\mathbb{P}}}}{\otimes}} \mathcal{D}b^{t}_{\mathbb{P}_{\mathbb{R}}})[-1]$.

2. $j^{-1}((\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P} \times \overline{\mathbb{P}}}}{\otimes}} \mathcal{D}b^{t}_{\mathbb{P}_{\mathbb{R}}}) \simeq [j^{-1}\mathcal{D}b^{t}_{\mathbb{P}_{\mathbb{R}}} \xrightarrow{(-\partial_z + 1, \overline{\partial}_z)^{t}} (j^{-1}\mathcal{D}b^{t}_{\mathbb{P}_{\mathbb{R}}})^{2} \xrightarrow{(-\overline{\partial}_z, -\partial_z + 1)} \underset{(0)}{j^{-1}\mathcal{D}b^{t}_{\mathbb{P}_{\mathbb{R}}}}]$

column vector matrix left multiplication

as illustrated below:

$\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} = [\mathcal{D}_{\mathbb{P}} \xrightarrow{\cdot(\partial_z + 1)} \overset{(0)}{\mathcal{D}_{\mathbb{P}}}](*Y)$

$\mathcal{O}_{\overline{\mathbb{P}}} = [\mathcal{D}_{\overline{\mathbb{P}}} \underset{\mathcal{O}_{\overline{\mathbb{P}}}}{\otimes} \Theta_{\overline{\mathbb{P}}} \longrightarrow \overset{(0)}{\mathcal{D}_{\overline{\mathbb{P}}}}] = [\mathcal{D}_{\overline{\mathbb{P}}} \xrightarrow{\cdot\overline{\partial}_z} \overset{(0)}{\mathcal{D}_{\overline{\mathbb{P}}}}]$

Spencer resolution

$\Rightarrow$ (double complex associates simple complex)

$\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}} = [\mathcal{D}_{\mathbb{P} \times \overline{\mathbb{P}}} \xrightarrow{\cdot(\partial_z + 1, -\overline{\partial}_z)^{t}} \mathcal{D}^{2}_{\mathbb{P} \times \overline{\mathbb{P}}} \xrightarrow{\cdot(\overline{\partial}_z, \partial_z + 1)} \overset{(0)}{\mathcal{D}_{\mathbb{P} \times \overline{\mathbb{P}}}}](*Y)$

$(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}})^{r} = [\mathcal{D} \xrightarrow{(-\partial_z + 1, \overline{\partial}_z)^{t}} \mathcal{D}^{2} \xrightarrow{(-\overline{\partial}_z, -\partial_z + 1)} \overset{(0)}{\mathcal{D}}](*Y)$

will be absorbed by $\mathcal{D}b^{t}$

(Proof of Prop II.1, cont'd.)

(Proof of Step3, cont'd.)

3. Applying f^{-1} on the complex in 2. and noticing that $f^{-1}j^{-1}Db^{t}_{\mathbb{P}_{\mathbb{R}}} \simeq k^{-1}Db^{t}_{P^2}$ ([DK15, Prop. 5.4.3])

we get: $\partial_z = \frac{1}{2}(\partial_x - \sqrt{-1}\partial_y)$, $\overline{\partial_z} = \frac{1}{2}(\partial_x + \sqrt{-1}\partial_y)$, and some identifications

$$k^{-1}((\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\overline{\mathbb{P}}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}\times\overline{\mathbb{P}}}}{\otimes}} Db^{t}_{\mathbb{P}_{\mathbb{R}}}) \simeq [\,k^{-1}Db^{t}_{P^2} \xrightarrow{(-\partial_x+1,\ +\partial_y-\sqrt{-1})^{t}} (k^{-1}Db^{t}_{P^2})^{2} \xrightarrow{(\partial_y-\sqrt{-1},\ -\partial_x-1)} \overset{(0)}{k^{-1}Db^{t}_{P^2}}\,]$$

$$\simeq k^{-1}((\mathcal{E}^{-u}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{E}^{-\sqrt{-1}v}_{\mathbb{C}|\mathbb{P}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}^2}}{\otimes}} Db^{t}_{P^2}) \quad (\mathbb{C}^2 \ni (u,v) \mapsto (u,v)|_{\mathbb{R}} = (x,y) \in \mathbb{R}^2)$$

$$\simeq k^{-1}((\mathcal{E}^{-u}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\mathbb{P}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}^2}}{\otimes}} Db^{t}_{P^2}).$$

$$k^{-1}Db^{t}_{P^2} \xrightarrow[\sim]{e^{-\sqrt{-1}y}} k^{-1}Db^{t}_{P^2}$$

$\partial_y \leftrightarrow \partial_y - \sqrt{-1}$

4. By 1. and 3.,

$$Db^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq Rj_{*}Rf_{*}k^{-1}((\mathcal{E}^{-u}_{\mathbb{C}|\mathbb{P}} \overset{D}{\boxtimes} \mathcal{O}_{\mathbb{P}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}^2}}{\otimes}} Db^{t}_{P^2})[-1]$$

$$\simeq Rj_{*}Rf_{*}k^{-1}((\mathcal{E}^{-u}_{\mathbb{C}|\mathbb{P}})^{r} \overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}} p_1^{!}Db^{t}_{P})[-1] \quad ([DK15, Lem. 5.3.2])$$

$$\simeq Rj_{*}Rf_{*}k^{-1}p_1^{-1}DR^{t}_{P}(\mathcal{E}^{-u}_{\mathbb{C}|\mathbb{P}}) \simeq Rj_{*}Rf_{*}k^{-1}p_1^{-1}\mathbb{C}_{x<*}[1] \quad (\text{Step2})$$

$$\simeq Rj_{*}j^{-1}\mathbb{C}_{\mathrm{Re}(z)<*}[1] \simeq R\mathcal{H}om(\mathbb{C}_{\mathbb{C}}, \mathbb{C}_{\mathrm{Re}(z)<*})[1] = E^{z}_{\mathbb{C}|\mathbb{P}}[1]. \qquad \#$$

([DK15, Lem. 3.3.7])

<u>Step4</u>: $\mathbb{C}^2 \ni (u,v)$: coordinates $\Rightarrow$ In $D^{b}(I\mathbb{C}_{\mathbb{C}^2})$, $DR^{t}_{\mathbb{C}^2}(\mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}) \simeq E^{u/v}_{v\neq 0|\mathbb{C}^2}[2]$. [DK15, Lem. 6.2.6]

(Proof.) Consider the blow-up

$$\mathbb{C}^2 \times \mathbb{P} \supset \widetilde{\mathbb{C}}^2 := Bl_0\,\mathbb{C}^2 \equiv \{((u,v),(z_0:z_1)) \in \mathbb{C}^2 \times \mathbb{P} \mid uz_0 = vz_1\}.$$

$$\begin{tikzcd} & \widetilde{\mathbb{C}}^2 \arrow[dl, "p"'] \arrow[d, "q"] & \\ (u,v) \in \mathbb{C}^2 & \mathbb{P} \ni (z_0:z_1) \leftrightarrow z = \frac{z_1}{z_0} = \frac{u}{v} & \end{tikzcd}$$

(p, q: projections)

$$\mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2} \simeq \mathcal{O}_{\mathbb{C}^2}(*\{v=0\}) \overset{D}{\otimes} Dp_{*}Dq^{*}\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}$$

$$\Rightarrow DR^{t}_{\mathbb{C}^2}(\mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}) \simeq DR^{t}_{\mathbb{C}^2}(\mathcal{O}_{\mathbb{C}^2}(*\{v=0\}) \overset{D}{\otimes} Dp_{*}Dq^{*}\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})$$

$$\simeq R\mathcal{H}om(\mathbb{C}_{v\neq 0}, DR^{t}_{\mathbb{C}^2}(Dp_{*}Dq^{*}\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}})). \quad (\text{Thm. I.3, \#4})$$

Also, $DR^{t}_{\mathbb{C}^2}(Dp_{*}Dq^{*}\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}) \simeq Rp_{*}q^{!}(DR^{t}_{\mathbb{P}}(\mathcal{E}^{-z}_{\mathbb{C}|\mathbb{P}}))[-1]$ (Thm I.3, #2 & #3)

$$\simeq Rp_{*}q^{!}R\mathcal{H}om(\mathbb{C}_{\mathbb{C}}, \mathbb{C}_{\mathrm{Re}z<*}) \quad (\text{Step3})$$

$$\simeq Rp_{*}R\mathcal{H}om(q^{-1}\mathbb{C}_{\mathbb{C}}, q^{!}\mathbb{C}_{\mathrm{Re}z<*}) \quad (\text{Prop. I.1, \#2})$$

$$\simeq Rp_{*}R\mathcal{H}om(q^{-1}\mathbb{C}_{\mathbb{C}}, q^{-1}\mathbb{C}_{\mathrm{Re}z<*})[2] \quad (q \text{ is smooth with fiber } \mathbb{C}). \ (?)$$

Therefore,

$$DR^{t}_{\mathbb{C}^2}(\mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}) \simeq R\mathcal{H}om(\mathbb{C}_{v\neq 0}, Rp_{*}R\mathcal{H}om(q^{-1}\mathbb{C}_{\mathbb{C}}, q^{-1}\mathbb{C}_{\mathrm{Re}z<*}))[2]$$

$$\simeq Rp_{*}R\mathcal{H}om(p^{-1}\mathbb{C}_{v\neq 0}, q^{-1}\mathbb{C}_{\mathrm{Re}z<*})[2] \quad (p^{-1}(v\neq 0) \subset q^{-1}(\mathbb{C}))$$

$$\simeq Rp_{*}R\mathcal{H}om(p^{-1}\mathbb{C}_{v\neq 0}, p^{-1}\mathbb{C}_{\mathrm{Re}(u/v)<*})[2] \quad (q^{-1}(\mathrm{Re}z<c) \cap p^{-1}(v\neq 0) = p^{-1}(\mathrm{Re}(u/v)<c), c\in\mathbb{R})$$

$$\simeq R\mathcal{H}om(\mathbb{C}_{v\neq 0}, \mathbb{C}_{\mathrm{Re}(u/v)<*})[2] \quad (p|_{v\neq 0} \text{ is an isomorphism})$$

$$= E^{u/v}_{v\neq 0|\mathbb{C}^2}[2].$$

\#

(Proof of Prop. II.1, cont'd.)

Step 5: End of proof of Prop. II.1.

Write $\varphi = a/b$ where $a, b \in \mathcal{O}_X$ such that $Y = b^{-1}(0)$.

Consider $f = (a,b) : X \to \mathbb{C}^2 \ni (u,v)$: coordinates. Then

$$f^{-1}(v=0) = b^{-1}(0) = Y \Rightarrow \begin{cases} f^{-1}(v \neq 0) = \mathcal{U} \\ \mathcal{E}^{-\varphi}_{\mathcal{U}|X} \simeq Df^* \mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}. \end{cases}$$

Also, $DR^t_X(Df^* \mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}) \simeq f^!(DR^t_{\mathbb{C}^2}(\mathcal{E}^{-u/v}_{v\neq 0|\mathbb{C}^2}))[2-d_X]$ (Thm. I.3 #2)

$\simeq f^! R\mathcal{H}om(\mathbb{C}_{v\neq 0}, \mathbb{C}_{\mathrm{Re}(u/v)<*})[4-d_X]$ (Step 4).

Thus we deduce:

$$\begin{aligned} DR^t_X(\mathcal{E}^{-\varphi}_{\mathcal{U}|X}) &\simeq f^! R\mathcal{H}om(\mathbb{C}_{v\neq 0}, \mathbb{C}_{\mathrm{Re}(u/v)<*})[4-d_X] \\ &\simeq R\mathcal{H}om(\mathbb{C}_{\mathcal{U}}, f^! \mathbb{C}_{\mathrm{Re}(u/v)<*})[4-d_X] \quad (\text{Prop. I.1 \#2 \& } f^{-1}(v\neq 0) = \mathcal{U}) \\ &\simeq R\mathcal{H}om(\mathbb{C}_{\mathcal{U}}, f^{-1} \mathbb{C}_{\mathrm{Re}(u/v)<*})[d_X] \quad ([\text{DK15, Prop. 2.2.4}]) \ (?) \\ &\simeq R\mathcal{H}om(\mathbb{C}_{\mathcal{U}}, \mathbb{C}_{\mathrm{Re}\varphi<*})[d_X] = E^{\varphi}_{\mathcal{U}|X}[d_X]. \quad \# \end{aligned}$$

§III. ENHANCED IND-SHEAVES (k=field)

(A) CONVOLUTIONS

Notation: $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{\pm\infty\} \simeq [-1,1]$ (2-point compatification) $\rightsquigarrow \mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ bordered space

$$\begin{tikzcd} & \mathbb{R}_\infty \times \mathbb{R}_\infty \arrow[r, "\mu:\text{addition}"] \arrow[dl, "q_1"', "\text{(1st)}"] \arrow[dr, "q_2", "\text{projections (2nd)}"'] & \mathbb{R}_\infty \\ \mathbb{R}_\infty & & \mathbb{R}_\infty \end{tikzcd} ; \qquad \begin{tikzcd} M \times \mathbb{R}_\infty \arrow[r, "j_M"] \arrow[dr, "\pi"'] & M \times \overline{\mathbb{R}} \arrow[d, "\overline{\pi}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd} \quad \text{(canonical)}$$

M : good topological space

Remark. $\mathbb{R}_\infty \equiv (\mathbb{R}, \overline{\mathbb{R}}) \simeq (\mathbb{R}, P \equiv \mathbb{RP}^1)$ as bordered spaces.

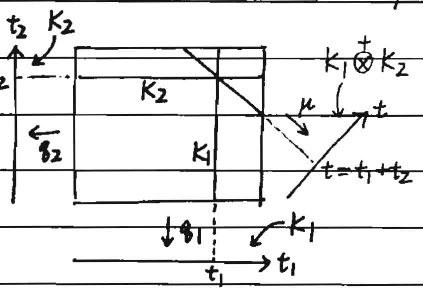
Definition (Convolutions). [DK15, Def.4.1.2]

$$\overset{+}{\otimes} : D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \times D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \to D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}),\quad K_1 \overset{+}{\otimes} K_2 := R\mu_{!!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2).$$

$$\mathcal{I}hom^+ : D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})^{op} \times D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \to D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}),\ \mathcal{I}hom^+(K_1,K_2) := Rq_{1*}R\mathcal{I}hom(q_2^{-1}K_1, \mu^! K_2).$$

Remarks.

1. We call $\overset{+}{\otimes}$ the "convolution operator." This is perhaps motivated by its "correspondence" to the classical convolution operator on usual functions:



$$(K_1 \overset{+}{\otimes} K_2)_t \text{ "="} \int_{t_1+t_2=t} K_1(t_1)\cdot K_2(t_2)$$
$$\text{"="} \int_{-\infty}^{\infty} K_1(t-t_2)\cdot K_2(t_2)dt_2 \leftarrow \text{convolution of } K_1 \text{ and } K_2.$$

2. $\mathcal{I}hom^+$ is the right adjoint of $\overset{+}{\otimes}$ under $\mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})}$; more precisely, for $K_1, K_2, K_3 \in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, ($\mathrm{Hom} \equiv \mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})}$ below)

$$\begin{aligned} \mathrm{Hom}(K_1 \overset{+}{\otimes} K_2, K_3) &= \mathrm{Hom}(R\mu_{!!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2), K_3) \simeq \mathrm{Hom}(q_1^{-1}K_1, q_2^{-1}K_2, \mu^! K_3) \\ &\simeq \mathrm{Hom}(q_1^{-1}K_1, R\mathcal{I}hom(q_2^{-1}K_2, \mu^! K_3)) \quad (\text{"Hom}-\otimes \text{ adjunction"}) \\ &\simeq \mathrm{Hom}(K_1, Rq_{1*}R\mathcal{I}hom(q_2^{-1}K_2, \mu^! K_3)) = \mathrm{Hom}(K_1, \mathcal{I}hom^+(K_2,K_3)). \end{aligned}$$

[DK15, Prop.4.1.5]

Basic properties. [DK15, Lem.4.1.4, Prop.4.1.5, Lem.4.3.1] Let $K, K_i \in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ and $L \in D^b(\mathrm{I}k_M)$.

1. $K_1 \overset{+}{\otimes} K_2 \simeq K_2 \overset{+}{\otimes} K_1$; $(K_1 \overset{+}{\otimes} K_2) \overset{+}{\otimes} K_3 \simeq K_1 \overset{+}{\otimes} (K_2 \overset{+}{\otimes} K_3)$.
2. $\mathcal{I}hom^+(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathcal{I}hom^+(K_1, \mathcal{I}hom^+(K_2,K_3))$.
3. $\pi^{-1}L \otimes (K_1 \overset{+}{\otimes} K_2) \simeq (\pi^{-1}L \otimes K_1) \overset{+}{\otimes} K_2$;

   $R\mathcal{I}hom(\pi^{-1}L, \mathcal{I}hom^+(K_1,K_2)) \simeq \mathcal{I}hom^+(\pi^{-1}L \otimes K_1, K_2) \simeq \mathcal{I}hom^+(K_1, R\mathcal{I}hom(\pi^{-1}L, K_2))$.

(Proof of 2.) By $\overset{+}{\otimes}-\mathcal{I}hom^+$ adjunction and 1., one can deduce that for any $K \in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, $\mathrm{Hom}(K, \mathcal{I}hom^+(K_1 \overset{+}{\otimes} K_2, K_3)) \simeq \mathrm{Hom}(K, \mathcal{I}hom^+(K_1, \mathcal{I}hom^+(K_2,K_3)))$. Then 2. follows from Yoneda lemma.

#

Notation: $k_{t\geq0} := k_{\{(x,t)\in M\times\mathbb{R} \mid t\geq0\}}$; $k_{t=0} := k_{\{(x,t)\in M\times\mathbb{R} \mid t=0\}}$;

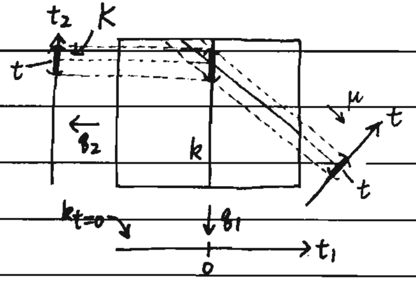
other notations like $k_{t\geq a}$, $k_{t<a}$ are similarly defined.

Lemma III.1. [DK15, ~~Lem.4.2.1~~ Cor.4.2.2] For $K\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$,

$$k_{t=0}\overset{+}{\otimes}K \simeq K \simeq \mathcal{I}hom^+(k_{t=0},K).$$

Thus $(D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}), \overset{+}{\otimes})$ is a commutative tensor category with unit object $k_{t=0}$.

(Proof)



The figure on the left shows that

$(k_{t=0}\overset{+}{\otimes}K)_t \simeq (k\otimes K)_t = K_t$ for each $t\in\mathbb{R}$

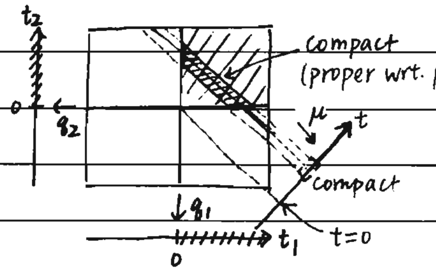
$\Rightarrow k_{t=0}\overset{+}{\otimes}K\simeq K$. The other isomorphism is similarly obtained.

#

Lemma III.2. [DK15, Lem.4.2.3] The following are isomorphisms in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$:

1. $k_{t\geq0}\overset{+}{\otimes}k_{t\geq0}\simeq k_{t\geq0}$
2. $k_{t\geq0}\overset{+}{\otimes}k_{t\leq0}\simeq 0$
3. $k_{t\geq0}\overset{+}{\otimes}k_{t>0}[1]\simeq 0$
4. $k_{t\geq0}\overset{+}{\otimes}k_{t<0}[1]\simeq k_{t\geq0}$.

(Proof) 1.



