COMPLEX ANALYSIS - NTU 2010
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APRIL 27, PM 1:00 - 3:25

There are 7 problems, each deserves 15 points. Give your works in details. No partial credit will be assigned to non substantial solutions.

1. State and prove Cauchy's theorem for a triangle and then for a circle.

2. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

3. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}},$$

whenever $a > 1$.

4. Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,dx = \frac{\pi}{2a}\log a.$$

5. Let $\Omega \subset \mathbb{C}$ be a connected bounded open set containing 0, and $\phi : \Omega \to \Omega$ be holomorphic such that $\phi(0) = 0$ and $\phi'(0) = 1$. Show that ϕ is a linear function on the whole Ω.

6. Determine the number of roots of the equation

$$z^6 + 6z + 10 = 0$$

in each quadrant of the complex plane. Determine also the number of zeros inside each annulus $k < |z| < k+1$ with $k \in \mathbb{Z}_{\geq 0}$.

7. Deduce from Hadamard's theorem that if F is entire of growth order $\rho \notin \mathbb{Z}$, then F assumes every value $w \in \mathbb{C}$ infinitely many times.