CALCULUS - NTU 2011
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APRIL 14, PM 12:30 - 3:15

1. Let $f(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$ for $x \neq 0$ and $y \neq 0$, and $f(x,y) = 0$ otherwise.

(a) Evaluate $f_x(x,y)$ and $f_y(x,y)$ for $x \neq 0$ and $y \neq 0$.

(b) Evaluate $f_x(0,0)$ and $f_y(0,0)$, and show that f is differentiable at $(0,0)$. What is the tangent plane of the surface $z = f(x,y)$ at $(0,0,0)$?

(c) Evaluate $f_x(0,y)$ and $f_y(x,0)$. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

2. Calculate the line integrals:

(a) $\int z\,dx + x\,dy + y\,dz$ over the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1,0,0)$ to $(1,0,2\pi)$.

(b) $\int \dfrac{y\,dx + x\,dy}{1 + x^2 y^2}$ over the arc of $y = \sin \frac{1}{x}$ from $(1/2\pi, 0)$ to $(1/\pi, 0)$.

3. Let $k > -1$. Calculate $f(k) = \displaystyle\int_0^1 (x-1)\frac{x^k}{\log x}\,dx$.

4. Let $F(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Suppose that $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) > 0$, show that there exists a $\delta > 0$ and a unique C^1 function $f(x) : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ with $F(x, f(x)) = 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. That is, y can be solved uniquely as a C^1 function of x near x_0.

5. Consider the composite mapping $H = F \circ G : (x,y) \overset{G}{\mapsto} (\xi, \eta) \overset{F}{\mapsto} (u,v)$ with

$$\begin{cases} u = e^{\xi} \cos \eta, \\ v = e^{\xi} \sin \eta. \end{cases} \qquad \begin{cases} \xi = x/(x^2+y^2), \\ \eta = -y/(x^2+y^2). \end{cases}$$

(a) If $(x,y) \neq (0,0)$, show that the local inverse mapping $H^{-1}(u,v)$ exists and compute its Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(b) Compute the matrix $D(H^{-1})$ at $(u,v) = (e,0)$.

6. (a) Let A be a 2×2 matrix. When does A defines a conformal mapping on $\mathbb{R}^2$? (b) Let $(\xi, \eta) = (\phi(x,y), \psi(x,y))$ be a C^2 conformal mapping for $(x,y) \in U \subset \mathbb{R}^2$. If $f(\xi, \eta)$ satisfies the Laplace equation, show that $g(x,y) := f(\phi(x,y), \psi(x,y))$ also satisfies the Laplace equation.

7. Characterize all the critical points of the function $f(x,y) = (x^2/2 + y^2)e^{-x^2-y^2}$ on $\mathbb{R}^2$ into maximum, minimum and saddle points.

8. Find the values of a and b for the ellipse $x^2/a^2 + y^2/b^2 = 1$ of least area containing the circle $(x-1)^2 + y^2 = 1$.