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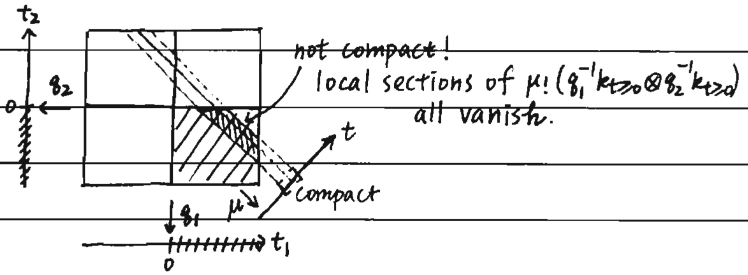


3. distinguished Δ in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$:

$$\begin{tikzcd}
k_{t\geq0} \arrow[r] & k_{t=0} \arrow[r] & k_{t>0}[1] \arrow[r, "+1"] & {} \arrow[r, Rightarrow, "k_{t\geq0}\overset{+}{\otimes}(\cdot)"] & k_{t\geq0}\overset{+}{\otimes}k_{t\geq0} \arrow[r, "\sim"] \arrow[d, equal, "1."] & k_{t\geq0}\overset{+}{\otimes}k_{t=0} \arrow[r] \arrow[d, equal, "\mathrm{III}.1."] & k_{t\geq0}\overset{+}{\otimes}k_{t>0}[1] \arrow[r, "+1"] \arrow[d, equal, "\text{thus}"] & {} \\
& & & & k_{t\geq0} & k_{t\geq0} & 0 &
\end{tikzcd}$$

4. $k_{t\geq0}\overset{+}{\otimes}[k_{t\geq0}\oplus k_{t\leq0}\to k_{t=0}\to k_{\mathbb{R}}[1]\xrightarrow{+1}] = [k_{t\geq0}\xrightarrow{\sim}k_{t\geq0}\to k_{t\geq0}\overset{+}{\otimes}k_{\mathbb{R}}[1]\xrightarrow{+1}]$

$\Rightarrow k_{t\geq0}\overset{+}{\otimes}k_{\mathbb{R}}[1]=0.$

$k_{t\geq0}\overset{+}{\otimes}[k_{t\geq0}\to k_{t<0}[1]\longrightarrow k_{\mathbb{R}}[1]\xrightarrow{+1}] = [k_{t\geq0}\to k_{t\geq0}\overset{+}{\otimes}k_{t<0}[1]\to 0\xrightarrow{+1}]$

$\Rightarrow k_{t\geq0}\overset{+}{\otimes}k_{t<0}[1]\simeq k_{t\geq0}$. #

Remark. In the proofs of Lemma III.1 & III.2, we have implicitly used the fact (or the "guess"?) that $k_{t=0}$ and $k_{t\geq0}$ are quasi-injective in $\mathrm{I}k_{M\times\mathbb{R}_\infty}$, so that we can drop the "R" in "$R\mu_{!!}$" directly; but from Lemma III.2 items 3. & 4., it seems that $k_{t\geq0}$ is not $\mu_{!!}$-injective (or not quasi-injective in $\mathrm{I}k_{M\times\mathbb{R}_\infty}$).

Lemma III.3. [DK15, Lem.4.3.2] For $K\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ and $L\in D^b(\mathrm{I}k_M)$,

$\pi^{-1}L\otimes K\simeq(\pi^{-1}L\otimes k_{t=0})\overset{+}{\otimes}K$ and $R\mathcal{I}hom(\pi^{-1}L,K)\simeq\mathcal{I}hom^+(\pi^{-1}L\otimes k_{t=0},K)$ in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$.

(Proof.) This follows from Basic property 3. & Lemma III.1. #

Proposition III.4. [DK15, Prop.4.3.10, modified] For $K \in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, there is a distinguished Δ:

$$\pi^{-1}L \to k_{t\geqslant -a} \overset{+}{\otimes} K \to \mathcal{I}hom^+(k_{t\geqslant a}, K) \xrightarrow{+1} \quad (a \in \mathbb{R})$$

where $L \in D^b(\mathrm{I}k_M)$ is given by (cf. the notation at the beginning of §III(A).)

$$L := R\overline{\pi}_*(k_{t\neq -\infty} \otimes Rj_{M*}K) \simeq R\pi_{!!}\,\mathcal{I}hom^+(k_{t\geqslant 0}, K) \simeq R\pi_*(k_{t\geqslant 0} \overset{+}{\otimes} K).$$

(Proof.) The proof consists of 3 steps.

Step 1:

$$\begin{tikzcd} M\times\mathbb{R}_\infty\times\overline{\mathbb{R}} \arrow[r, "\overline{q}_1"] \arrow[d, "\overline{q}_2"'] \arrow[dr, phantom, "\square"] & M\times\mathbb{R}_\infty \arrow[d, "\pi"] \\ M\times\overline{\mathbb{R}} \arrow[r, "\overline{\pi}"'] & M \end{tikzcd}$$

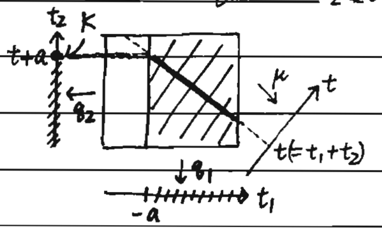
$(\overline{\mathbb{R}}^2 \supset \mathbb{R}^2 \ni (t_1, t_2)$: coordinates$)$

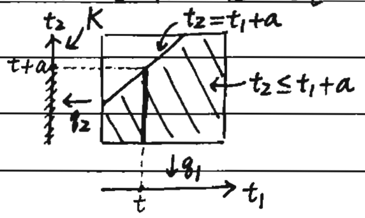
There are two isomorphisms in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$:

1. $k_{t\geqslant -a} \overset{+}{\otimes} K \simeq R\overline{q}_{1!!}(k_{t_2\leq t_1+a} \otimes \overline{q}_2^{\,-1}\widetilde{K})$.
2. $\mathcal{I}hom^+(k_{t\geqslant a}, K) \simeq R\overline{q}_{1*}R\mathcal{I}hom(k_{t_1+a\leq t_2}, \overline{q}_2^{\,!}\widetilde{K})$.

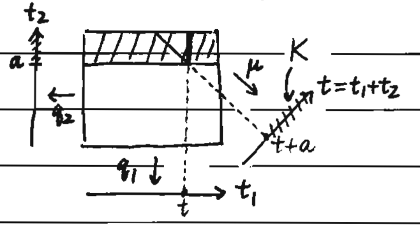
Here, $\widetilde{K} := Rj_{M*}K \in D^b(\mathrm{I}k_{M\times\overline{\mathbb{R}}})$.

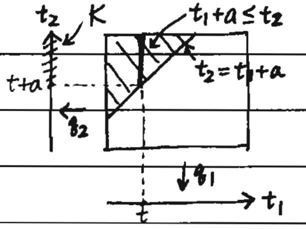
(Proof.) 1. $[k_{t\geqslant -a} \overset{+}{\otimes} K]_t \longleftrightarrow K_{t_2\leq t+a} \longleftrightarrow [R\overline{q}_{1!!}(k_{t_2\leq t_1+a} \otimes \overline{q}_2^{\,-1}\widetilde{K})]_t$





2. $[\mathcal{I}hom^+(k_{t\geqslant a}, K)]_t \longleftrightarrow K_{t_2\geq t+a} \longleftrightarrow [R\overline{q}_{1*}R\mathcal{I}hom(k_{t_1+a\leq t_2}, \overline{q}_2^{\,!}\widetilde{K})]$





#

Step 2: There are two isomorphisms in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$:

1. $k_{t\geqslant -a} \overset{+}{\otimes} K \simeq R\overline{q}_{1*}(k_{M\times\mathbb{R}\times(\overline{\mathbb{R}}\setminus\{-\infty\})} \otimes R\mathcal{I}hom(k_{t_2<t_1+a}[1], \overline{q}_2^{\,!}\widetilde{K}))$.
2. $\mathcal{I}hom^+(k_{t\geqslant a}, K) \simeq R\overline{q}_{1*}(k_{M\times\mathbb{R}\times(\overline{\mathbb{R}}\setminus\{-\infty\})} \otimes R\mathcal{I}hom(k_{t_1+a\leq t_2}, \overline{q}_2^{\,!}K))$.

(Proof.) We remark first that $\overline{q}_2^{\,!}F \simeq \overline{q}_2^{\,-1}F[1]$ for $F \in D^b(\mathrm{I}k_{M\times\overline{\mathbb{R}}})$.

1. $k_{t\geqslant -a} \overset{+}{\otimes} K \simeq R\overline{q}_{1!!}(k_{t_2\leq t_1+a} \otimes \overline{q}_2^{\,-1}\widetilde{K})$ (Step 1, #1)

$\simeq R\overline{q}_{1*}(k_{M\times\mathbb{R}^2} \otimes R\mathcal{I}hom(k_{t_2<t_1+a}, k_{M\times\mathbb{R}^2}) \otimes \overline{q}_2^{\,!}\widetilde{K}[-1])$ (?)

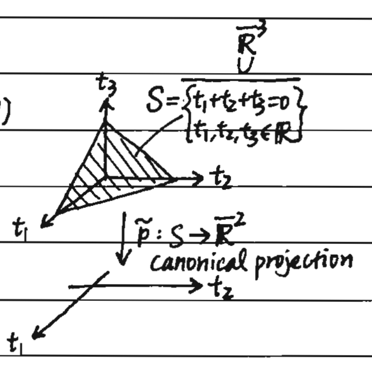
$\simeq R\overline{q}_{1*}(k_{M\times\mathbb{R}^2} \otimes R\mathcal{I}hom(k_{t_2<t_1+a}[1], \overline{q}_2^{\,!}\widetilde{K}))$. ([DK15, Prop.2.3.4])

But for any $F \in D^b(\mathrm{I}k_{M\times\overline{\mathbb{R}}^2})$ (in particular for $F = \overline{q}_2^{\,!}\widetilde{K}$),

$k_{M\times\mathbb{R}\times\{+\infty\}} \otimes [R\mathcal{I}hom(k_{t_2<t_1+a}, F) \simeq R\widetilde{p}_*R\mathcal{I}hom(k_{\widetilde{p}^{-1}(t_2<t_1+a)}, \widetilde{p}^{\,!}F)]$

$\simeq R\widetilde{p}_*(k_{\widetilde{p}^{-1}(M\times\mathbb{R}\times\{+\infty\})} \otimes R\mathcal{I}hom(k_{\widetilde{p}^{-1}(t_2<t_1+a)}, \widetilde{p}^{\,!}F)) \simeq 0.$

$\because \widetilde{p}^{-1}(M\times\mathbb{R}\times\{+\infty\}) \cap \overline{\widetilde{p}^{-1}(t_2<t_1+a)} = \emptyset$



So 1. follows.

2. The proof is similar to #1 (first use Step.1 #2, etc.). #

(Proof of Prop. III.4, cont'd.)

Step 3: End of the proof:

1. Apply $R\bar{q}_{1*}(k_{M\times\mathbb{R}\times(\overline{\mathbb{R}}\setminus\{-\infty\})}\otimes R\mathcal{I}hom(\cdot, \bar{q}_2^{!}\widetilde{K}))$ to the distinguished Δ

$$k_{t_1+a\leq t_2} \to k_{t_2<t_1+a}[1] \to k_{M\times\mathbb{R}^2}[1] \xrightarrow{+1}$$

and use the result from Step 2, we get a distinguished Δ

$$\widetilde{L} \to k_{t\geq -a}\overset{+}{\otimes} K \to \mathcal{I}hom^{+}(k_{t\geq a}, K) \xrightarrow{+1}$$

with $\widetilde{L} = R\bar{q}_{1*}(k_{M\times\mathbb{R}\times(\overline{\mathbb{R}}\setminus\{-\infty\})}\otimes R\mathcal{I}hom(k_{M\times\mathbb{R}^2}[1], \bar{q}_2^{!}\widetilde{K})) \in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$.

2. $R\mathcal{I}hom(k_{M\times\mathbb{R}^2}, \bar{q}_2^{!}\widetilde{K}) \simeq R\mathcal{I}hom(\bar{q}_2^{-1}k_{M\times\mathbb{R}}, \bar{q}_2^{!}\widetilde{K}) \simeq \bar{q}_2^{!}R\mathcal{I}hom(k_{M\times\mathbb{R}}, \widetilde{K}) \simeq \bar{q}_2^{!}\widetilde{K}$

$\Rightarrow k_{M\times\mathbb{R}\times(\overline{\mathbb{R}}\setminus\{-\infty\})}\otimes R\mathcal{I}hom(k_{M\times\mathbb{R}^2}[1], \bar{q}_2^{!}\widetilde{K}) \simeq \bar{q}_2^{-1}k_{t\neq-\infty}\otimes\bar{q}_2^{!}\widetilde{K}[-1]$

$\simeq \bar{q}_2^{-1}k_{t\neq-\infty}\otimes\bar{q}_2^{-1}\widetilde{K} \simeq \bar{q}_2^{-1}(k_{t\neq-\infty}\otimes\widetilde{K}) \simeq \bar{q}_2^{!}(k_{t\neq-\infty}\otimes\widetilde{K})[-1]$

$\Rightarrow$ from 1., $\widetilde{L} \simeq \underset{\underset{\pi^{!}R\bar{\pi}_*}{\wr|}}{\boxed{R\bar{q}_{1*}\bar{q}_2^{!}}}(k_{t\neq-\infty}\otimes\widetilde{K})[-1] \simeq \pi^{-1}L$ with $L = R\bar{\pi}_*(k_{t\neq-\infty}\otimes Rj_{M*}K)$.

(base change, cf. the commutative diagram in Step 1)

3. $L \equiv R\bar{\pi}_*(k_{t\neq-\infty}\otimes Rj_{M*}K) \simeq R\pi_*(k_{t\geq0}\overset{+}{\otimes}K)$ (similarly $L \simeq R\pi_{!!}\mathcal{I}hom^{+}(k_{t\geq0}, K)$):

$R\pi_*[\pi^{-1}L \to k_{t\geq-a}\overset{+}{\otimes}K \to \mathcal{I}hom^{+}(k_{t\geq a}, K)\xrightarrow{+1}]_{a=0}$

$= [\underset{\underset{L}{\wr|}}{R\pi_*\pi^{-1}L} \to R\pi_*(k_{t\geq0}\overset{+}{\otimes}K) \to \underset{\underset{R\mathcal{I}hom(R\pi_{!!}\underset{0}{\cancel{k_{t\geq0}}},\, R\pi_* K_2) = 0}{\wr|\,[\text{DK15, Lem. 4.3.4}]}}{R\pi_*\mathcal{I}hom^{+}(k_{t\geq0}, K)} \xrightarrow{+1}]$

$\Rightarrow L \simeq R\pi_*(k_{t\geq0}\overset{+}{\otimes}K)$. #

(B) ENHANCED IND-SHEAVES AND OPERATIONS ON THEM

Definition (Enhanced ind-sheaves). [DK15, Def.4.4.1] M: good topological space.

The triangulated category of enhanced ind-sheaves on M over k is

$$E^b(\mathrm{I}k_M) := D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})/\mathrm{I}\mathcal{C}_{t^*=0}$$

where $\mathrm{I}\mathcal{C}_{t^*=0} := \{K\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \mid (k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}K\simeq 0\}$.

We also set $E^b_{\pm}(\mathrm{I}k_M) := \mathrm{I}\mathcal{C}_{\pm t^*\geq 0}/\mathrm{I}\mathcal{C}_{t^*=0}$ where $\mathrm{I}\mathcal{C}_{\pm t^*\geq 0} := \{K\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \mid k_{\pm t\leq 0}\overset{+}{\otimes}K\simeq 0\}$.

Lemma III.5 (Characterization of $\mathrm{I}\mathcal{C}_{t^*=0}$). [DK15, Lem.4.4.3] (Recall $\pi: M\times\mathbb{R}_\infty\to M$)

$$\mathrm{I}\mathcal{C}_{t^*=0} \equiv \{K \mid (k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}K\simeq 0\} \overset{①}{=} \{K \mid \mathcal{I}hom^+(k_{t\geq 0}\oplus k_{t\leq 0},K)\simeq 0\}$$

$$\overset{②}{=} \{K \mid \pi^{-1}R\pi_*K \xrightarrow{\sim} K\} = \{K \mid K\xrightarrow{\sim}\pi^! R\pi_{!!}K\}$$

$$\overset{③}{=} \{K \mid K\simeq\pi^{-1}L \text{ for some } L\in D^b(\mathrm{I}k_M)\} = \{K \mid K\simeq\pi^!L \text{ for some } L\in D^b(\mathrm{I}k_M)\}$$

$$= \{K \mid K\xrightarrow{\sim} k_{M\times\mathbb{R}}[1]\overset{+}{\otimes}K\} = \{K \mid \mathcal{I}hom^+(k_{M\times\mathbb{R}}[1],K)\xrightarrow{\sim}K\}.$$

(Proof.) Let's state the following lemma whose proof can be found in [DK15, Lem.4.3.6 & Cor.4.3.7]:

Lemma: For $K\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ and $L\in D^b(\mathrm{I}k_M)$,

1. $(\pi^{-1}L)\overset{+}{\otimes}K\simeq\pi^{-1}(L\otimes R\pi_{!!}K)$; $\mathcal{I}hom^+(\pi^{-1}L,K)\simeq\pi^!R\mathcal{I}hom(L,R\pi_*K)$; $\mathcal{I}hom^+(K,\pi^!L)\simeq\pi^!R\mathcal{I}hom(R\pi_{!!}K,L)$.
2. $(k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}\pi^{-1}L\simeq 0$; $\mathcal{I}hom^+(k_{t\geq 0}\oplus k_{t\leq 0},\pi^{-1}L)\simeq 0$.
3. $k_{M\times\mathbb{R}}\overset{+}{\otimes}K\simeq\pi^{-1}R\pi_{!!}K$; $\mathcal{I}hom^+(k_{M\times\mathbb{R}},K)\simeq\pi^!R\pi_*K$.

Back to the proof:

① Prop. III.4 with $a=0$ $\Rightarrow$ $\exists$ dist. Δ: $\mathcal{T}=[\pi^{-1}L\to k_{t\geq 0}\overset{+}{\otimes}K\to\mathcal{I}hom^+(k_{t\geq 0},K)\xrightarrow{+1}]$.

$$\mathcal{I}hom^+(k_{t\geq 0},\mathcal{T}) = [\underset{\underset{0}{\wr\ \text{Lemma \#2}}}{\mathcal{I}hom^+(k_{t\geq 0},\pi^{-1}L)} \to \mathcal{I}hom^+(k_{t\geq 0},k_{t\geq 0}\overset{+}{\otimes}K) \xrightarrow{\sim} \underset{\underset{\mathcal{I}hom^+(k_{t\geq 0},K)\ \text{(Basic Prop.\#2)}}{\wr\ \text{by adjunction}}}{\mathcal{I}hom^+(k_{t\geq 0},\mathcal{I}hom^+(k_{t\geq 0},K))} \xrightarrow{+1}]$$

$$k_{t\geq 0}\overset{+}{\otimes}\mathcal{T} = [\underset{\underset{0}{\wr\ \text{Lemma \#2}}}{k_{t\geq 0}\overset{+}{\otimes}\pi^{-1}L} \to k_{t\geq 0}\overset{+}{\otimes}K \xrightarrow{\sim} k_{t\geq 0}\overset{+}{\otimes}\mathcal{I}hom^+(k_{t\geq 0},K)\xrightarrow{+1}].$$ So ① holds.

