COMPLEX ANALYSIS 2014
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Give your works in details. No partial credit will be assigned to non substantial solutions.

1. (15) Prove Goursat's theorem for a triangle and then deduce from it Cauchy's theorem for arbitrary piecewise C^1 curves inside a disk.

2. (15) Prove that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

3. (15) Prove that for $\xi \in \mathbb{R}$

$$\int_{-\infty}^\infty \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}.$$

4. (15) Prove that

$$\int_0^1 \log(\sin \pi x)\, dx = -\log 2.$$

5. (15) Determine the number of roots of the equation

$$z^6 + 8z^4 + z^3 + 2z + 3 = 0$$

in each quadrant of the complex plane. Determine also the number of zeros inside each annulus $k \le |z| < k+1$ with $k \in \mathbb{Z}_{\ge 0}$.

6. (10) Let f be an entire function such that for each $a \in \mathbb{C}$ at least one Taylor coefficient at $z = a$ is zero. Prove that f is a polynomial.

7. (10) If f is entire of growth order $\rho \notin \mathbb{Z}$, show that f assumes every value $w \in \mathbb{C}$ infinitely many times. Give an example of such f.

8. (10) Does Cauchy's residue theorem hold for functions with not just poles but also isolated essential singularities? Justify your answer.