COMPLEX ANALYSIS - NTU 2014
FINAL EXAM

There are four problem sets, each problem set deserves 30 points. You may work on each part independently by assuming the previous parts.

1. The gamma function is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$ for $\operatorname{Re} s > 0$.

(a) For $0 < a < 1$, show that

$$\int_0^\infty \frac{v^{a-1}}{1+v}\, dv = \frac{\pi}{\sin \pi a}.$$

(b) Extend $\Gamma(s)$ to all $s \in \mathbb{C}$ and derive the functional equation:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \qquad \forall s \in \mathbb{C}.$$

(c) Use the fact that the growth order of $1/\Gamma$ is one to show that

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^\infty \Big(1 + \frac{s}{n}\Big) e^{-s/n},$$

where $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$. And then derive

$$\frac{d^2}{ds^2} \log \Gamma(s) = \sum_{n=0}^\infty \frac{1}{(s+n)^2}.$$

2. The Riemann zeta function is define by $\zeta(s) = \sum_{n=1}^\infty 1/n^s$ for $\operatorname{Re} s > 1$.

(a) Let $f(x) = e^{-\pi x^2}$. Show that it has Fourier transform $\hat{f}(\xi) = e^{-\pi \xi^2}$. Then use the Poisson summation formula to derive the functional equation for $\vartheta(t) := \sum_{n=-\infty}^\infty e^{-\pi n^2 t}$ (defined for $t > 0$):

$$\vartheta(1/t) = \sqrt{t}\, \vartheta(t).$$

(b) Define $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ for $\operatorname{Re} s > 1$. Show that

$$\xi(s) = \int_0^\infty u^{\frac{s}{2}-1} \frac{\vartheta(u) - 1}{2}\, du,$$

and $\xi(s)$ has analytic continuation to all $s \in \mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Moreover $\xi(s) = \xi(1-s)$.

(c) Show that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$, and all zeros of $\zeta(s)$ for $\operatorname{Re} s < 0$ are precisely $s = -2, -4, -6, \cdots$.

A course by Chin-Lung Wang. Date: January 8, pm 12:30-3:20.

3. Conformal mappings. Let $\mathbb{D} = \{ z \mid |z| < 1 \}$ and $\mathbb{H} = \{ z \mid \operatorname{Im} z > 0 \}$.

(a) State and prove the Schwarz Lemma for $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$.

(b) Use (a) to show that the group $\operatorname{Aut} \mathbb{D}$ of bi-holomorphic maps on $\mathbb{D}$ consists of Möbius transformations of the form ($\theta \in \mathbb{R}$, $a \in \mathbb{D}$)

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}.$$

(c) Give an explicit conformal map $g : \mathbb{D} \to \mathbb{H}$. Assuming the Schwarz–Christoffel formula from $\mathbb{H}$ to a polygon domain P. Show that a similar formula holds for a conformal map $\mathbb{D} \to P$.

4. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice with $\tau = \omega_2/\omega_1 \in \mathbb{H}$, and $\wp(z)$ be the Weierstrass elliptic function, doubly periodic with respect to Λ.

(a) Let $\omega_3 := \omega_1 + \omega_2$. Show that $\wp'(z)$ has precisely 3 roots $\omega_i/2$ (mod Λ), $i = 1, 2, 3$, and all of them are simple. Moreover, $e_i := \wp(\omega_i/2)$, $i = 1, 2, 3$, are all distinct and

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

(b) Using Laurent expansion of $\wp(z)$ near $z = 0$ to show that

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for some $g_2, g_3 \in \mathbb{C}$.

(c) Let $\Omega \subset \mathbb{C}$ be a simply connected domain not containing any e_i, and $w_0 \in \Omega$. Show that

$$I(w) := \int_{w_0}^{w} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}, \qquad w \in \Omega$$

defines a inverse of $\wp(z + a)$ for some a.