② It follows from the dist. Δ

$$[k_{t\geq 0}\oplus k_{t\leq 0}\to k_{t=0}\to k_{M\times\mathbb{R}}[1]\xrightarrow{+1}]\overset{+}{\otimes}K \overset{\text{Lemma III.1, Lemma \#3 above}}{=} [(k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}K\to K\to\pi^{-1}R\pi_{!!}K[1]\xrightarrow{+1}].$$

③ "$\subset$" follows from ②; "$\supset$" follows from Lemma #2. #

Definition/Proposition (Adjoint functors of the quotient functor $D^b\to E^b$). [DK15, 4.4.4–4.4.6]

$L^E: E^b(\mathrm{I}k_M)\to {}^{\perp}\mathrm{I}\mathcal{C}_{t^*=0}\equiv\{K \mid (k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}K\xrightarrow{\sim}K\}\subset D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, $K\mapsto(k_{t\geq 0}\oplus k_{t\leq 0})\overset{+}{\otimes}K$.

$R^E: E^b(\mathrm{I}k_M)\to \mathrm{I}\mathcal{C}_{t^*=0}^{\perp}\equiv\{K \mid \mathcal{I}hom^+(k_{t\geq 0}\oplus k_{t\leq 0},K)\xrightarrow{\sim}K\}\subset D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, $K\mapsto\mathcal{I}hom^+(k_{t\geq 0}\oplus k_{t\leq 0},K)$.

They satisfy: 1. L^E (resp. R^E) is the left adjoint (resp. right adjoint) of $Q: D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})\xrightarrow{\text{quotient}}E^b(\mathrm{I}k_M)$.

2. $\mathcal{I}hom^+(L^EQF_1,L^EQF_2)\simeq\mathcal{I}hom^+(L^EQF_1,F_2)\simeq\mathcal{I}hom^+(F_1,R^EQF_2)$ $(F_1,F_2\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}))$.

3. $\mathrm{Hom}_{E^b(\mathrm{I}k_M)}(QF_1,QF_2)\simeq\mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})}(L^EQF_1,F_2)\simeq\mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})}(F_1,R^EQF_2)$

$(F_1,F_2\in D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}))$.

Definition/Proposition (Operations on enhanced ind-sheaves). [DK15, §4.5] $f\colon \overset{\text{good topological spaces}}{M \longrightarrow N}$ continuous

We have the following functors induced by their counterpart on $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$:

$$\begin{cases} \overset{+}{\otimes}\colon E^b(\mathrm{I}k_M)\times E^b(\mathrm{I}k_M)\longrightarrow E^b(\mathrm{I}k_M) \\ \mathcal{I}hom^+\colon E^b(\mathrm{I}k_M)^{op}\times E^b(\mathrm{I}k_M)\to E^b(\mathrm{I}k_M) \end{cases}$$

$$\begin{cases} Ef_{!!},\ Ef_*\colon E^b(\mathrm{I}k_M)\longrightarrow E^b(\mathrm{I}k_N) & (\text{from } Rf_{!!} \text{ and } Rf_*) \\ Ef^{-1},\ Ef^!\colon E^b(\mathrm{I}k_N)\to E^b(\mathrm{I}k_M) & (\text{from } f^{-1}, f^!). \end{cases}$$

Here are some of their properties:

1. For $K_1, K_2, K_3 \in E^b(\mathrm{I}k_M)$, $\mathrm{Hom}_{E^b(\mathrm{I}k_M)}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(K_1, \mathcal{I}hom^+(K_2, K_3))$.
2. For $K \in E^b(\mathrm{I}k_M)$ and $L \in E^b(\mathrm{I}k_N)$, $\begin{cases} Ef_{!!}K \simeq Rf_{!!}L^EK \simeq Rf_{!!}R^EK \text{ in } E^b(\mathrm{I}k_N) \text{ (may replace } f_{!!} \text{ by } f_*); \\ Ef^{-1}L \simeq f^{-1}L^EL \simeq f^{-1}R^EL \text{ in } E^b(\mathrm{I}k_M) \text{ (may replace } f^{-1} \text{ by } f^!). \end{cases}$
3. (Base change) $$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & N' \arrow[d, "g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$ $\begin{cases} Eg^{-1}\circ Ef_{!!} \simeq Ef'_{!!}\circ Eg'^{-1}\colon E^b(\mathrm{I}k_M)\longrightarrow E^b(\mathrm{I}k_{N'}) \\ Eg^{!}\circ Ef_{*} \simeq Ef'_{*}\circ Eg'^{!}\colon E^b(\mathrm{I}k_M)\to E^b(\mathrm{I}k_{N'}). \end{cases}$

We also define the "exterior convolution" $\overset{+}{\boxtimes}$ by:

$\overset{+}{\boxtimes}\colon E^b(\mathrm{I}k_M)\times E^b(\mathrm{I}k_N)\longrightarrow E^b(\mathrm{I}k_{M\times N})$, $K \overset{+}{\boxtimes} L := Ep_1^{-1}K \overset{+}{\otimes} Ep_2^{-1}L$ (where $\begin{tikzcd} & M\times N \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & N \end{tikzcd}$: projections).

(C) STABLE OBJECTS

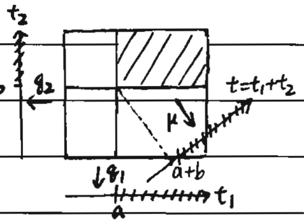
Notation: $k_{t\gg 0} := \text{"}\varinjlim_{a\to+\infty}\text{"}\, k_{t\geq a}$; $k_{t<*} := \text{"}\varinjlim_{a\to+\infty}\text{"}\, k_{t<a}$. They are objects in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$.

There is a distinguished Δ in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$: $k_{M\times\mathbb{R}} \to k_{t\gg 0} \to k_{t<*}[1] \xrightarrow{+1}$

Basic property: In $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$, $k_{t\gg 0}\overset{+}{\otimes} k_{t\gg 0} \xrightarrow{\sim} k_{t\gg 0}$ and $k_{t\geq a}\overset{+}{\otimes} k_{t\gg 0} \simeq k_{t\gg 0}$ $(a\in\mathbb{R})$.

(Proof of 1st isom.) $\forall \ell\in\mathbb{Z}$,

$[k_{t\geq a}\overset{+}{\otimes} k_{t\geq b} \simeq k_{t\geq a+b}]$



$$\begin{aligned}
H^\ell(k_{t\gg 0}\overset{+}{\otimes} k_{t\gg 0}) &= R^\ell\mu_{!!}\big(q_1^{-1}\,\text{"}\varinjlim_{a\to+\infty}\text{"}\, k_{t\geq a}\otimes q_2^{-1}\,\text{"}\varinjlim_{b\to+\infty}\text{"}\, k_{t\geq b}\big)\\
&\simeq \text{"}\varinjlim_{a,b\to+\infty}\text{"}\, R^\ell\mu_{!!}(q_1^{-1}k_{t\geq a}\otimes q_2^{-1}k_{t\geq b}) = \text{"}\varinjlim_{a,b\to+\infty}\text{"}\, H^\ell(\underbrace{k_{t\geq a}\overset{+}{\otimes} k_{t\geq b}})\\
&\simeq \text{"}\varinjlim_{a,b\to+\infty}\text{"}\, H^\ell(\underbrace{k_{t\geq a+b}}) \simeq H^\ell\big(\text{"}\varinjlim_{a,b\to+\infty}\text{"}\, k_{t\geq a+b}\big) = H^\ell(k_{t\gg 0}).
\end{aligned}$$ #

Notation: $k_M^E :=$ (the image of $k_{t\gg 0}$ via the quotient functor $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})\to E^b(\mathrm{I}k_M)$) $\in E^b(\mathrm{I}k_M)$.

$F^E := k_M^E \otimes \pi^{-1}F$ for $F\in D^b(k_M)$.

Basic property: $L^E k_M^E \simeq k_{t\gg 0}$; $R^E k_M^E \simeq k_{t<*}[1]$.

(Proof of 2nd isom.) Consider $R^E[k_{M\times\mathbb{R}} \to k_{t\gg 0} \to k_{t<*}[1] \xrightarrow{+1}]$ and note $R^E(\overset{\overset{\text{already}}{\pi^{-1}k_M = 0 \text{ in } E^b(\mathrm{I}k_M)}}{\overset{\parallel}{k_{M\times\mathbb{R}}}}) = 0$. #

Proposition III.6 (Stable objects). [DK15, Prop.4.7.5] For $K\in E^b(\mathrm{I}k_M)$, the following are equivalent:

1. $K\simeq k_{t\geq a}\overset{+}{\otimes} K$ for any $a\in\mathbb{R}_{\geq 0}$
2. $K\simeq \mathcal{I}hom^+(k_{t\geq a}, K)$ for any $a\in\mathbb{R}_{\geq 0}$
3. $K\simeq k_M^E\overset{+}{\otimes} K$
4. $K\simeq \mathcal{I}hom^+(k_M^E, K)$
5. $K\simeq k_M^E\overset{+}{\otimes} L$ for some $L\in E^b(\mathrm{I}k_M)$
6. $K\simeq \mathcal{I}hom^+(k_M^E, L)$ for some $L\in E^b(\mathrm{I}k_M)$.

If K satisfies one of these equivalent conditions, we call K a "stable object."

(Proof.) Observation: by Prop.III.4 and Lemma III.5, $[\underset{0 \text{ in } E^b(\mathrm{I}k_M)}{\pi^{-1}L} \to k_{t\geq -a}\overset{+}{\otimes} K \underset{\text{in } E^b(\mathrm{I}k_M)}{\xrightarrow{\sim}} \mathcal{I}hom^+(k_{t\geq a}, K) \xrightarrow{+1}]$ in $D^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ $(a\in\mathbb{R})$

$\Rightarrow \mathcal{I}hom^+(k_{t\geq a}, K) \overset{\text{in } E^b(\mathrm{I}k_M)}{\simeq} k_{t\geq -a}\overset{+}{\otimes} K$ for any $a\in\mathbb{R}$.

Therefore:

$$\begin{tikzcd}
 & K\simeq k_{t\geq a}\overset{+}{\otimes} K\ (a\in\mathbb{R}) \arrow[dl, Rightarrow] \arrow[ddd, Leftrightarrow, bend left] & \\
1 \arrow[r, Rightarrow] & 3. \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & 5. \\
2. \arrow[r, Rightarrow] & 4. \arrow[d, Rightarrow] \arrow[r, Leftrightarrow] & 6. \\
 & K\simeq \mathcal{I}hom^+(k_{t\geq a}, K)\ (a\in\mathbb{R}) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

(cf. [DK15, Prop.2.2.1 & Cor.2.2.3] for $\begin{smallmatrix}1.\Rightarrow 3.\\ 2.\Rightarrow 4.\end{smallmatrix}$)

#

Examples of stable objects (in $E^b(\mathrm{I}k_M)$):

1. k_M^E.
2. $k_M^E\overset{+}{\otimes} K$ $(K\in E^b(\mathrm{I}k_M))$.
3. $\mathcal{I}hom^+(k_M^E, K)$ $(K\in E^b(\mathrm{I}k_M))$.

Proposition III.7. [DK15, Prop.4.7.9] For $F \in D^b(k_{M\times \mathbb{R}_\infty})$ and $K \in E^b(\mathrm{I}k_M)$ such that $\overline{\pi}(\mathrm{Supp}(Rj_{M!}F))$ is compact,

$$\mathrm{Hom}_{E^b(\mathrm{I}k_M)}(k_M^E \overset{+}{\otimes} F, k_M^E \overset{+}{\otimes} K) \simeq \varinjlim_{a\to+\infty} \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(F, k_{t\geq a} \overset{+}{\otimes} K)$$

$$\simeq \varinjlim_{a\to+\infty} \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(k_{t\geq -a} \overset{+}{\otimes} F, K).$$

(Proof of 1st isom.)

$$\mathrm{Hom}_{E^b(\mathrm{I}k_M)}(k_M^E \overset{+}{\otimes} F, k_M^E \overset{+}{\otimes} K) \simeq \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(k_{t\geq 0} \overset{+}{\otimes} F, k_M^E \overset{+}{\otimes} K)$$

$$\simeq \mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\overline{\mathbb{R}}})}(Rj_{M!!}(k_{t\geq 0} \overset{+}{\otimes} F), Rj_{M*}(k_{t\gg 0} \overset{+}{\otimes} L^E K)) \quad (?)$$

$$\simeq \varinjlim_{a\to+\infty} \mathrm{Hom}_{D^b(\mathrm{I}k_{M\times\overline{\mathbb{R}}})}(Rj_{M!!}(k_{t\geq 0} \overset{+}{\otimes} F), Rj_{M*}(k_{t\geq a} \overset{+}{\otimes} L^E K)) \quad ([\text{DK15, Cor.2.2.3}])$$

$$\simeq \varinjlim_{a\to+\infty} \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(F, k_{t\geq a} \overset{+}{\otimes} K). \quad \#$$

Lemma III.8. [DK15, Lem.4.7.10] For $F \in D^b(k_{M\times \mathbb{R}_\infty})$ and $K \in E^b(\mathrm{I}k_M)$,

$$k_M^E \overset{+}{\otimes} \mathcal{I}hom^+(F, K) \xrightarrow{\sim} \mathcal{I}hom^+(F, k_M^E \overset{+}{\otimes} K) \text{ in } E^b(\mathrm{I}k_M).$$

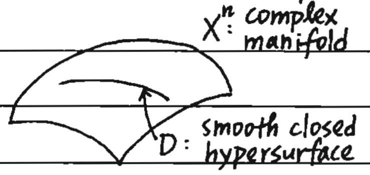
Lemma III.9. [DK15, Cor.4.7.11] For $K \in E^b(\mathrm{I}k_M)$ and $F \in D^b(k_M)$,

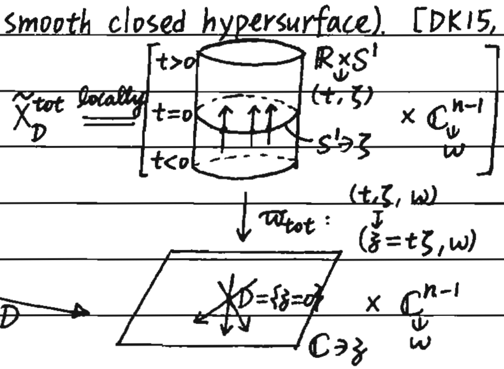
$$k_M^E \overset{+}{\otimes} R\mathcal{I}hom(\pi^{-1}F, K) \simeq R\mathcal{I}hom(\pi^{-1}F, k_M^E \overset{+}{\otimes} K).$$

§ IV. NORMAL FORMS OF HOLONOMIC D-MODULES

(A) REAL BLOW-UP

Definition (Real blow-up along a smooth closed hypersurface). [DK15, §7.1]





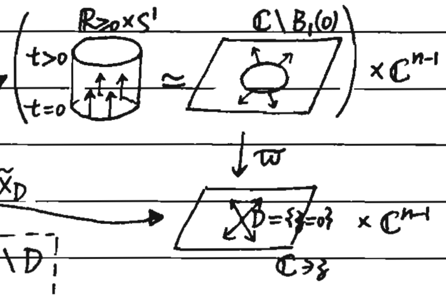
$\xrightarrow{\text{gluing}}$ get the "total real blow-up" (of X along D) $\begin{array}{c}\widetilde{X}_D^{tot}\\ \downarrow \varpi_{tot}\\ X\end{array}$ ← unramified 2-sheeted covering over $X\setminus D$

Also, locally define

$\widetilde{X}_D^{>0} := \{(t,\zeta,w)\in \widetilde{X}_D^{tot} \mid t>0\}$

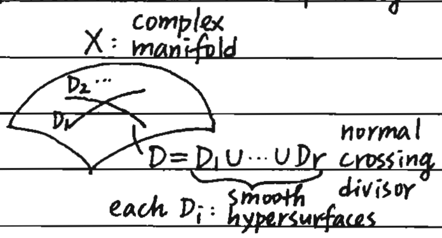
$\widetilde{X}_D := \{(t,\zeta,w)\in \widetilde{X}_D^{tot} \mid t\geq 0\}$ ⟶ "real blow-up" (of X along D) $\begin{array}{c}\widetilde{X}_D\\ \downarrow \varpi\equiv\varpi_{tot}|_{\widetilde{X}_D}\\ X\end{array}$

$\widetilde{X}_D^{0} := \{(t,\zeta,w)\in \widetilde{X}_D^{tot} \mid t=0\}$



Note: $\begin{cases}\varpi|_{\widetilde{X}_D^{>0}} : \widetilde{X}_D^{>0} \xrightarrow{\sim} X\setminus D\\ \widetilde{X}_D^{0} = \varpi^{-1}(D)\end{cases}$

Definition (Real blow-up along a normal crossing divisor). [DK15, §7.1]

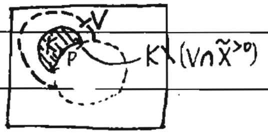


⟶ ~~total real blow-up~~ & ~~real blow-up~~

$\begin{array}{c}\widetilde{X}_D^{tot} := \widetilde{X}_{D_1}^{tot} \times_X \cdots \times_X \widetilde{X}_{D_r}^{tot}\\ \downarrow \varpi_{tot}\\ X\end{array}$ $\begin{array}{c}\widetilde{X}_D := \overline{\widetilde{X}_{D_1}^{>0} \times_X \cdots \times_X \widetilde{X}_{D_r}^{>0}}\\ \downarrow \varpi=\varpi_{tot}|_{\widetilde{X}_D}\\ X\end{array}$ ← closure in $\widetilde{X}_D^{tot}$

Note: $\varpi: \widetilde{X}_D^{>0} = \widetilde{X}_{D_1}^{>0} \times_X \cdots \times_X \widetilde{X}_{D_r}^{>0} \xrightarrow{\sim} X\setminus D.$

Definition (Sheaves of functions on the real blow-up). [DK15, Not. 7.2.1 & §5.1] Not. 7.2.4, 7.2.6



X: complex manifold
$\cup$
D: normal crossing divisor $\rightsquigarrow \widetilde{X}\equiv\widetilde{X}_D$ (notation) $\underset{\text{closed embedding}}{\overset{i}{\hookrightarrow}} \widetilde{X}^{tot} = \widetilde{X}_D^{tot}$.

1. $\mathcal{C}_{\widetilde{X}}^{\infty,temp}: [\widetilde{X} \underset{open}{\supset} V \mapsto \{u\in \mathcal{C}^{\infty}_{\widetilde{X}^{tot}}(V\cap\widetilde{X}^{>0}) \mid u \text{ is tempered at any point } p\in V\cap\widetilde{X}^{0}\}]$ sheaf of $\mathbb{C}$-algebras on $\widetilde{X}$.
   ↳ i.e. ∀ derivatives $\tilde{u}$ of u, $\exists\ p\in K \underset{compact}{\subset} \widetilde{X}$ & const. $\begin{array}{l}C>0\\ r\in\mathbb{Z}_{>0}\end{array}$, s.t. $|\tilde{u}(x)| \leq C\cdot \mathrm{dist}(K\setminus(V\cap\widetilde{X}^{>0}), x)^{-r}$ for any $x\in K\cap(V\cap\widetilde{X}^{>0})$.
2. $\mathcal{A}_{\widetilde{X}}: [\widetilde{X} \underset{open}{\supset} V \mapsto \{u\in \mathcal{C}^{\infty,temp}_{\widetilde{X}}(V) \mid u \text{ is holomorphic on } V\cap\widetilde{X}^{>0}\}]$ sheaf of rings on $\widetilde{X}$.
3. $\mathcal{D}_{\widetilde{X}}^{\mathcal{C}^{\infty,temp}} := \mathcal{C}^{\infty,temp}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X$
   $\mathcal{D}_{\widetilde{X}}^{\mathcal{A}} := \mathcal{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X.$
4. $\mathcal{D}b^t_{\widetilde{X}} := i^{-1}\mathcal{H}om(\mathbb{C}_{\widetilde{X}^{>0}}, \mathcal{D}b^t_{\widetilde{X}^{tot}}) \simeq R\mathcal{H}om(\mathbb{C}_{\widetilde{X}^{>0}}, i^{!}\mathcal{D}b^t_{\widetilde{X}^{tot}})$: a $\mathcal{D}_{\widetilde{X}}^{\mathcal{C}^{\infty,temp}}$-module.
   $\mathcal{O}^t_{\widetilde{X}} := R\mathcal{H}om_{\varpi^{-1}\mathcal{D}_{\widetilde{X}}}(\varpi^{-1}\mathcal{O}_{\widetilde{X}}, \mathcal{D}b^t_{\widetilde{X}}) \in D^b(\mathrm{I}\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}).$

Theorem IV.1. [DK15, Thm.7.2.7 & Rmk 7.2.8 & Cor. 7.2.9] $\mathrm{for}: D^b(I\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}) \to D^b(I\varpi^{-1}\mathcal{D}_X)$ the forgetful functor.

