CALCULUS FINAL EXAM - NTU 2011
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1. Consider the function $\mathbf{U} = \mathbf{F}(\mathbf{X}) = (x^2 - y^2, xy)$.

(a) Obtain an iterative approximation $\mathbf{G}(\mathbf{X})$, which depends on given $\mathbf{U}$, for the inverse transformation $\mathbf{F}^{-1}(\mathbf{U})$ near $\mathbf{X}_0 = (1,1)$ or $\mathbf{U}_0 = (0,1)$. Verify that the fixed point $\mathbf{X}_{\text{fixed}}$ of $\mathbf{G}$ satisfies $\mathbf{U} = \mathbf{F}(\mathbf{X}_{\text{fixed}})$.

(b) Show that there exists a $\delta > 0$ s.t. for any $\mathbf{U} \in B_\delta(\mathbf{U}_0)$ the iteration $\mathbf{X}_{n+1} = \mathbf{G}(\mathbf{X}_n)$ with initial value $\mathbf{X}_0$ converges to a limit, denoted by $\mathbf{X}(\mathbf{U})$.

2. Evaluate the integrals:

$$\text{(a)} \int_0^1 \int_y^1 e^{x^2}\, dx\, dy, \qquad \text{(b)} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} (x^2+y^2+z^2)xyz\, dx\, dy\, dz.$$

3. Evaluate the integral $\displaystyle\int_{\{x^2+y^2+z^2\le 1\}} e^{x+y+z}\, dx\, dy\, dz$.

4. Evaluate the improper integral

$$\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x} \cos x\, dx.$$

(State explicitly the theorem you use and check all the required conditions.)

5. Calculate

$$\int_S z\, dx \wedge dy - x\, dy \wedge dz,$$

where S is the spherical cap $x^2 + y^2 + z^2 = 1$, $x > 1/2$, oriented positively with respect to the normal pointing to infinity.

6. Prove Green's theorem on $\mathbb{R}^2$. Use it to derive one of the following: (1) The change of variable formula of double integrals with a C^2 transformation. (2) Stoke's theorem for oriented surface S with $\partial S = C$ in $\mathbb{R}^3$.

7. Derive the formula for Laplace operator on $\mathbb{R}^3$ in spherical coordinates. Use it to show that the only radial vector field $\mathbf{F}$ (i.e. $\mathbf{F}(\mathbf{r}) = a(r)\mathbf{r}$ for some function a in $r = |\mathbf{r}|$) with $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} = 0$, except possibility at $\mathbf{r} = 0$, is given by

$$\mathbf{F} = \frac{c\mathbf{r}}{r^3}.$$

8. It is known experimentally that a charge conducting spherical lamina exerts zero force on a point charge inside the sphere. Assume that point charges repel or attract each other with a force dependent only only the distance between them, prove that this experiment implies Columb's law.