COMPLEX ANALYSIS - NTU 2010

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JUNE 22, PM 12:20 - 3:20

There are 7 problems, each is of 15 pts except the first one which is of 10 pts.

1. Show that
$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

2. (a) For $0 < a < 1$, show that
$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \frac{\pi}{\sin \pi a}.$$
(b) Use it to derive the functional equation for Γ:
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \qquad \forall s \in \mathbb{C}.$$

3. (a) Prove that for $\operatorname{Re} s > 1$,
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$
(b) Show that $\zeta(s)$ is continuable to $s \in \mathbb{C}$ with only singularity a simple pole at $s = 1$.

4. Let $\psi(x) := \sum_{p^m < x} \log p = \sum_{n \le x} \Lambda(n)$ and $\psi_1(x) := \int_1^x \psi(u)\,du$. Show that
$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$
for any $c > 1$.

5. State and prove the Schwartz Lemma for $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$. Use it to determine the group $\operatorname{Aut} \mathbb{D}$ of bi-holomorphic maps on $\mathbb{D}$.

6. Describe the conformal mapping
$$f(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}}, \qquad z \in \mathbb{H}$$
in details. Describe also the inverse map of f.

7. (a) Show that $\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ where $e_i := \wp(\omega_i/2)$.
(b) Let $\Omega \subset \mathbb{C}$ be a simply connected domain not containing e_i. Show that
$$I(w) := \int_{w_0}^{w} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}, \qquad w_0, w \in \Omega$$
defines a inverse of $\wp(z + a)$ for some a.