1. $\mathrm{for}(\mathcal{O}^t_{\widetilde{X}}) \simeq \varpi^! R\mathcal{H}om(\mathbb{C}_{X\setminus D}, Db^t_X)$ in $D^b(I\varpi^{-1}\mathcal{D}_X)$. ($\Rightarrow \varpi^! R\mathcal{H}om(\mathbb{C}_{X\setminus D}, Db^t_X)$ is a $\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}$-module.)
2. $R\varpi_* \mathcal{O}^t_{\widetilde{X}} \simeq R\mathcal{H}om(\mathbb{C}_{X\setminus D}, \mathcal{O}^t_X)$ in $D^b(I\mathcal{D}_X)$.

(Proof.) We need a lemma (and we shall assume it):

Lemma [DK15, Lem.5.3.2] $f: M \to N$ real analytic map $\rightsquigarrow$ complexification $X \to Y$

$\rightsquigarrow f^! Db^t_N \simeq \mathcal{D}_{N\leftarrow M} \overset{L}{\otimes}_{\mathcal{D}_M} Db^t_M$ (where $\mathcal{D}_{N\leftarrow M} = \mathcal{D}_{Y\leftarrow X}|_M \otimes or_M \otimes f^{-1} or_N$).

1. $\varpi^! R\mathcal{H}om(\mathbb{C}_{X\setminus D}, Db^t_X) \simeq R\mathcal{H}om(\mathbb{C}_{\widetilde{X}>0}, \varpi^! Db^t_X)$ (Prop. I.1 #2)

$$\simeq R\mathcal{H}om(\mathbb{C}_{\widetilde{X}>0}, \mathcal{D}_{X_{\mathbb{C}}\leftarrow \widetilde{X}^{tot}_{\mathbb{C}}} \overset{L}{\otimes}_{\mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}} Db^t_{\widetilde{X}}) \quad (\underline{\text{Lemma}})$$

$$\simeq \mathcal{D}_{X_{\mathbb{C}}\leftarrow \widetilde{X}^{tot}_{\mathbb{C}}} \overset{L}{\otimes}_{\mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}} Db^t_{\widetilde{X}} \simeq \underbrace{\mathcal{D}_{X_{\mathbb{C}}\leftarrow \widetilde{X}^{tot}_{\mathbb{C}}} \overset{L}{\otimes}_{\mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}} \mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}(*\widetilde{X}^o_{\mathbb{C}})}_{\substack{\wr\ (?) \\ \mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}(*\widetilde{X}^o_{\mathbb{C}})}} \overset{L}{\otimes}_{\mathcal{D}_{\widetilde{X}^{tot}_{\mathbb{C}}}(*\widetilde{X}^o_{\mathbb{C}})} Db^t_{\widetilde{X}}$$

$$\simeq Db^t_{\widetilde{X}}.$$ #

2. $R\varpi_* \mathcal{O}^t_{\widetilde{X}} \overset{1.+\text{Prop. I.1\#2}}{\simeq} R\varpi_* R\mathcal{H}om(\varpi^{-1}\mathbb{C}_{X\setminus D}, \varpi^!(\mathcal{O}^t_X)) \overset{\text{[KS01, Cor.5.3.5]}}{\simeq} R\mathcal{H}om(R\varpi_{!!}\varpi^{-1}\mathbb{C}_{X\setminus D}, \mathcal{O}^t_X) \simeq R\mathcal{H}om(\mathbb{C}_{X\setminus D}, \mathcal{O}^t_X)$. #

Proposition IV.2. [DK15, Prop.7.2.10] $\mathcal{A}_{\widetilde{X}} \simeq \alpha_{\widetilde{X}} \mathcal{O}^t_{\widetilde{X}}$. $\leftarrow$

(We omit the proof because it is somewhat involved.)

Definition: For M = good topological space
[KS01, §3.3 & Def.4.1.2] $\mathcal{A}$ = sheaf of k-algebras on M,
define

$\alpha_M : I\mathcal{A} \equiv \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})) \to \mathrm{Mod}(\mathcal{A})$

$\quad \text{“}\varinjlim_i\text{”} F_i \mapsto \varinjlim_i F_i$

which is exact and is the left adjoint of the natural inclusion functor

$\iota_M : \mathrm{Mod}(\mathcal{A}) \to I\mathcal{A}$

$\quad F \mapsto \mathcal{H}om_{\mathcal{A}}(\cdot|_{\mathrm{Mod}^c(\mathcal{A})}, F)$.

(B) NORMAL FORMS

Setting: $\begin{array}{l} X: \text{complex manifold} \\ \cup \\ D: \text{normal crossing divisor (NCD)} \end{array} \xrightarrow[\text{a point of } D]{\text{locally near}} \begin{array}{l} X \simeq \mathbb{C}^n \ni (z_1,\cdots,z_n): \text{coordinates} \\ \cup \\ D = \{z_1 \cdots z_r = 0\}. \end{array}$

Recall the real blow-up
$$\begin{tikzcd} \tilde{X} \equiv \tilde{X}_D . \arrow[d, "\varpi"] \\ X \end{tikzcd}$$

Notation. [DK15, Not. 7.3.1] For $M \in D^b(\mathcal{D}_X)$, $M^{\mathcal{A}} := \mathcal{D}^{\mathcal{A}}_{\tilde{X}} \overset{L}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M$.

Lemma IV.3. [DK15, Lem. 7.3.2] M: holonomic $\mathcal{D}_X$-module, $\text{sing.supp}(M) \subset D$, and $M \xrightarrow{\sim} M(*D)$.

Then $M^{\mathcal{A}} \simeq \mathcal{D}^{\mathcal{A}}_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M$.

(Proof.) $M^{\mathcal{A}} = \mathcal{D}^{\mathcal{A}}_{\tilde{X}} \overset{L}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M \simeq (\mathcal{A}_{\tilde{X}} \overset{L}{\otimes}_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X) \overset{L}{\otimes}_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M \simeq \mathcal{A}_{\tilde{X}} \overset{L}{\otimes}_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}M$

$\wr|$ ← M is flat/$\mathcal{O}_X$ since $\begin{tikzcd} M \simeq M(*D) \arrow[d, dash, "\text{flat}"] \\ \mathcal{O}_X(*D) \arrow[d, dash, "\text{flat}"] \\ \mathcal{O}_X \end{tikzcd}$

$\mathcal{D}^{\mathcal{A}}_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M \simeq (\mathcal{A}_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{D}_X) \otimes_{\varpi^{-1}\mathcal{D}_X} \varpi^{-1}M \simeq \mathcal{A}_{\tilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}M$ #

Definition (Normal form). [DK15, Def. 7.3.3]

A holonomic $\mathcal{D}_X$-module M has a normal form along D if the following statements all hold:

1. $M \simeq M(*D)$ → i.e. the set of points of X where
2. $\boxed{\text{sing.supp}(M)} \subset D$ char(M) is not contained in the zero-section of T^*X [DK15, §2.5]
3. $\forall x \in \tilde{X}^0$, $\exists\, \varpi(x) \in \mathcal{U} \underset{\text{open}}{\subset} X$ and $\varphi_1,\cdots,\varphi_s \in \Gamma(\mathcal{U}, \mathcal{O}_X(*D))$ $(r \geq 0)$ and $x \in V \underset{\text{open}}{\subset} \varpi^{-1}(\mathcal{U})$ such that $(M^{\mathcal{A}})|_V \simeq \left(\bigoplus_{i=1}^{s} (\mathcal{E}^{\varphi_i}_{\mathcal{U}\setminus D|\mathcal{U}})^{\mathcal{A}}\right)|_V$.

Example: It is clear that each $\mathcal{E}^{\varphi}_{X\setminus D|X}$ $(\varphi \in \mathcal{O}_X(*D))$ has a normal form along D.

Definition (Ramification). [DK15, §7.3]

A ramification of X along D on $\mathcal{U}$: neighborhood of x in X, $x \in D$, is a finite map $\begin{tikzcd} X' \arrow[d, "p"'] \\ \mathcal{U} \end{tikzcd}$ locally looking like (near x)

$$\begin{tikzcd} X' = \mathbb{C}^r \times \mathbb{C}^{n-r} \arrow[d, "p"'] \ni & (w_1,\cdots,w_r;\, w_{r+1},\cdots,w_n) \arrow[d, mapsto] \\ \mathcal{U} = \mathbb{C}^r \times \mathbb{C}^{n-r} \ni & (z_1,\cdots,z_r;\, z_{r+1},\cdots,z_n) \arrow[d, equal] \\ D = \{z_1 \cdots z_r = 0\} \arrow[u] & (w_1^{m_1},\cdots,w_r^{m_r};\, w_{r+1},\cdots,w_n),\ (m_1,\cdots,m_r) \in (\mathbb{Z}_{>0})^{\oplus r}. \end{tikzcd} \quad (*)$$

Definition (Quasi-normal form). [DK15, Def. 7.3.4 & Rmk. 7.3.5]

A holonomic $\mathcal{D}$-module M has a quasi-normal form along D if 1. & 2. below all hold:

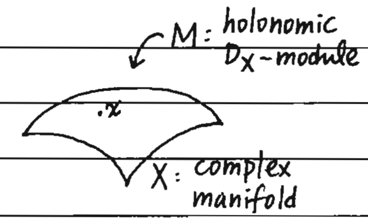
1. $M \simeq M(*D)$, $\text{sing.supp}(M) \subset D$ (cf. def. of normal form)
2. $\forall x \in D$, $\exists$ ramification $\begin{tikzcd} X' \supset p^{-1}(D\cap\mathcal{U}) \arrow[d, "p"'] \\ X \underset{\text{open}}{\supset} \mathcal{U} \supset D\cap\mathcal{U} \end{tikzcd}$ such that $Dp^*(M|_{\mathcal{U}})$ has a normal form along $p^{-1}(D\cap\mathcal{U})$.

In this case, we have the following properties:

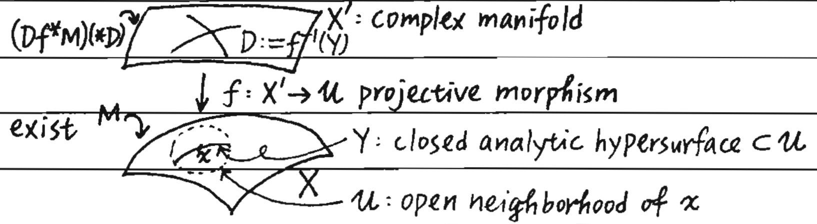
- $Dp^*(M|_{\mathcal{U}})$ & $Dp_*Dp^*(M|_{\mathcal{U}})$ both concentrate at degree 0.
- $M|_{\mathcal{U}}$ is a direct summand of $Dp_*Dp^*(M|_{\mathcal{U}})$: For example, if $\begin{tikzcd} X' \arrow[d, "p"'] \\ \mathcal{U} \end{tikzcd}$ is given in (*), and if $Dp^*(M|_{\mathcal{U}}) \simeq \mathcal{E}^{\varphi}_{X'\setminus p^{-1}(D\cap\mathcal{U})|X'}$, then $Dp_*Dp^*(M|_{\mathcal{U}}) \simeq (M|_{\mathcal{U}})^{\oplus m_1\cdots m_r}$. (?)

The following deep theorem will be used to prove an important reduction tool (Lemma IV.5).

Theorem IV.4. [DK15, Thm.7.3.6]; cf. also [Ke10] and [Ke11]



$\Rightarrow$ there exist



such that: 1. $\mathrm{sing.supp}(M) \cap \mathcal{U} \subset Y$.

2. $D = f^{-1}(Y)$ is a normal crossing divisor of X'.

3. $f|_{X' \setminus D} : X' \setminus D \xrightarrow{\sim} \mathcal{U} \setminus Y$.

4. $(Df^*M)(*D)$ has a quasi-normal form along D.

Remark. This theorem, quoted from [DK15, Thm.7.3.6], seems to be derived from the "formal version" discussed in [Ke10] and [Ke11], but the mechanism from "formal" to "analytic" is not very clear and needs more discussion, at least for me; nevertheless we still use this thm. first.

Lemma IV.5 ("Reduction to normal form along a NCD"). [DK15, Lem.7.3.7]

Let $P_X(M)$ be a statement concerning a complex manifold X and a holonomic object $M \in D^b_{hol}(D_X)$.

Then $P_X(M)$ is true for any X: complex manifold and $M \in D^b_{hol}(D_X)$, if the following all hold:

(i) For $X = \bigcup_{i \in I} \mathcal{U}_i$ an open covering, we have: $P_X(M)$ is true $\Leftrightarrow$ $P_{\mathcal{U}_i}(M|_{\mathcal{U}_i})$ is true for all $i \in I$.

(ii) $P_X(M)$ is true $\Rightarrow$ $P_X(M[n])$ is true for any $n \in \mathbb{Z}$.

(iii) For a distinguished Δ $[M' \to M \to M'' \xrightarrow{+1}]$ in $D^b_{hol}(D_X)$, $P_X(M')$ and $P_X(M'')$ are true $\Rightarrow$ $P_X(M)$ is true.

(iv) For $M, M' \in \mathrm{Mod}_{hol}(D_X)$, $P_X(M \oplus M')$ is true $\Rightarrow$ $P_X(M)$ is true.

(v) For $f: X \to Y$ a projective morphism and M: $\overset{\text{good}}{\text{holonomic}}$ D_X-module, $P_X(M)$ is true $\Rightarrow$ $P_Y(Df_*M)$ is true.

(vi) If M is a holonomic D_X-module with a normal form along a normal crossing divisor of X, then $P_X(M)$ is true.

(Proof) 1. The case "$M =$ good holonomic D_X-module with a quasi-normal form along a NCD $D \subset X$" is true: $\exists$ (locally) a ramification $\begin{array}{c} X' \\ \downarrow p \\ X \end{array}$ s.t. Dp^*M has a normal form; then $P_{X'}(Dp^*M)$ is true by (vi) $\Rightarrow$ $P_X(Dp_*Dp^*M)$ is true by (v) $\Rightarrow$ $P_X(M)$ is true since M is a direct summand of Dp_*Dp^*M.

2. General $M \in D^b_{hol}(D_X)$: By (ii)(iii) & the dist. Δ $[\tau^{\leq a}M \to M \to \tau^{\geq a+1}M \xrightarrow{+1}]$, may assume $M \in \mathrm{Mod}_{hol}(D_X)$; by (i) may also assume M is good.

Now we prove this $P_X(M)$ $(M \in \mathrm{Mod}_{hol,good}(D_X))$ is true by induction on $\begin{cases} \dim X\ (\uparrow) \\ \dim(S := \mathrm{supp}(M))\ (\uparrow) \end{cases}$:

- Initial case "$\dim X = 0$": This reduces to 1.

(See the next page.)

(Proof of Lemma IV.5, 2. General $M \in D^b_{hol}(D_X)$, cont'd.)

- The case "$S=X$": We apply Thm. IV.4 and its notations to any point $x \in X$.

  Then $P_{X'}(Df^*M(*D))$ is true by 1. (We may shrink the neighborhood $\mathcal{U}$ of x so that X' is small enough to ensure that $Df^*M(*D)$ is good.)

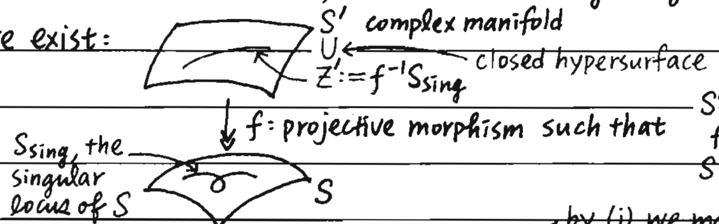
  So $M(*Y) \simeq Df_*Df^*M(*D)$ is good holonomic $\Rightarrow P_{\mathcal{U}}(M(*Y))$ is true by (v).

  Consider a distinguished Δ: $[M \to \underset{\substack{\uparrow \\ \text{true}}}{M(*Y)} \to \underset{\substack{\uparrow \\ \text{true!}}}{N} \xrightarrow{+1}]$ (dim supp$N <$ dim$X \Rightarrow$ apply induction hypothesis on dim$X$)

  $\Rightarrow$ by (ii)(iii), $P_{\mathcal{U}}(M)$ is true; since $\mathcal{U}$ runs over X, by (i), $P_X(M)$ is true.

- The other case "$S \subsetneq X$": By Hironaka's desingularization theorem,

  there exist:



  f: projective morphism such that $\begin{array}{c} S'\setminus Z' \\ f\downarrow\wr \\ S\setminus S_{sing} \end{array}$.

  Then $N := Df^*M(*Z')[d_{S'}-d_X]$ is a $\boxed{\text{good}}$ holonomic $D_{Y'}$-module ← by (i) ~~we may treat the problem locally~~, so ~~goodness~~ can always be assumed once we have holonomicity

  $\Rightarrow P_{Y'}(N)$ is true ($\because$ dim$S' <$ dim$X \Rightarrow$ apply induction hypothesis on dim$X$)

  $\Rightarrow P_X(Df_*N)$ is true by (v).

  Consider another distinguished Δ: $[M \to \underset{\substack{\uparrow \\ \text{true}}}{Df_*N} \to \underset{\substack{\uparrow \\ \text{true!}}}{L} \xrightarrow{+1}]$ (supp$L \subset S_{sing} \Rightarrow$ apply induction hypothesis on dim$S$)

  $\Rightarrow$ by (ii)(iii), $P_X(M)$ is true.

#

§V. ENHANCED DE RHAM AND SOLUTION FUNCTORS

(A) DEFINITIONS AND FIRST PROPERTIES

Definition (Enhanced tempered distributions). [DK15, Def. 8.1.1] M: real analytic manifold

$$\left.\begin{array}{l} P\equiv\mathbb{RP}^1\supset\mathbb{R}\ni t=\tau|_{\mathbb{R}} \\ \cap \qquad\quad\ \cap \\ \mathbb{P}\equiv\mathbb{CP}^1\supset\mathbb{C}\ni\tau\end{array} \ \& \ \begin{array}{c} M\times\mathbb{R}_\infty \\ \downarrow j:\text{natural morphism}\\ M\times P\end{array}\right\} \begin{array}{l} Db_M^T := j^! R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, Db^{t}_{M\times P})[1] \in D^b(I\mathbb{C}_{M\times\mathbb{R}_\infty}). \\ Db_M^E := (\text{image of } Db_M^T \text{ via } D^b(I\mathbb{C}_{M\times\mathbb{R}_\infty})\to E^b(I\mathbb{C}_M)).\end{array}$$

Remark. By a similar argument as "Step 2" in the proof of Prop. II.1, we get $H^k(Db_M^T)=0$ for $k\neq-1$.

Remark. By [DK15, Prop. 8.1.3], for any $a\geq0$, $Db_M^T\simeq\mathcal{I}hom^+(\mathbb{C}_{t\geq0}, Db_M^T)\simeq\mathcal{I}hom^+(\mathbb{C}_{t\geq a}, Db_M^T)$ in $D^b(I\mathbb{C}_{M\times\mathbb{R}_\infty})$; therefore Db_M^E is a stable object (cf. Prop. III.6).

Definition (Enhanced tempered holomorphic functions). [DK15, $\overset{\text{Thm. 8.2.2}}{\text{Def. 8.2.1}}$] X: complex manifold.

