COMPLEX ANALYSIS - NTU 2014
FINAL EXAM

There are four problems, each problem deserves 25 points.

1. Define the gamma function $\Gamma(s)$ for $\operatorname{Re} s > 0$ and prove its analytic continuation to $s \in \mathbb{C}$ and functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$. Find its zeros and poles as well as the corresponding residues. Hint: Prove first that

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \frac{\pi}{\sin \pi a}.$$

2. (a) Define the Riemann zeta function $\zeta(s)$ for $\operatorname{Re} s > 1$ and discuss its analytic continuation to $s \in \mathbb{C}$ with the only simple pole at $s = 1$. (b) Show that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$, and all zeros of $\zeta(s)$ for $\operatorname{Re} s < 0$ are precisely $s = -2n$, $n \in \mathbb{N}$.

(Hint: On way to do it is to consider $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ for $\operatorname{Re} s > 1$ and prove its functional equations $\xi(s) = \xi(1-s)$ thorough the one for $\vartheta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ where $t > 0$.)

3. (a) Prove the Schwarz Lemma for $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ and use it to determine the group $\operatorname{Aut} \mathbb{D}$. (b) Assuming the Schwarz–Christoffel formula from $\mathbb{H}$ to a polygon domain P. Deduce the formula for a conformal map $f : \mathbb{D} \to P$ by way of an explicit conformal map $g : \mathbb{D} \to \mathbb{H}$.

4. (a) Show that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for some $g_2, g_3 \in \mathbb{C}$. (b) Let $\Omega \subset \mathbb{C}$ be a simply connected domain not containing any root of $4x^3 - g_2 x - g_3$. Show that

$$I(w) := \int_{w_0}^{w} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}, \qquad w_0, w \in \Omega$$

defines a inverse of $\wp(z + a)$ for some a.

(Bonus) Write down something you have well prepared but not shown above.

A course by Chin-Lung Wang. Date: January 8, pm 12:30-3:20.