$\mathbb{C}\ni\tau\leftrightarrow t=\tau|_{\mathbb{R}}\in\mathbb{R}$ & $i: X\times\mathbb{R}_\infty\to X\times\mathbb{P}$ natural morphism. ($r: M\mapsto M^r=\Omega_{\mathbb{P}}\overset{L}{\underset{\mathcal{O}_{\mathbb{P}}}{\otimes}}M$ side-changing)

$$\rightsquigarrow \mathcal{O}_X^E := i^!((\mathcal{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^r\overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}}\mathcal{O}^t_{X\times\mathbb{P}})[1]\simeq i^! R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}},\mathcal{O}^t_{X\times\mathbb{P}})[2]\simeq R\mathcal{H}om_{\pi^{-1}\mathcal{D}_{\overline{X}}}(\pi^{-1}\mathcal{O}_{\overline{X}}, Db^E_{X_{\mathbb{R}}})\in E^b(I\mathcal{D}_X)$$

$$\Omega_X^E := \Omega_X\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathcal{O}_X^E\simeq i^!(\Omega^t_{X\times\mathbb{P}}\overset{L}{\underset{\mathcal{D}_{\mathbb{P}}}{\otimes}}\mathcal{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})[1]\in E^b(I\mathcal{D}_X^{op}).$$

Remark. (cf. [G17 §5.2]) In terms of complexes,

$$Db_M^T = [Db^t_{M\times P}\xrightarrow{\partial_t-1}Db^t_{M\times P}] \quad ((-1)^{st}\to0^{th})$$

$$\mathcal{O}_X^E = [Db^T_{X_{\mathbb{R}}}\xrightarrow{\overline{\partial}_t}Db^{T,(0,1)}_{X_{\mathbb{R}}}\xrightarrow{\overline{\partial}_t}\cdots\xrightarrow{\overline{\partial}_t}Db^{T,(0,\dim_{\mathbb{C}}X)}_{X_{\mathbb{R}}}] \quad (0^{th}\to1^{st}\to\cdots\to(\dim_{\mathbb{C}}X)^{th}).$$

Remark. By [DK15, Thm. 8.2.2 & Cor. 8.2.3], $\mathcal{O}_X^E$ is a stable object in $E^b(I\mathbb{C}_X)$.

Definition (Enhanced de Rham and solution functors). [DK15, Def. 9.1.1] (cf. the last def. for notations)

$$\left.\begin{array}{l} X:\text{ complex manifold} \\ i: X\times\mathbb{R}_\infty\to X\times\mathbb{P}\text{ canonical}\end{array}\right\} \begin{array}{l} DR_X^E: D^b(\mathcal{D}_X)\to E^b(I\mathbb{C}_X),\ M\mapsto\Omega_X^E\overset{L}{\underset{\mathcal{D}_X}{\otimes}}M\simeq i^! DR^t_{X\times\mathbb{P}}(M\overset{D}{\boxtimes}\mathcal{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})[1] \\ Sol_X^E: D^b(\mathcal{D}_X)^{op}\to E^b(I\mathbb{C}_X),\ M\mapsto R\mathcal{H}om_{\mathcal{D}_X}(M,\mathcal{O}_X^E)\simeq i^! Sol^t_{X\times\mathbb{P}}(M\overset{D}{\boxtimes}\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}})[2].\end{array}$$

Notation: [DK15, §9.3]

$$\left.\begin{array}{l} X\supset Y:\text{ complex analytic hypersurface} \\ \mathcal{U}:=X\setminus Y;\quad \varphi\in\mathcal{O}_X(*Y) \\ \{t=\mathrm{Re}\,\varphi\}=\{(x,t)\in\mathcal{U}\times\mathbb{R}\mid t=\mathrm{Re}\,\varphi(x)\}\subset X\times P\end{array}\right\} \begin{array}{l} E^E_{\mathcal{U}|X}(\varphi):=\mathbb{C}^E_X\overset{+}{\otimes}R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}},\mathbb{C}_{t=\mathrm{Re}\varphi})\in E^b(I\mathbb{C}_X). \\ \qquad (\text{cf. } E^{\varphi}_{\mathcal{U}|X}\text{ before Prop. II.1.})\end{array}$$

The following proposition reveals (one of) the mission(s) of the auxiliary variable $t\in\mathbb{R}$ in $E^b(I\mathbb{C}_X)$:

Proposition V.1. [DK15, Lem. 9.3.1] Settings as in the above notation. Then ($d_X=\dim_{\mathbb{C}}X$)

$$DR_X^E(\mathcal{E}^{\varphi}_{\mathcal{U}|X})\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}},\mathbb{C}^E_X\overset{+}{\otimes}\mathbb{C}_{t=\mathrm{Re}\varphi})[d_X]\simeq E^E_{\mathcal{U}|X}(\varphi)[d_X] \text{ in } E^b(I\mathbb{C}_X).$$

(Proof.) $DR_X^E(\mathcal{E}^{\varphi}_{\mathcal{U}|X})\simeq i^! DR^t_{X\times\mathbb{P}}(\mathcal{E}^{\varphi}_{\mathcal{U}|X}\overset{D}{\boxtimes}\mathcal{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})[1]\simeq i^! DR^t_{X\times\mathbb{P}}(\mathcal{E}^{\varphi-\tau}_{\mathcal{U}\times\mathbb{C}|X\times\mathbb{P}})[1] \leftarrow \because \mathcal{E}^{\varphi}_{\mathcal{U}|X}\overset{D}{\boxtimes}\mathcal{E}^{-\tau}_{\mathbb{C}|\mathbb{P}}\simeq\mathcal{E}^{\varphi-\tau}_{\mathcal{U}\times\mathbb{C}|X\times\mathbb{P}}$

$\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}}, i^! \text{"}\varinjlim_{a\to+\infty}\text{"}\,\mathbb{C}_{\mathrm{Re}(\tau-\varphi)<a}[d_X+2])$ (Prop. II.1 + Prop. I.1 #2)

$\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}}, \text{"}\varinjlim_{a\to+\infty}\text{"}\,\mathbb{C}_{t-\mathrm{Re}\varphi<a}[d_X+1])$ ($\because \mathbb{C}_{\mathcal{U}\times\mathbb{R}}\otimes i^!\mathbb{C}_{\mathrm{Re}(\tau-\varphi)<a}\simeq\mathbb{C}_{t-\mathrm{Re}\varphi<a}[-1]$)

$\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}}, \text{"}\varinjlim_{a\to+\infty}\text{"}\,\mathbb{C}_{t-\mathrm{Re}\varphi\geq a}[d_X]) \leftarrow \because$ in $E^b(I\mathbb{C}_X)$, $[\overset{0}{\cancel{\mathbb{C}_{t-\mathrm{Re}\varphi\in\mathbb{R}}}}\to\mathbb{C}_{t-\mathrm{Re}\varphi\geq a}\xrightarrow{\sim}\mathbb{C}_{t-\mathrm{Re}\varphi<a}[1]\xrightarrow{+1}]$

$\simeq R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}}, \mathbb{C}^E_X\overset{+}{\otimes}\mathbb{C}_{t=\mathrm{Re}\varphi})[d_X] \leftarrow \because \mathbb{C}_{t\geq a}\overset{+}{\otimes}\mathbb{C}_{t=\mathrm{Re}\varphi}\simeq\mathbb{C}_{t-\mathrm{Re}\varphi\geq a}$

$\simeq\mathbb{C}^E_X\overset{+}{\otimes}R\mathcal{I}hom(\mathbb{C}_{\mathcal{U}\times\mathbb{R}},\mathbb{C}_{t=\mathrm{Re}\varphi})[d_X]$ (Lemma III.9 ; $\mathbb{C}_{\mathcal{U}\times\mathbb{R}}\simeq\pi^{-1}\mathbb{C}_{\mathcal{U}}$)

$= E^E_{\mathcal{U}|X}(\varphi)[d_X]$. #

Remark. Later we will prove a similar formula for $Sol^{E}_{X}(\mathcal{E}^{\varphi}_{U|X})$; the formula for $Sol^{E}_{X}(\mathcal{E}^{\varphi}_{U|X})$ is somewhat more concise in view of calculations, but to prove this formula we need more preparations on the notions "constructibility" and "duality" which are to be discussed in (B) and (C).

We will need the following Theorems V.2 and V.4 in (C).

Theorem V.2. [DK15, Thm.9.1.2] $f : \overset{d_X}{X} \to \overset{d_Y}{Y}$ complex analytic map. ($\pi : M \times \mathbb{R}_\infty \to M$)

1. $Ef^{!}\mathcal{O}^{E}_{Y}[d_Y] \simeq \mathcal{D}_{Y \leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}^{E}_{X}[d_X]$ in $E^b(I f^{-1}\mathcal{D}_Y)$.
2. $\forall N \in D^b(\mathcal{D}_Y)$, $DR^{E}_{X}(Df^{*}N)[d_X] \simeq Ef^{!}DR^{E}_{Y}(N)[d_Y]$ in $E^b(I\mathbb{C}_X)$.
3. $\forall M \in D^b_{good}(\mathcal{D}_X)$ (cf. Thm.I.3#3) s.t. $\operatorname{supp} M$ is proper over Y, $DR^{E}_{Y}(Df_{*}M) \simeq Ef_{!!}DR^{E}_{X}(M)$ in $E^b(I\mathbb{C}_Y)$.
4. $\forall L \in D^b_{rh}(\mathcal{D}_X)$ and $M \in D^b(\mathcal{D}_X)$, $DR^{E}_{X}(L \overset{D}{\otimes} M) \simeq R\mathcal{I}hom(\pi^{-1}Sol_X(L), DR^{E}_{X}(M))$.

   In particular, if Y is a closed hypersurface of X, $DR^{E}_{X}(M(*Y)) \simeq R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X \setminus Y}, DR^{E}_{X}(M))$.

(Proof.) They follow from Thm.I.3. #

Proposition V.3. [DK15, Prop.9.1.3] X : complex manifold.

For $L \in D^b_{rh}(\mathcal{D}_X)$, $DR^{E}_{X}(L) \simeq \mathbb{C}^{E}_{X} \otimes \pi^{-1}DR_X(L)$; in particular, $DR^{E}_{X}(\mathcal{O}_X) \simeq \mathbb{C}^{E}_{X}[d_X]$.

(Proof.) Nevertheless we shall prove $DR^{E}_{X}(\mathcal{O}_X) \simeq \mathbb{C}^{E}_{X}[d_X]$ first: With the map $a_X : X \to \{pt\}$,

$$DR^{E}_{X}(\mathcal{O}_X) = DR^{E}_{X}(Da_X^{*}\mathbb{C}_{\{pt\}}) \simeq Ea_X^{!}DR^{E}_{\{pt\}}(\mathbb{C}_{\{pt\}})[-d_X] \quad (\text{Thm.V.2}\#2)$$

$$\underset{\text{Step 2 of Prop.II.1}}{\simeq} Ea_X^{!}\mathbb{C}_{t<*}[1][-d_X] \underset{\text{in } E^b,\ [\mathbb{C}_{\mathbb{R}} \xrightarrow{0} \mathbb{C}^{E}_{\{pt\}} \xrightarrow{\sim} \mathbb{C}_{t<*}[1] \xrightarrow{+1}]\ (\text{cf. III(C)})}{\simeq} Ea_X^{!}\mathbb{C}^{E}_{\{pt\}}[-d_X] \simeq \mathbb{C}^{E}_{X}[d_X].$$

For general L: $DR^{E}_{X}(L) \simeq R\mathcal{I}hom(\pi^{-1}Sol_X(L), \mathbb{C}^{E}_{X}[d_X])$ (Thm.V.2#4 with $M = \mathcal{O}_X$)

$\simeq \mathbb{C}^{E}_{X} \overset{+}{\otimes} R\mathcal{I}hom(\pi^{-1}Sol_X(L), \mathbb{C}_{t=0}[d_X])$ (Lemma III.9)

$\simeq \mathbb{C}^{E}_{X} \otimes \pi^{-1}D_X(Sol_X(L)[d_X])$ ($D_X := R\mathcal{H}om(\cdot, \underset{\text{dualizing complex}}{\omega_X})$; $\omega_X \simeq \mathbb{C}_X[2d_X]$ here)

$\simeq \mathbb{C}^{E}_{X} \otimes \pi^{-1}DR_X(L)$. ($D_X(Sol_X(L)[d_X]) \simeq DR_X(L)$) #

Theorem V.4 (Enhanced de Rham and real blow-up). [DK15, §9.2] (cf. §IV for notations)

X : complex manifold
$\cup$
D : normal crossing divisor

$$\rightsquigarrow \begin{tikzcd} \widetilde{X} \equiv \widetilde{X}_D \arrow[d, "\varpi"] \\ X \end{tikzcd} \text{ and } \begin{tikzcd} \widetilde{X} \times \mathsf{P} \\ \widetilde{X} \times \mathbb{R}_\infty \arrow[u, "i"] \end{tikzcd}$$

(cf. §IV.(A)) Also, $\overset{\text{forgetful}}{for} : E^b(I(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}})^{op}) \to E^b(I\varpi^{-1}\mathcal{D}^{op}_{X})$.

Set $\Omega^{E}_{\widetilde{X}} = i^{!}(\Omega^{t}_{\widetilde{X}\times\mathsf{P}} \overset{L}{\underset{\mathcal{D}_{\mathsf{P}}}{\otimes}} \mathcal{E}^{-\tau}_{\mathbb{C}|\mathsf{P}})[1] \in E^b(I(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}})^{op})$ and $DR^{E}_{\widetilde{X}}(L) := \Omega^{E}_{\widetilde{X}} \overset{L}{\underset{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}}{\otimes}} L \in E^b(I\mathbb{C}_{\widetilde{X}})$ for $L \in D^b(\mathcal{D}^{\mathcal{A}}_{\widetilde{X}})$. Then

1. $E\varpi_{*}\Omega^{E}_{\widetilde{X}} \simeq R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \Omega^{E}_{X})$ in $E^b(I\mathcal{D}^{op}_{X})$.
2. $for(\Omega^{E}_{\widetilde{X}}) \simeq E\varpi^{!}R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \Omega^{E}_{X})$ in $E^b(I\varpi^{-1}\mathcal{D}^{op}_{X})$.
3. For $M \in D^b_{hol}(\mathcal{D}_X)$ such that $M \xrightarrow{\sim} M(*D)$, we have $DR^{E}_{X}(M) \simeq E\varpi_{*}DR^{E}_{\widetilde{X}}(M^{\mathcal{A}})$ and $DR^{E}_{\widetilde{X}}(M^{\mathcal{A}}) \simeq E\varpi^{!}DR^{E}_{X}(M)$.

(Proof.) 1. follows from Thm.IV.1#2, and 2. follows from Thm.IV.1#1.

3. We prove the 2nd isom. here, and the proof of the 1st isom. is similar (cf. [DK15, Cor.9.2.3]):

$$DR^{E}_{\widetilde{X}}(M^{\mathcal{A}}) = \Omega^{E}_{\widetilde{X}} \overset{L}{\underset{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}}{\otimes}} M^{\mathcal{A}} \simeq \Omega^{E}_{\widetilde{X}} \overset{L}{\underset{\varpi^{-1}\mathcal{D}_X}{\otimes}} \varpi^{-1}M \overset{2.}{\simeq} E\varpi^{!}(R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \Omega^{E}_{X}) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} M)$$

$$\simeq E\varpi^{!}(\Omega^{E}_{X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} (\mathcal{O}_X(*D) \overset{D}{\otimes} M)) \simeq E\varpi^{!}DR^{E}_{X}(M). \ \#$$

(B) DUALITY AND $\mathbb{R}$-CONSTRUCTIBILITY IN ENHANCED IND-SHEAVES

This section serves as a preparation for later results, so the pace will be rather rapid and many proofs will be refered to [DK15] and omitted. However, as a first application of these new tools, we will prove, following [DK15], the "$\mathbb{R}$-constructibility" of $DR^E_X(M)$ for $M \in D^b_{hol}(D_X)$ in the end of this section (cf. Thm. V.12).

Definition (Enhanced duality functor). [DK15, Def. 4.8.1] M: good topological space, k: field.

$$D^E_M : E^b(Ik_M) \to E^b(Ik_M)^{op},\ K \mapsto D^E_M K := \mathcal{I}hom^+(K, \omega^E_M). \quad (\text{cf. the "classical" } D_M := R\mathcal{H}om(\cdot, \underset{\substack{\uparrow\\ \text{put in objects in } D^b(k_M)}}{\omega_M}))$$

Proposition V.5. [DK15, Prop. 4.8.3 & Cor. 4.8.4]

1. For $F \in D^b(k_{M\times\mathbb{R}_\infty})$, $D^E_M(k^E_M \overset{+}{\otimes} F) \simeq k^E_M \overset{+}{\otimes} a^{-1} D_{M\times\mathbb{R}} F$ ($a : M\times\mathbb{R} \circlearrowleft$, $a(x,t) := (x,-t)$).
2. For $F \in D^b(k_M)$, $D^E_M(k^E_M \otimes \pi^{-1}F) \simeq k^E_M \otimes \pi^{-1} D_M F$.

Definition ($\mathbb{R}$-constructible objects). [Ka84 §2, KS90 §8.3, DK15 Def. 4.9.1 & 4.9.2] M: subanalytic space or real analytic manifold. (similar to complex analytic sp., cf. [DK15, Def 2.3.1])

1. Sheaf version: $F \in \mathrm{Mod}(k_M)$ is called $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigsqcup_{i\in I} M_i$ (i.e. each M_i subanalytic, $\{M_i\}$ locally finite, $M_i \subset \overline{M_j}$ if $M_i \cap \overline{M_j} \neq \emptyset$) such that each $F|_{M_i}$ is a local system (i.e. locally constant of finite rank).

   Let $D^b_{\mathbb{R}\text{-}c}(k_M) := \{F^\cdot \in D^b(k_M) \mid \text{each } H^i(F)\ (i\in\mathbb{Z}) \text{ is } \mathbb{R}\text{-constructible}\}$.

   subanalyticity: $Z \subset M$ is subanalytic if $\forall p \in M$, $\exists p \in W \underset{open}{\subset} M$, $r \in \mathbb{Z}_{>0}$, real analytic maps $f^{(\nu)}_j : N^{(\nu)}_j \to W$ ($\nu = 1,2$, $j = 1,\dots,r$), s.t. $Z \cap W = \bigcup_{j=1}^r (f^{(1)}_j(N^{(1)}_j) \setminus f^{(2)}_j(N^{(2)}_j))$.

2. Ind-sheaf version: (recall $\mathbb{R}_\infty = (\mathbb{R}, \overline{\mathbb{R}} = \mathbb{R} \sqcup \{\pm\infty\})$ & $M\times\mathbb{R}_\infty \xrightarrow{j_M} M\times\overline{\mathbb{R}}$)

   $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty}) := \{F \in D^b(k_{M\times\mathbb{R}_\infty}) \mid Rj_{M!}F \in D^b_{\mathbb{R}\text{-}c}(k_{M\times\overline{\mathbb{R}}})\}$.

   We have $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty}) \overset{full}{\subset} D^b(k_{M\times\mathbb{R}_\infty}) \overset{full}{\hookrightarrow} D^b(Ik_{M\times\mathbb{R}_\infty})$.

   Also, $D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty})$ is stable by $\overset{+}{\otimes}$, $\mathcal{I}hom^+$, $\otimes$ and $R\mathcal{I}hom$.

3. Enhanced ind-sheaf version:

$$\begin{array}{l} E^b_{\mathbb{R}\text{-}c}(Ik_M) := \{K \in E^b(Ik_M) \mid \forall \mathcal{U} : \overset{\text{relatively compact}}{\text{open subanalytic}} \subset M,\ \exists F \in D^b_{\mathbb{R}\text{-}c}(k_{M\times\mathbb{R}_\infty}) \text{ s.t. } \pi^{-1}k_{\mathcal{U}} \otimes K \simeq k^E_M \overset{+}{\otimes} F\}. \\ \cap \text{ full} \\ E^b(Ik_M) \end{array}$$

   Note: $K \in E^b_{\mathbb{R}\text{-}c}(Ik_M) \Rightarrow K$ stable (cf. §III(C)).

Remark (Characterization of $E^b_{\mathbb{R}\text{-}c}(Ik_M)$). [DK15, Lemma 4.9.9]

$K \in E^b(Ik_M)$ is $\mathbb{R}$-constructible if and only if

$\exists \{Z_i\}_{i\in I}$: locally finite family of locally closed subanalytic subsets of M,

$\exists$ finite sets A_i ($i \in I$), $\exists$ continuous subanalytic $\begin{cases} \varphi_{i,a} : Z_i \to \mathbb{R} \\ \psi_{i,a} : Z_i \to \mathbb{R} \cup \{+\infty\} \end{cases}$ with $\varphi_{i,a} < \psi_{i,a}$ ($i \in I, a \in A_i$),

↳ i.e. their graphs are subanalytic in $M \times \overline{\mathbb{R}}$

$\exists m_{i,a} \in \mathbb{Z}$ ($i \in I, a \in A_i$),

s.t. $M = \bigsqcup_{i\in I} Z_i$ and $\pi^{-1}k_{Z_i} \otimes K \simeq \bigoplus_{a\in A_i} k^E_M \overset{+}{\otimes} k_{\{(x,t)\in Z_i\times\mathbb{R} : \varphi_{i,a}(x) \le t < \psi_{i,a}(x)\}}[m_{i,a}]$.

Proposition V.6. [DK15, Prop.4.9.3 & 4.9.6]

1. If $K' \to K \to K'' \xrightarrow{+1}$ is a distinguished Δ in $E^b(\mathrm{I}k_M)$ and if $K', K \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$, then $K'' \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$.

2. For $K_1, K_2 \in E^b(\mathrm{I}k_M)$, $K_1 \oplus K_2 \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M) \Leftrightarrow K_1, K_2 \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$.

Notation (Enhanced support). [DK15, Not.4.9.10]

For $K \in E^b(\mathrm{I}k_M)$, $\mathrm{supp}^E(K) := \overline{\pi}(\mathrm{supp}(Rj_{M!!}L^E K)) \subset M$. (cf. §III (A) 1st notation)

Proposition V.7. [DK15, Prop.4.7.14] In this proposition $f: M \to N$ is a continuous map of good topological spaces. Then 1. For $K \in E^b(\mathrm{I}k_M)$, $Ef_{!!}(k^E_M \overset{+}{\otimes} K) \simeq k^E_N \overset{+}{\otimes} Ef_{!!}K$.

2. For $L \in E^b(\mathrm{I}k_N)$, $Ef^{-1}(k^E_N \overset{+}{\otimes} L) \simeq k^E_N \overset{+}{\otimes} Ef^{-1}L$ and $Ef^!(k^E_N \overset{+}{\otimes} L) \simeq k^E_M \overset{+}{\otimes} Ef^! L$.

Proposition V.8. [DK15, Prop.4.9.11] $f: M \to N$ continuous subanalytic map.

1. $Ef^{-1}, Ef^! : E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_N) \to E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ are well-defined.

2. For $K \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ such that $\mathrm{supp}^E(K)$ is proper over N, $Ef_{!!}K \simeq Ef_* K$ in $E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_N)$.

(This is a consequence of Prop. V.7.)

Theorem V.9. [DK15, Thm.4.9.12]

If $K \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$, then $D^E_M K \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ and $K \xrightarrow{\sim} D^E_M D^E_M K$ canonically.

(Proof.) 1. $D^E_M K \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$: May assume $K = k^E_M \overset{+}{\otimes} F$ for some $F \in D^b_{\mathbb{R}\text{-}c}(k_{M \times \mathbb{R}_\infty})$. Then

$$D^E_M K \simeq D^E_M(k^E_M \overset{+}{\otimes} F) \underset{\text{Prop.V.5\#1}}{\simeq} k^E_M \overset{+}{\otimes} a^{-1} \underbrace{D_{M \times \mathbb{R}} F}_{\in D^b_{\mathbb{R}\text{-}c}(k_{M \times \mathbb{R}_\infty})} \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M).$$

2. $K \xrightarrow{\sim} D^E_M D^E_M K$: The morphism is given by the image of id through

$$\mathrm{Hom}_{E^b}(\mathcal{I}hom^+(K, \omega^E_M), \mathcal{I}hom^+(K, \omega^E_M)) \simeq \mathrm{Hom}_{E^b}(\mathcal{I}hom^+(K, \omega^E_M) \overset{+}{\otimes} K, \omega^E_M)$$
$$\simeq \mathrm{Hom}_{E^b}(K, \mathcal{I}hom^+(\mathcal{I}hom^+(K, \omega^E_M), \omega^E_M)) = \mathrm{Hom}_{E^b}(K, D^E_M D^E_M K).$$

Then note that $D^E_M D^E_M K \simeq D^E_M(k^E_M \overset{+}{\otimes} a^{-1} D_{M \times \mathbb{R}} F) \simeq k^E_M \overset{+}{\otimes} D^2_{M \times \mathbb{R}} F \simeq k^E_M \overset{+}{\otimes} F \simeq K$. #

Prop. V.5#1

Proposition V.10. [DK15, Prop.4.9.13] For $K, K' \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$, the following are true:

1. $K \overset{+}{\otimes} K', \mathcal{I}hom^+(K, K') \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$

2. $D^E_M(K \overset{+}{\otimes} K') \simeq \mathcal{I}hom^+(K, D^E_M K')$; $D^E_M \mathcal{I}hom^+(K, K') \simeq K \overset{+}{\otimes} D^E_M K'$; $\mathcal{I}hom^+(K, K') \simeq \mathcal{I}hom^+(D^E_M K', D^E_M K)$.

(Proof.) $D^E_M(K \overset{+}{\otimes} K') = \mathcal{I}hom^+(K \overset{+}{\otimes} K', \omega^E_M) \simeq \mathcal{I}hom^+(K, \mathcal{I}hom^+(K', \omega^E_M)) = \mathcal{I}hom^+(K, D^E_M K')$. (*)

Thus $\mathcal{I}hom^+(K, K') \simeq D^E_M(K \overset{+}{\otimes} D^E_M K')$ (by Thm.V.9 & (*)). (**)

$K \overset{+}{\otimes} K' \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ is immediate, so $\mathcal{I}hom^+(K, K') \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ by (**) and Thm.V.9.

The rest can be done with the help of Thm.V.9. #

Proposition V.11. [DK15, Prop.4.9.22] $\begin{cases} f_1 : M_1 \to N_1 \\ f_2 : M_2 \to N_2 \end{cases}$ subanalytic, $f = (f_1, f_2) : M_1 \times M_2 \to N_1 \times N_2$, $\begin{matrix} L_1 \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_{N_1}) \\ L_2 \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_{N_2}) \end{matrix}$

$\Rightarrow Ef^{-1}(L_1 \overset{+}{\boxtimes} L_2) \simeq Ef_1^{-1} L_1 \overset{+}{\boxtimes} Ef_2^{-1} L_2$ and $Ef^!(L_1 \overset{+}{\boxtimes} L_2) \simeq Ef_1^! L_1 \overset{+}{\boxtimes} Ef_2^! L_2$. (cf. §III.(B) for def. of $\overset{+}{\boxtimes}$.)

Now we arrive at our first application of these tools.

Theorem V.12 (DR^E_X and $\mathbb{R}$-constructibility). [DK15, Thm. 9.3.2] X: complex manifold.

If $M \in D^b_{hol}(D_X)$, then $DR^E_X(M) \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$.

(Proof.) We apply Lemma IV.5 to the statement $P_X(M) =$ "$DR^E_X(M)$ is $\mathbb{R}$-constructible."

(i) and (ii) are obviously true.

(iii) $[M' \to M \to M'' \xrightarrow{+1}]$ in $D^b_{hol}(D_X) \Rightarrow [DR^E_X(M') \to DR^E_X(M) \to DR^E_X(M'') \xrightarrow{+1}]$ in $E^b(I\mathbb{C}_X)$.

Then apply Prop. V.6 #1.

(iv) $DR^E_X(M \oplus M') \simeq DR^E_X(M) \oplus DR^E_X(M')$. Then apply Prop. V.6 #2.

(v) $DR^E_Y(Df_* M) \simeq Ef_{!!} DR^E_X(M)$ by Thm. V.2 #3.

Then apply Prop. V.8 #2.

(vi) Let M be a holonomic D_X-module with a normal form along a normal crossing divisor D. Then $M^{\mathcal{A}}$ is locally a finite direct sum of some $(\mathcal{E}^{\varphi}_{X\setminus D|X})^{\mathcal{A}}$ for $\varphi \in \mathcal{O}_X(*D)$ (cf. §IV.(B)).

Now each $DR^E_{\tilde{X}}((\mathcal{E}^{\varphi}_{X\setminus D|X})^{\mathcal{A}}) \underset{\text{Thm.V.4\#3}}{\simeq} E\varpi^! DR^E_X(\mathcal{E}^{\varphi}_{X\setminus D|X}) \underset{\text{Prop.V.1}}{\simeq} E\varpi^! \underbrace{E^E_{X\setminus D|X}(\varphi)[d_X]}_{\overset{\cap}{E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)} \text{ by definition}} \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_{\tilde{X}})$ by Prop. V.8 #1

$\Rightarrow DR^E_{\tilde{X}}(M^{\mathcal{A}}) \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_{\tilde{X}})$ (by (i) or by noticing that $\mathbb{R}$-constructibility is a "local property")

cf. [DK15, 4.9.7–4.9.8]

$\Rightarrow DR^E_X(M) \underset{\text{Thm.V4\#3}}{\simeq} E\varpi_* DR^E_{\tilde{X}}(M^{\mathcal{A}}) \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$ by Prop. V.8 #2.

#

(C) DUALITY OF ENHANCED DE RHAM AND SOLUTION FUNCTORS

In this section we will focus on the interplay of enhanced de Rham and solution functors via duality; again, some fundamental but "functorial" properties will be stated but not be proved here, and their proofs will once more be refered to [DK15].

Proposition V.13. [DK15, Prop.8.2.4] X, Y : complex manifolds.

There is a canonical morphism $\mathcal{O}^E_X \overset{+}{\boxtimes} \mathcal{O}^E_Y \to \mathcal{O}^E_{X\times Y}$.

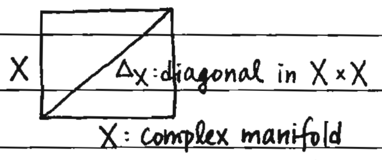
Theorem V.14 (DR^E_X and $\overset{+}{\boxtimes}$). [DK15, Thm.9.3.3] X, Y : complex manifolds.

For $M \in D^b_{hol}(\mathcal{D}_X)$ and $N \in D^b_{hol}(\mathcal{D}_Y)$, the morphism $DR^E_X(M) \overset{+}{\boxtimes} DR^E_Y(N) \to DR^E_{X\times Y}(M \overset{D}{\boxtimes} N)$ induced by Prop. V.13 is an isomorphism.

(Sketch of proof.)

1. Since in the isomorphism to be proved M and N are "independent with each other functorially," by Lemma IV.5 we may assume M and N are holonomic D-modules along normal crossing divisors $D_X \subset X$ and $D_Y \subset Y$ respectively.
2. By Thm. V.4 #3, $DR^E_X(M) \overset{+}{\boxtimes} DR^E_Y(N) \simeq E\varpi_{(X\times Y)*}(DR^E_{\widetilde{X}}(M^A) \overset{+}{\boxtimes} DR^E_{\widetilde{Y}}(N^A))$ and $DR^E_{X\times Y}(M \overset{D}{\boxtimes} N) \simeq E\varpi_{(X\times Y)*} DR^E_{\widetilde{X}\times\widetilde{Y}}(M^A \overset{D}{\boxtimes} N^A)$, so it suffices to show $DR^E_{\widetilde{X}}(M^A) \overset{+}{\boxtimes} DR^E_{\widetilde{Y}}(N^A) \simeq DR^E_{\widetilde{X}\times\widetilde{Y}}(M^A \overset{D}{\boxtimes} N^A)$ for $M = \mathcal{E}^{\varphi}_{X\setminus D_X|X}$ and $N = \mathcal{E}^{\psi}_{Y\setminus D_Y|Y}$ ($\varphi \in \mathcal{O}_X(*D_X)$, $\psi \in \mathcal{O}_Y(*D_Y)$). By Thm. V.4 #3 and Prop. V.11, $DR^E_{\widetilde{X}}(M^A) \overset{+}{\boxtimes} DR^E_{\widetilde{Y}}(N^A) \simeq E\varpi^!_{X\times Y}(DR^E_X(M) \overset{+}{\boxtimes} DR^E_Y(N))$ and $DR^E_{\widetilde{X}\times\widetilde{Y}}(M^A \overset{D}{\boxtimes} N^A) \simeq E\varpi^!_{X\times Y} DR^E_{X\times Y}(M \overset{D}{\boxtimes} N)$, so now it suffices to prove the original theorem for $M = \mathcal{E}^{\varphi}_{X\setminus D_X|X}$ and $N = \mathcal{E}^{\psi}_{Y\setminus D_Y|Y}$.
3. $DR^E_X(\mathcal{E}^{\varphi}_{X\setminus D_X|X}) \overset{+}{\boxtimes} DR^E_Y(\mathcal{E}^{\psi}_{Y\setminus D_Y|Y}) \simeq DR^E_{X\times Y}(\mathcal{E}^{\varphi}_{X\setminus D_X|X} \overset{D}{\boxtimes} \mathcal{E}^{\psi}_{Y\setminus D_Y|Y})$
$\overset{\text{Prop.V.1.}}{\Longleftrightarrow} E^E_{X\setminus D_X|X}(\varphi) \overset{+}{\boxtimes} E^E_{Y\setminus D_Y|Y}(\psi) \simeq E^E_{(X\setminus D_X)\times(Y\setminus D_Y)|X\times Y}(\varphi+\psi)$ (*).
Then prove $D^E[(*)]$ by utilizing Prop. V.5 and by noticing that D^E commutes with $\overset{+}{\boxtimes}$. #
cf. [DK15, Prop.4.9.21]

Definition (Adjunction in $E^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)$). [DK15, Def.9.4.5 & Lem.9.4.4]



An adjunction in $E^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)$ is a datum

$$\left(\underset{E^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)}{K_1, K_2}, \quad \begin{array}{c}\mathbb{C}^E_{\Delta_X}\\ \eta\downarrow\\ K_1\overset{+}{\boxtimes}K_2\end{array}, \quad \begin{array}{c}K_2\overset{+}{\boxtimes}K_1\\ \varepsilon\downarrow\\ \omega^E_{\Delta_X}\end{array}\right)$$ such that:

cf. [DK15, Lem.9.4.4]

- $[\mathbb{C}^E_{\Delta_X} \overset{+}{\boxtimes} K_1 \xrightarrow{\eta} K_1 \overset{+}{\boxtimes} K_2 \overset{+}{\boxtimes} K_1 \xrightarrow{\varepsilon} K_1 \overset{+}{\boxtimes} \omega^E_{\Delta_X}] \leftrightarrow \mathrm{id}_{K_1}$ via $\mathrm{Hom}_{E^b}(\mathbb{C}^E_{\Delta_X} \overset{+}{\boxtimes} K_1, K_1 \overset{+}{\boxtimes} \omega^E_{\Delta_X}) \simeq \mathrm{Hom}_{E^b}(K_1, K_1)$;
- $[K_2 \overset{+}{\boxtimes} \mathbb{C}^E_{\Delta_X} \xrightarrow{\eta} K_2 \overset{+}{\boxtimes} K_1 \overset{+}{\boxtimes} K_2 \xrightarrow{\varepsilon} \omega^E_{\Delta_X} \overset{+}{\boxtimes} K_2] \leftrightarrow \mathrm{id}_{K_2}$ via $\mathrm{Hom}_{E^b}(K_2 \overset{+}{\boxtimes} \mathbb{C}^E_{\Delta_X}, \omega^E_{\Delta_X} \overset{+}{\boxtimes} K_2) \simeq \mathrm{Hom}_{E^b}(K_2, K_2)$.

Proposition V.15 (Adjunction and duality). [DK15, Prop.9.4.6]

1. For $K \in E^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)$, there is a natural adjunction $(K, D^E_X K, \eta, \varepsilon)$.
2. If $(K_1, K_2, \eta, \varepsilon)$ is an adjunction in $E^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)$, then $K_2 \simeq D^E_X K_1$.

Theorem V.16. [DK15, Thm.9.4.8] X: complex manifold. ($d_X = \dim_{\mathbb{C}} X$)

For $M \in D^b_{hol}(\mathcal{D}_X)$, $D^E_X DR^E_X(M) \simeq DR^E_X(\mathbb{D}_X M)$. (Recall: $\mathbb{D}_X M := R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X]$)

(Sketch of proof.) Denote $\mathcal{B}_{\Delta_X} = D\delta_* \mathcal{O}_{\Delta_X}$ where $\delta : \Delta_X \hookrightarrow X \times X$ is the diagonal embedding.

Then there is a natural adjunction $(M, \mathbb{D}_X M, \mathcal{B}_{\Delta_X}[-d_X] \xrightarrow{\eta} M \overset{D}{\boxtimes} \mathbb{D}_X M, \mathbb{D}_X M \overset{D}{\boxtimes} M \xrightarrow{\varepsilon} \mathcal{B}_{\Delta_X}[d_X])$ in $D^b_{hol}(\mathcal{D}_X)$, i.e. they satisfy: ([DK15, 9.4.1–9.4.3])

$$\left.\begin{cases} [\mathcal{B}_{\Delta_X}[-d_X] \overset{D}{\boxtimes} M \xrightarrow{\eta} M \overset{D}{\boxtimes} \mathbb{D}_X M \overset{D}{\boxtimes} M \xrightarrow{\varepsilon} M \overset{D}{\boxtimes} \mathcal{B}_{\Delta_X}[d_X]] \leftrightarrow id_M \text{ and} \\ [\mathbb{D}_X M \overset{D}{\boxtimes} \mathcal{B}_{\Delta_X}[-d_X] \xrightarrow{\eta} \mathbb{D}_X M \overset{D}{\boxtimes} M \overset{D}{\boxtimes} \mathbb{D}_X M \xrightarrow{\varepsilon} \mathcal{B}_{\Delta_X}[d_X] \overset{D}{\boxtimes} \mathbb{D}_X M] \leftrightarrow id_{\mathbb{D}_X M} \\ \text{via } Hom_{D^b}(\mathcal{B}_{\Delta_X}[-d_X] \overset{D}{\boxtimes} M, M \overset{D}{\boxtimes} \mathcal{B}_{\Delta_X}[d_X]) \simeq Hom_{D^b}(M, M) \text{ and} \\ \quad Hom_{D^b}(\mathbb{D}_X M \overset{D}{\boxtimes} \mathcal{B}_{\Delta_X}[-d_X], \mathcal{B}_{\Delta_X}[d_X] \overset{D}{\boxtimes} \mathbb{D}_X M) \simeq Hom_{D^b}(\mathbb{D}_X M, \mathbb{D}_X M) \text{ respectively.} \end{cases}\right\} (*)$$

Applying $DR^E(\cdot)$ to $(*)$, using Thm.V.14 and substituting $\omega_X \simeq \mathbb{C}_X[2\dim_{\mathbb{C}} X]$ if X = complex manifold

$$DR^E_{X\times X}(\mathcal{B}_{\Delta_X}[-d_X]) \simeq \mathbb{C}^E_{\Delta_X} \text{ and } DR^E_{X\times X}(\mathcal{B}_{\Delta_X}[d_X]) \simeq \mathbb{C}^E_{\Delta_X}[2d_X] \simeq \omega^E_{\Delta_X}$$ (by Thm.V.2 #3 and Prop.V.3),

we find that $(DR^E_X(M), DR^E_X(\mathbb{D}_X M), DR^E_{X\times X}(\eta), DR^E_{X\times X}(\varepsilon))$ is an adjunction in $E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$

(note that $DR^E_X(M), DR^E_X(\mathbb{D}_X M) \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$ by Thm. V.12).

So Prop.V.15 #2 implies $DR^E_X(\mathbb{D}_X M) \simeq D^E_X DR^E_X(M)$. #

From the natural isomorphism "$DR^E_X(\mathbb{D}_X M) \simeq Sol^E_X(M)[d_X]$ for $M \in D^b(\mathcal{D}_X)$" and from the properties of DR^E_X derived so far, we can prove the following properties for Sol^E_X:

Proposition V.17 (Properties for Sol^E_X). [DK15, Cor.9.4.9–9.4.10] $f : \overset{d_X}{X} \to \overset{d_Y}{Y}$ complex analytic map.

1. For $M \in D^b_{hol}(\mathcal{D}_X)$, $Sol^E_X(M)[d_X] \simeq D^E_X(DR^E_X(M))$. ($\Rightarrow$ by Thm.V.9 & V.12, $Sol^E_X(M) \in E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$.)
2. For $N \in D^b_{hol}(\mathcal{D}_Y)$, $Sol^E_X(Df^* N) \simeq Ef^{-1} Sol^E_Y(N)$ in $E^b(I\mathbb{C}_X)$.
3. For $M \in D^b_{hol}(\mathcal{D}_X) \cap D^b_{good}(\mathcal{D}_X)$ such that $\operatorname{supp} M$ is proper over Y, $Sol^E_Y(Df_* M)[d_Y] \simeq Ef_* Sol^E_X(M)[d_X]$.
4. For $M \in D^b_{hol}(\mathcal{D}_X)$ and $N \in D^b_{hol}(\mathcal{D}_Y)$, $Sol^E_X(M) \overset{+}{\boxtimes} Sol^E_Y(N) \xrightarrow{\sim} Sol^E_{X\times Y}(M \overset{D}{\boxtimes} N)$.

Proposition V.18 (More properties for Sol^E_X). [DK15, Cor.9.4.11–12] $Y \subset X$ a closed hypersurface (X: complex analytic).

1. If $M \in D^b_{hol}(\mathcal{D}_X)$, then $Sol^E_X(M(*Y)) \simeq \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes Sol^E_X(M)$.
2. If $\varphi \in \mathcal{O}_X(*Y)$, then $Sol^E_X(\mathcal{E}^{\varphi}_{X\setminus Y|X}) \simeq \mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=-\mathrm{Re}\varphi}$.

(Proof.) 1. $Sol^E_X(M(*Y)) \simeq D^E_X DR^E_X(M(*Y))[-d_X]$ (Prop.V.17 #1)

$\simeq \mathcal{I}hom^+(R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus Y}, DR^E_X(M)), \omega^E_X)[-d_X]$ (Thm.V.2 #4)

$\simeq \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes \mathcal{I}hom^+(DR^E_X(M), \omega^E_X)[-d_X] = \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes D^E_X DR^E_X(M)[-d_X] \overset{\text{(Prop.V.17 \#1)}}{\simeq} \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes Sol^E_X(M)$. #

2. $Sol^E_X(\mathcal{E}^{\varphi}_{X\setminus Y|X}) \simeq D^E_X DR^E_X(\mathcal{E}^{\varphi}_{X\setminus Y|X})[-d_X]$ (Prop. V.17 #1)

$\simeq D^E_X R\mathcal{I}hom(\mathbb{C}_{(X\setminus Y)\times\mathbb{R}}, \mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=\mathrm{Re}\varphi})$ (Prop.V.1)

$\simeq \mathcal{I}hom^+(R\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus Y}, \mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=\mathrm{Re}\varphi}), \omega^E_X)$

$\simeq \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes \mathcal{I}hom^+(\mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=\mathrm{Re}\varphi}, \omega^E_X) = \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes D^E_X(\mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=\mathrm{Re}\varphi})$

$\simeq \pi^{-1}\mathbb{C}_{X\setminus Y} \otimes (\mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=-\mathrm{Re}\varphi})$ (Prop.V.5 #1)

$\simeq \mathbb{C}^E_X \overset{+}{\otimes} (\pi^{-1}\mathbb{C}_{X\setminus Y} \otimes \mathbb{C}_{t=-\mathrm{Re}\varphi}) \simeq \mathbb{C}^E_X \overset{+}{\otimes} \mathbb{C}_{t=-\mathrm{Re}\varphi}$. #

§ VI. RIEMANN-HILBERT CORRESPONDENCE FOR HOLONOMIC D-MODULES (MAIN THEOREM)

We shall use the techniques and results developed so far to give a quick sketch to the Riemann-Hilbert correspondence introduced in [DK15, Thm.9.5.3].

Definition VI.1 (The functor $\mathcal{H}om^E$). [DK15, Def.4.5.13] M = good topological space, $\Bbbk$ = field.

$\mathcal{H}om^E : E^b(I\Bbbk_M)^{op} \times E^b(I\Bbbk_M) \to D^b(\Bbbk_M)$ is defined by (cf. § III (A) 1st notation & § III (B) def. of L^E/R^E)

$$\left.\begin{aligned}\mathcal{H}om^E(K_1, K_2) &\simeq \alpha_M R\pi_* R\mathcal{I}hom(L^E K_1, L^E K_2)\\ &\simeq R\bar{\pi}_* R\mathcal{H}om(Rj_{M*}L^E K_1, Rj_{M*}L^E K_2).\end{aligned}\right\}$$

one may replace any L^E by R^E and any Rj_{M*} by $Rj_{M!!}$ here.

cf. [DK15, Lem.3.3.7 (iv)] and note $\alpha \circ R\bar{\pi}_* \cong R\bar{\pi}_* \circ \alpha$ (cf. Prop. IV.2 for def. of α)

Proposition VI.2. [DK15, Prop.9.5.1]

There is a functorial morphism in $D^b(D_X)$:

$$[D^b(D_X) \ni M \longrightarrow \mathcal{H}om^E(\mathcal{S}ol^E_X(M), \mathcal{O}^E_X)].$$

(Proof.)

$$\begin{tikzcd} X \times \mathbb{R}_\infty \arrow[d, "j_X"] \\ X \times \bar{\mathbb{R}} \arrow[d, "\bar{\pi}_X"] \\ X \end{tikzcd}$$

Observe that $Rj_{X*}R^E\mathcal{S}ol^E_X(M) \simeq R\mathcal{H}om_{\bar{\pi}_X^{-1}D_X}(\bar{\pi}_X^{-1}M, Rj_{X*}R^E\mathcal{O}^E_X)$

$\Rightarrow \exists$ morphism $\bar{\pi}_X^{-1}M \to R\mathcal{H}om(Rj_{X*}R^E\mathcal{S}ol^E_X(M), Rj_{X*}R^E\mathcal{O}^E_X)$ which induces the desired one through the following adjunction:

$$\begin{aligned}&\mathrm{Hom}(\bar{\pi}_X^{-1}M, R\mathcal{H}om(Rj_{X*}R^E\mathcal{S}ol^E_X(M), Rj_{X*}R^E\mathcal{O}_X))\\ &\simeq \mathrm{Hom}(M, R\bar{\pi}_*R\mathcal{H}om(Rj_{X*}R^E\mathcal{S}ol^E_X(M), Rj_{X*}R^E\mathcal{O}_X)) = \mathrm{Hom}(M, \mathcal{H}om^E(\mathcal{S}ol^E_X(M), \mathcal{O}^E_X)).\end{aligned}$$ #

Main theorem VI.3 (Riemann-Hilbert correspondence for holonomic D-modules). [DK15, Thm.9.5.3]

1. For $M \in D^b_{hol}(D_X)$, the morphism $[M \to \mathcal{H}om^E(\mathcal{S}ol^E_X(M), \mathcal{O}^E_X)]$ defined in Prop. VI.2 is an isomorphism.
   Thus we can reconstruct $M^{\in D^b_{hol}(D_X)}$ from $\mathcal{S}ol^E_X(M)$ or from $DR^E_X(M)$.
2. The functor $DR^E_X : D^b_{hol}(D_X) \to E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$ is fully faithful.
   Therefore, $\mathcal{S}ol^E_X : D^b_{hol}(D_X)^{op} \to E^b_{\mathbb{R}\text{-}c}(I\mathbb{C}_X)$ is also fully faithful.

(Proof of Main thm. VI.3 #1.)

Step 1: Reduction to the case "M is a holonomic D-module with a normal form along a normal crossing divisor." [DK15, $\overset{\text{Thm.9.6.1}}{\text{Lem.9.6.2}}$]

We would like to employ Lemma IV.5. Consider the statement $P_X(M) :=$ "$M \xrightarrow{\sim} \mathcal{H}om^E(\mathcal{S}ol^E_X(M), \mathcal{O}^E_X)$" where the morphism is defined by Prop. VI.2. Then $P_X(M)$ verifies Lemma IV.5 (i)~(iv) obviously.

Verify Lemma IV.5 (v): If $f : X \to Y$ projective and M : good holonomic D_X-module such that $P_X(M)$ is true, then $P_Y(Df_*M)$ is true because

$$\begin{aligned}\mathcal{H}om^E(\mathcal{S}ol^E_Y(Df_*M), \mathcal{O}^E_Y) &\simeq Rf_*\mathcal{H}om^E(\mathcal{S}ol^E_X(M), D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}\mathcal{O}^E_X) \quad \text{(Prop.V.17\#3, Thm.V.2\#1)}\\ &\simeq Rf_*(D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}\mathcal{H}om^E(\mathcal{S}ol^E_X(M), \mathcal{O}^E_X))\\ &\simeq Rf_*(D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}M) \quad (\because P_X(M) \text{ is true}) = Df_*M.\end{aligned}$$

Thus it remains to verify Lemma IV.5 (vi), i.e. our reduction is sucessful.

(Proof of Main thm. VI.3 #1, cont'd.)

Step 2: Prove Main thm. VI.3 #1 for M : holonomic D_X-module with a normal form along a normal crossing divisor $D \subset X$. (Set $\mathcal{U} = X \setminus D$) [DK15, 9.6.3–9.6.6]

(Sketch of proof.)

1. For $Y \subset X$ a complex analytic hypersurface and $\varphi \in \mathcal{O}_X(*Y)$, we have

$$R\pi_* R\mathcal{I}hom(L^E Sol^E_X(\mathcal{E}^\varphi_{X\setminus Y|X}), R^E\mathcal{O}^E_X) \simeq \mathcal{E}^\varphi_{X\setminus Y|X} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{O}^t_X \quad (\Rightarrow \mathcal{H}om^E(Sol^E_X(\mathcal{E}^\varphi_{X\setminus Y|X}), \mathcal{O}^E_X) \simeq \mathcal{E}^\varphi_{X\setminus Y|X});$$

the proof requires Prop. V.18 #2, the correspondence of ind-sheaves/ subanalytic sheaves (cf. [KS01, §7]) and the stability of $\mathcal{O}^E_X \in E^b(I\mathbb{C}_X)$ (cf. [DK15, Cor. 8.2.3]); details can be found in [DK15, 9.6.3–9.6.5].

2. Set $\tilde{X} = \tilde{X}_D$ and then set $Sol^E_{\tilde X}(L) := R\mathcal{H}om_{D^{\mathcal{A}}_{\tilde X}}(L, \mathcal{O}^E_{\tilde X}) \in E^b(I\mathbb{C}_{\tilde X})$ for $L \in D^b(D^{\mathcal{A}}_{\tilde X})$.
Then by using "$\mathcal{O}^E_{\tilde X} \simeq E\varpi^! R\mathcal{I}hom(\pi^{-1}\mathbb{C}_\mathcal{U}, \mathcal{O}^E_X)$" and $\varpi|_{\tilde X^{>0}} : \tilde X^{>0} \xrightarrow{\sim} \mathcal{U}$ etc., one can show that $\tilde\varpi^{-1}\pi^{-1}\mathbb{C}_\mathcal{U} \otimes Sol^E_{\tilde X}(M^{\mathcal{A}}) \simeq E\varpi^{-1} Sol^E_X(M)$ ($\tilde\varpi : \tilde X \times \mathbb{R}_\infty \to X \times \mathbb{R}_\infty$) (cf. [DK15, Lem. 9.6.6 pf. (i)])

3. We can show $M^{\mathcal{A}} \xrightarrow{\sim} \mathcal{H}om^E(Sol^E_{\tilde X}(M^{\mathcal{A}}), \mathcal{O}^E_{\tilde X})$: Indeed, may assume $M = \mathcal{E}^\varphi_{\mathcal{U}|X}$ with $\varphi \in \mathcal{O}_X(*D)$, and then one can show: ($\pi_{\tilde X} : \tilde X \times \mathbb{R}_\infty \to \tilde X$)

$$\begin{aligned}
\mathcal{H}om^E(Sol^E_{\tilde X}(M^{\mathcal{A}}), \mathcal{O}^E_{\tilde X}) &\simeq \alpha_X R\pi_{\tilde X *} R\mathcal{I}hom(L^E Sol^E_{\tilde X}(M^{\mathcal{A}}) \otimes \tilde\varpi^{-1}\pi^{-1}\mathbb{C}_\mathcal{U}, \tilde\varpi^! R^E\mathcal{O}^E_X) \\
&\overset{2.}{\simeq} \alpha_X R\pi_{\tilde X *} R\mathcal{I}hom(\tilde\varpi^{-1} L^E Sol^E_X(M), \tilde\varpi^! R^E\mathcal{O}^E_X) \\
&\simeq \alpha_X \varpi^! R\pi_* R\mathcal{I}hom(L^E Sol^E_X(M), R^E\mathcal{O}^E_X) \quad (\text{Prop. I.1 \#2 \& } \overset{\text{base change}}{\text{Prop. I.2}}) \\
&\overset{1.}{\simeq} \alpha_X \varpi^!(M \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{O}^t_X) \simeq \alpha_X \varpi^!(M \overset{L}{\underset{\mathcal{O}_X}{\otimes}} R\mathcal{I}hom(\mathbb{C}_\mathcal{U}, \mathcal{O}^t_X)) \quad (?) \\
&\simeq \alpha_X(\varpi^{-1}M \overset{L}{\underset{\varpi^{-1}\mathcal{O}_X}{\otimes}} \mathcal{O}^t_{\tilde X}) \simeq M^{\mathcal{A}} \quad (\text{Prop. IV.2}). \quad (\text{cf. [DK15, Lem. 9.6.6 pf. (ii)]})
\end{aligned}$$

4. End of proof: One can verify

$$M \simeq R\varpi_* M^{\mathcal{A}} \overset{3.}{\simeq} R\varpi_* \mathcal{H}om^E(Sol^E_{\tilde X}(M^{\mathcal{A}}), \mathcal{O}^E_{\tilde X}) \overset{\text{use 2. etc.}}{\simeq} \mathcal{H}om^E(\underset{\underset{Sol^E_X(M)}{\wr}}{\underbrace{E\varpi_{!!} E\varpi^{-1} \overset{\overset{\pi^{-1}\mathbb{C}_\mathcal{U} \otimes Sol^E_X(M)\ (\text{Prop. V.18 \#1})}{\wr}}{\boxed{Sol^E_X(M)}}}}, \mathcal{O}^E_X).$$

#

(Proof of Main thm. VI.3 #2.)

Lemma: For $K, K' \in E^b_{\mathbb{R}\text{-}c}(\mathrm{I}k_M)$ (M = good topological space, k = field),
[DK15, Prop. 4.9.13] $\mathcal{H}om^E(K, K') \simeq \mathcal{H}om^E(D^E_M K', D^E_M K)$.

(Proof.) $\mathcal{H}om^E(K, K') \simeq \mathcal{H}om^E(k^E_M, \mathcal{I}hom^+(K, K')) \overset{\text{Prop. V.10 \#2}}{\simeq} \mathcal{H}om^E(k^E_M, \mathcal{I}hom^+(D^E_M K', D^E_M K)) \simeq \mathcal{H}om^E(D^E_M K', D^E_M K)$.
$\mathcal{H}om^E(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathcal{H}om^E(K_1, \mathcal{I}hom^+(K_2, K_3))$ [DK15, Lem. 4.5.15] & $K \simeq k^E_M \overset{+}{\otimes} K$ #

Now, for $M, N \in D^b_{hol}(D_X)$, we have:

$$\begin{aligned}
\mathcal{H}om^E(DR^E_X M, DR^E_X N) &\simeq \mathcal{H}om^E(Sol^E_X N, Sol^E_X M) \quad (\text{Thm V.12, Lemma, Prop. V.17 \#1}) \\
&\simeq \mathcal{H}om^E(Sol^E_X N, R\mathcal{H}om_{D_X}(M, \mathcal{O}^E_X)) \\
&\simeq R\mathcal{H}om_{D_X}(M, \mathcal{H}om^E(Sol^E_X N, \mathcal{O}^E_X)) \overset{\text{Main thm. VI.3 \#1}}{\simeq} R\mathcal{H}om_{D_X}(M, N) \quad (*)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathrm{Hom}_{E^b}(DR^E_X M, DR^E_X N) &\simeq H^0 R\Gamma(X, \mathcal{H}om^E(DR^E_X M, DR^E_X N)) \leftarrow \text{For } K_1, K_2 \in E^b(\mathrm{I}k_M),\ \mathrm{Hom}_{E^b(\mathrm{I}k_M)}(K_1, K_2) \simeq H^0 R\Gamma(M, \mathcal{H}om^E(K_1, K_2)) \\
&\overset{(*)}{\simeq} H^0 R\Gamma(X, R\mathcal{H}om_{D_X}(M, N)) \simeq \mathrm{Hom}_{D^b}(M, N). \quad \text{([DK15, Lem. 4.5.14])}
\end{aligned}$$

#

§ VII. EXAMPLES

(A) SCALAR ORDINARY DIFFERENTIAL EQUATIONS (cf. [G17], [KS15, §4.5])

Scalar ordinary differential equations on $\mathbb{C}$ with a pole at $0 \in \mathbb{C}$ may be regarded as a $\mathcal{D}_{\mathbb{C}}$-module: Let $z \in \mathbb{C}$ be the coordinate. Then

$$\left[z^m \frac{dy}{dz} = -b(z)y \text{ with } b(z) \in \mathcal{O}_{\mathbb{C},0} \right] \longleftrightarrow \left[L^{m,b(z)} := \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} P^{m,b(z)} \text{ with } P^{m,b(z)} := z^m \partial_z + b(z) \in \mathcal{D}_{\mathbb{C}} \right].$$

In this example we want to use various de Rham/solution functors introduced in this survey to classify the "representatives" $L^{m,b}$ with $m \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{C}$ (constant function) up to $\mathcal{D}_{\mathbb{C}}$-isomorphism.

0. Removable singularity at 0: all $L^{m,0}$, $m \in \mathbb{Z}_{\geq 0}$, are isomorphic in $\mathrm{Mod}(\mathcal{D}_{\mathbb{C}})$; also, all $L^{0,b}$, $b \in \mathbb{C}$, are isomorphic in $\mathrm{Mod}(\mathcal{D}_{\mathbb{C}})$.

1. Regular singularity at 0 $[L^{1,b}]$: (cf. also [Ka84] for the "regular Riemann-Hilbert correspondence")
   - For $b \neq 1$, $\exists$ isomorphism $$\begin{tikzcd} L^{1,-b} \arrow[r, shift left, "Q \mapsto Q\partial_z", "\sim"'] & L^{1,-b+1} \arrow[l, shift left, "\frac{Qz}{-b+1} \mapsfrom Q"] \end{tikzcd}.$$
   - Applying $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\cdot, \mathcal{O}_{\mathbb{C}})$ to the short exact sequence $0 \to \mathcal{D}_{\mathbb{C}} \xrightarrow{\cdot P^{1,-b}} \mathcal{D}_{\mathbb{C}} \to L^{1,-b} \to 0$, one obtains $\mathcal{S}ol_{\mathbb{C}}(L^{1,-b}) \simeq \mathcal{E}xt^1_{\mathcal{D}_{\mathbb{C}}}(L^{1,-b}, \mathcal{O}_{\mathbb{C}}) \simeq \mathcal{O}_{\mathbb{C}}/P^{1,-b}\mathcal{O}_{\mathbb{C}}$
   $$\Rightarrow H^0(\mathcal{S}ol_{\mathbb{C}}(L^{1,-b})) \simeq \begin{cases} H^0_0 \simeq 0,\ H^0|_{\mathbb{C}^\times} = \text{locally constant sheaf of rank 1} & (b \in \mathbb{C} \setminus \mathbb{Z}) \\ \mathbb{C}_{\mathbb{C}^\times} & (b \in \mathbb{Z}_{>0}) \\ \mathbb{C}_{\mathbb{C}} & (b \in \mathbb{Z}_{\leq 0}) \end{cases}$$

   Therefore all $L^{1,b}$ are separated to the distinct isomorphism classes:

   ① $\{L^{1,b} \mid b \in \alpha + \mathbb{Z}\}$ ($\alpha \in \mathbb{C} \setminus \mathbb{Z}$; distinct α corresponds to distinct isom. class) ② $\{L^{1,b} \mid b \in \mathbb{Z}_{>0}\}$ ③ $\{L^{1,b} \mid b \in \mathbb{Z}_{\leq 0}\} \supset \{\text{these } L^{m,b} \text{ in item } 0.\}$.

2. Irregular (meromorphic) singularity at 0 $[L^{m,b}, m \geq 2, b \neq 0]$:

   2.1. Classical: $\mathcal{S}ol_{\mathbb{C}}(L^{m,b}) \simeq [\underset{(0)}{\mathcal{O}_X} \xrightarrow{P^{2,b}} \underset{(1)}{\mathcal{O}_X}]$ and $H^0(\mathcal{S}ol_{\mathbb{C}}(L^{m,b})) \simeq \mathbb{C}_{\mathbb{C}^\times}$, $H^1(\mathcal{S}ol_{\mathbb{C}}(L^{m,b})) \simeq \mathbb{C}_{\{0\}}$ $(b \in \mathbb{C}^\times)$

   $\rightsquigarrow$ $\mathcal{S}ol_{\mathbb{C}}$ provides no information.

   2.2. Tempered: Note that $\mathcal{S}ol^t_{\mathbb{C}} \simeq \mathcal{S}ol_{\mathbb{C}}$ on $D^b_{rh}(\mathcal{D}_{\mathbb{C}})$ (item 1) [KS01, Lem. 7.4.11].

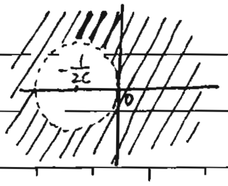
   $$\mathcal{DR}^t_{\mathbb{C}}(\underset{\underset{\mathcal{E}^{\varphi^{m,b}}_{\mathbb{C}^\times|\mathbb{C}}}{\wr}}{\boxed{L^{m,b}}}) \simeq R\mathcal{I}hom(\mathbb{C}_{\mathbb{C}^\times}, \mathbb{C}_{-\mathrm{Re}(\varphi^{m,b})<*})[1] \text{ by Prop. II.1.}$$
   with $\varphi^{m,b}(z) := \frac{1}{m-1}\frac{b}{z^{m-1}}$ $(m \geq 2, b \neq 0)$

   The only non-trivial cohomologies of $\mathcal{DR}^t_{\mathbb{C}}(L^{m,b})$ are: $(b \neq 0)$ $H^0(\mathcal{DR}^t_{\mathbb{C}}(L^{m,b})) \simeq \mathbb{C}_{\{0\}}$ (?);

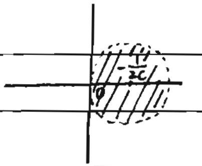
   $$H^{-1}(\mathcal{DR}^t_{\mathbb{C}}(L^{m,b})) \simeq \mathcal{I}hom(\mathbb{C}_{\mathbb{C}^\times}, \mathbb{C}_{-\mathrm{Re}(\varphi^{m,b})<*}) \simeq \underset{c \to +\infty}{\text{"lim"}} \mathcal{H}om(\mathbb{C}_{\mathbb{C}^\times}, \mathbb{C}_{\mathrm{Re}(\frac{b}{z^{m-1}}) > -(m-1)c})$$
   $$\simeq \underset{c \to +\infty}{\text{"lim"}} \mathbb{C}_{\mathrm{Re}(\frac{b}{z^{m-1}}) > -(m-1)c} \simeq \underset{c \to +\infty}{\text{"lim"}} \mathbb{C}_{\mathrm{Re}(\frac{\theta}{z^{m-1}}) > -c} \quad (\theta = \tfrac{b}{|b|} \in S^1).$$

   The graph of $\{\mathrm{Re}(\frac{1}{z^{m-1}}) > -c\}$ is illustrated in the following figures:

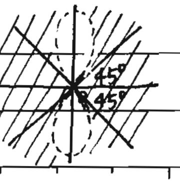
[m-1=1, c>0]



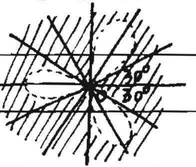
[m-1=1, c<0]



[m-1=2, c>0]



[m-1=3, c>0]



(2.2 Tempered case, cont'd.)

Therefore, $H^{-1}(DR^t_{\mathbb{C}}(L^{m,b})) \simeq H^{-1}(DR^t_{\mathbb{C}}(L^{m',b'})) \Leftrightarrow m=m',\ \arg(b)=\arg(b') \bmod 2\pi$;
thus, if we set $\mathcal{S}^{m,\theta} := \{L^{m,b} \mid \overset{b\neq 0 \&}{\arg(b)} = \theta \bmod 2\pi\}$ $(m \geq 2,\ 0 \leq \theta < 2\pi)$, then any two $L^{m,b}$ lying in two different $\mathcal{S}^{m,\theta}$ are non-isomorphic as $D_{\mathbb{C}}$-modules.

But we still don't know whether two $L^{m,b}$ lying in the same $\mathcal{S}^{m,\theta}$ are isomorphic or not.

2.3. Enhanced: Note first that by Prop. V.3, $DR^E_{\mathbb{C}} \simeq \mathbb{C}^E_{\mathbb{C}} \otimes \pi^{-1} DR_{\mathbb{C}}$ $(\pi : \mathbb{C} \times \mathbb{R}_\infty \to \mathbb{C})$ on $D^b_{rh}(D_{\mathbb{C}})$.

By Prop. V.18 #2,

$$Sol^E_{\mathbb{C}}(L^{m,b}) \simeq Sol^E_{\mathbb{C}}(\mathcal{E}^{\varphi^{m,b}}_{\mathbb{C}^\times|\mathbb{C}}) \simeq \mathbb{C}^E_{\mathbb{C}} \overset{+}{\otimes} \mathbb{C}_{t=-Re(\varphi^{m,b})} \simeq \underset{a\to+\infty}{\text{"}\varinjlim\text{"}} \mathbb{C}_{t \geq a - Re(\varphi^{m,b})} \in E^b(I\mathbb{C}_{\mathbb{C}}).$$

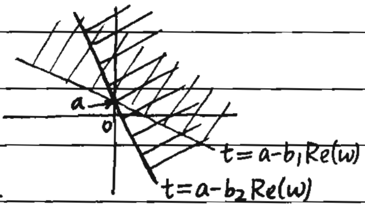
Suppose $L^{m,b} \in \mathcal{S}^{m,\theta}$. Then the above yields

$$Sol^E_{\mathbb{C}}(L^{m,b}) \simeq \underset{a\to+\infty}{\text{"}\varinjlim\text{"}} \mathbb{C}_{t \geq a - \frac{1}{m-1}|b| Re(\frac{e^{i\theta}}{z^{m-1}})} = \underset{a\to+\infty}{\text{"}\varinjlim\text{"}} \mathbb{C}_{t \geq a - \frac{|b|}{m-1} Re(e^{i\theta} w^{m-1})} \text{ with } w = \frac{1}{z}.$$

If $m=2$, $\theta=0$ $(\Rightarrow b \in \mathbb{R}_{>0})$ for example, the graph of

$$\{t \geq a - \frac{|b|}{m-1} Re(e^{i\theta} w^{m-1})\} = \{t \geq a - b\, Re(w)\}$$

looks like the right figure: (e.g. $a>0$)



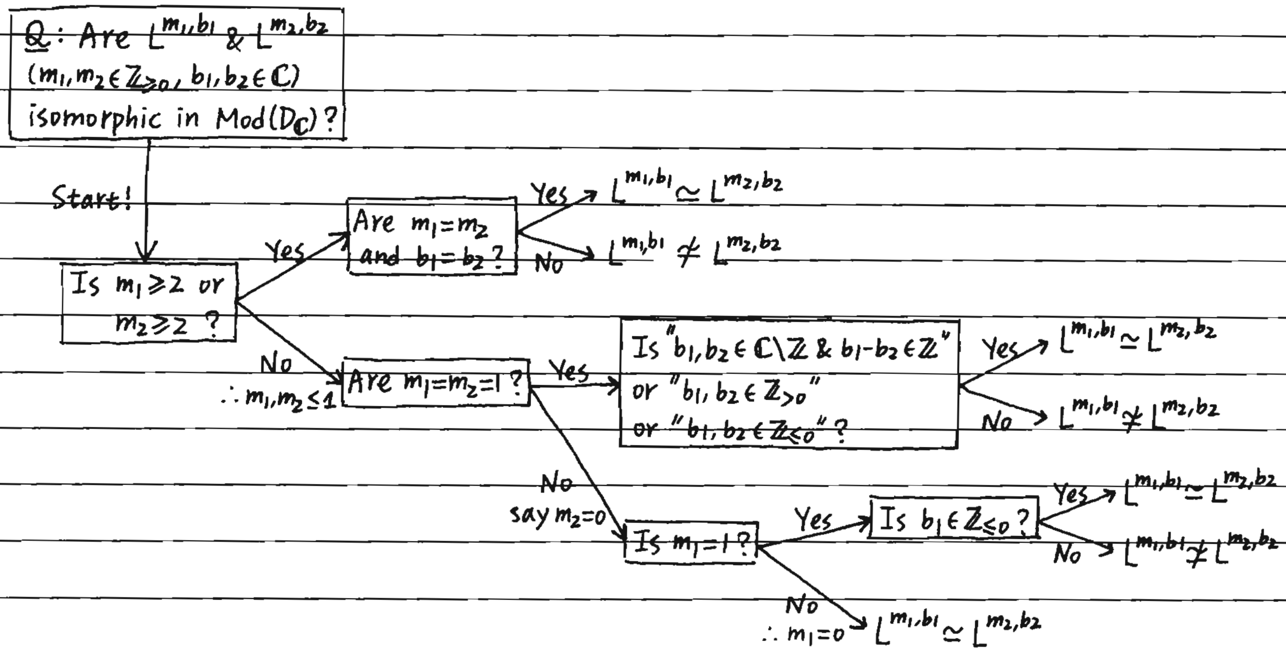
Thus for $L^{m,b_1}, L^{m,b_2} \in \mathcal{S}^{2,0}$, $Sol^E_{\mathbb{C}}(L^{m,b_1}) \simeq Sol^E_{\mathbb{C}}(L^{m,b_2}) \Leftrightarrow b_1 = b_2 (\in \mathbb{R}_{>0})$.

Similarly we can conclude:

"all elements in a fixed $\mathcal{S}^{m,\theta}$ $(m \geq 2,\ 0 \leq \theta < 2\pi)$ are mutually non-isomorphic."

Therefore, all elements in $\{L^{m,b} \mid m \geq 2,\ b \neq 0\}$ are mutually non-isomorphic as $D_{\mathbb{C}}$-modules.

Conclusion: We may summarize the above discussion as follows:



(B) STOKES PHENOMENA (cf. [DK15, §9.8] & [W65, §15])

For simplicity (and without loss of generality) we consider the following 2-dimensional case:

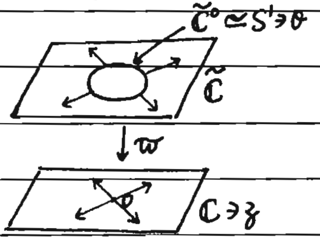
$\Big[$MODEL: $M_0 := \mathcal{E}^{\varphi}_{\mathbb{C}^\times|\mathbb{C}} \oplus \mathcal{E}^{\psi}_{\mathbb{C}^\times|\mathbb{C}}$ with $\varphi(z)=\alpha z^r$ and $\psi(z)=\beta z^r$ in $\mathcal{O}_{\mathbb{C}}(*0)$. $\Big] \leftarrow$ "irregular singularity at $z=0$"

$(\alpha\neq\beta\in\mathbb{C};\ z=$ coordinate on $\mathbb{C};\ 2\leq r\in\mathbb{Z})$

Now suppose M is a holonomic $D_{\mathbb{C}}$-module such that

$M\simeq M(*0)$, sing.supp$(M)=\{0\}$, and $\forall\,\theta\in\widetilde{\mathbb{C}}^0=S^1$, $\exists\,\theta\in I\underset{\text{open}}{\subset}S^1$

such that $M^{\mathsf{A}}|_{I\times\mathbb{R}_{\geq0}}\simeq M_0^{\mathsf{A}}|_{I\times\mathbb{R}_{\geq0}}$. $(*)$

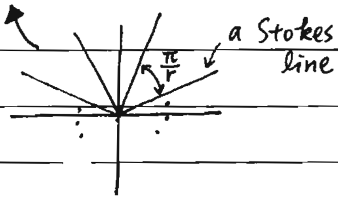


(We say loosely that "M has the normal form M_0.")

We "translate" the Stokes phenomena from the theory of ordinary differential equations into our framework of enhanced ind-sheaves introduced in this survey:

1. The Stokes lines (separation rays): Fix a $\theta_0\in\mathbb{R}$ so that $\alpha\overset{\overset{0}{\#}}{}-\beta=|\alpha-\beta|e^{i\theta_0}$.

They are the rays in $\{\mathrm{Re}(\varphi-\psi)=0\}=\{z\in\mathbb{C}^\times\mid\exists\,\theta\in\mathbb{R}\text{ s.t. }z=e^{i\theta}\text{ and }\cos(\theta_0+r\theta)=0\}\cup\{0\}\subset\mathbb{C}$.



Observation: For any open sector $S\subset\mathbb{C}^\times$ with vertex $0\in\mathbb{C}$,

$$\begin{cases} S\subset\{\pm\mathrm{Re}(\varphi-\psi)>0\} \Rightarrow \mathrm{End}_{E^b(I\mathbb{C}_X)}(\pi^{-1}\mathbb{C}_S\otimes(F\oplus G))\simeq b^{\pm} \\ S\text{ contains exactly one Stokes line} \Rightarrow \mathrm{End}_{E^b(I\mathbb{C}_X)}(\pi^{-1}\mathbb{C}_S\otimes(F\oplus G))\simeq t; \end{cases}$$

here, $F:=\mathbb{C}^E_X\overset{+}{\otimes}\mathbb{C}_{t=\mathrm{Re}\varphi}\simeq\underset{a\to+\infty}{\text{"}\varinjlim\text{"}}\mathbb{C}_{t-\mathrm{Re}\varphi\geq a}$;

$G:=\mathbb{C}^E_X\overset{+}{\otimes}\mathbb{C}_{t=\mathrm{Re}\psi}\simeq\underset{a\to+\infty}{\text{"}\varinjlim\text{"}}\mathbb{C}_{t-\mathrm{Re}\psi\geq a}$;

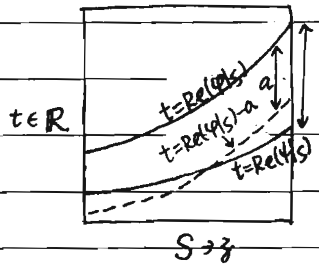
$b^{\pm}:=\{$upper(+)/lower(−) triangular matrices in $M_2(\mathbb{C})\}$;

$t:=b^+\cap b^-=\{$diagonal matrices in $M_2(\mathbb{C})\}$.

(Proof for the case $S\subset\{\mathrm{Re}(\varphi-\psi)>0\}$.)

$$\begin{aligned}\mathrm{End}_{E^b(I\mathbb{C}_X)}(\pi^{-1}\mathbb{C}_S\otimes(F\oplus G))&\simeq\mathrm{End}_{E^b(I\mathbb{C}_X)}(\mathbb{C}^E_X\overset{+}{\otimes}(\mathbb{C}_{t=\mathrm{Re}(\varphi|_S)}\oplus\mathbb{C}_{t=\mathrm{Re}(\psi|_S)}))\\ &\simeq\varinjlim_{a\to+\infty}\mathrm{Hom}_{E^b(I\mathbb{C}_X)}(\mathbb{C}_{t\geq-a}\overset{+}{\otimes}(\mathbb{C}_{t=\mathrm{Re}(\varphi|_S)}\oplus\mathbb{C}_{t=\mathrm{Re}(\psi|_S)}),\ \mathbb{C}_{t=\mathrm{Re}(\varphi|_S)}\oplus\mathbb{C}_{t=\mathrm{Re}(\psi|_S)})\ (\text{Prop.III.7})\\ &\simeq\varinjlim_{a\to+\infty}\mathrm{Hom}_{E^b(I\mathbb{C}_X)}(\mathbb{C}_{t\geq\mathrm{Re}(\varphi|_S)-a}\oplus\mathbb{C}_{t\geq\mathrm{Re}(\psi|_S)-a},\ \mathbb{C}_{t=\mathrm{Re}(\varphi|_S)}\oplus\mathbb{C}_{\mathrm{Re}(\psi|_S)})\\ &\simeq\varinjlim_{a\to+\infty}\left[M_a:=\begin{pmatrix}\mathrm{Hom}_{E^b}(\mathbb{C}_{t\geq\mathrm{Re}(\varphi|_S)-a},\mathbb{C}_{t=\mathrm{Re}(\varphi|_S)}) & \mathrm{Hom}_{E^b}(\mathbb{C}_{t\geq\mathrm{Re}(\psi|_S)-a},\mathbb{C}_{t=\mathrm{Re}(\varphi|_S)})\\ \mathrm{Hom}_{E^b}(\mathbb{C}_{t\geq\mathrm{Re}(\varphi|_S)-a},\mathbb{C}_{t=\mathrm{Re}(\psi|_S)}) & \mathrm{Hom}_{E^b}(\mathbb{C}_{t\geq\mathrm{Re}(\psi|_S)-a},\mathbb{C}_{t=\mathrm{Re}(\psi|_S)})\end{pmatrix}\right]\\ &\simeq b^+\end{aligned}$$

since $M_a=\begin{cases}b^+ & \text{if } a\geq0\\ 0 & \text{if } a<0\end{cases}$ by the following figure:



a gap of growth order $=|z|^r$ $(r\geq2>0)$

$\Downarrow$

$\exists\,z\in S$, $|z|\gg0$, s.t. $\mathrm{Re}(\psi(z))<\mathrm{Re}(\varphi(z))-a$

$\Downarrow$

$\mathrm{Re}(\varphi|_S)-a\geq\mathrm{Re}(\psi|_S)$ everywhere is impossible. #

From Thm.V.4 #3, Prop.V.1 & $(*)$,

$DR^E_{\mathbb{C}}(M)\simeq R\mathcal{H}om(\pi^{-1}\mathbb{C}_{\mathbb{C}^\times},H)[1]$ where

$H\in E^b(I\mathbb{C}_{\mathbb{C}})$ such that $\begin{cases}H\simeq\pi^{-1}\mathbb{C}_{\mathbb{C}^\times}\otimes H\\ \pi^{-1}\mathbb{C}_S\otimes H\simeq\pi^{-1}\mathbb{C}_S\otimes(F\oplus G)\text{ if } S\text{ sufficiently small}\end{cases}$

$\rightsquigarrow$ From Observation, Stokes lines are encoded in H, thus in $DR^E_{\mathbb{C}}(M)$.

2. The Stokes multipliers of M_0 correspond to the transition maps induced from gluing data of H.