Calculus A (Spring 2010) Quiz 1

March 10, 2010

Dept.____________________ ID No.____________________ Name:____________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Let $f(x, y)$ be a real-valued function. Show that if $f_x(x, y)$ and $f_y(x, y)$ are both continuous at $(0, 0)$, then $f(x, y)$ is differentiable at $(0, 0)$.

B. Consider the function $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

(a) Show that f is continuous at $(0, 0)$.

(b) Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist and evaluate their values.

(c) Show that f is not differentiable at $(0, 0)$.

C. Consider the function $f(x, y) = \begin{cases} x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y) & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Evaluate $f_x(x, y)$ and $f_y(x, y)$ for $x \neq 0$ and $y \neq 0$.

(b) Evaluate $f_x(0, 0)$ and $f_y(0, 0)$, and show that f is differentiable at $(0, 0)$. What is the tangent plane of the surface $z = f(x, y)$ at $(0, 0, 0)$?

(c) Evaluate $f_x(0, y)$ for $y \neq 0$ and $f_y(x, 0)$ for $x \neq 0$. Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

D. Let $f(x, y)$, $u(x, y)$ and $v(x, y)$ be C^2 functions. Suppose that $f_{xx} + f_{yy} = 0$ and $u_x = v_y$, $u_y = -v_x$, show that the function $\phi(x, y) = f(u(x, y), v(x, y))$ also satisfies $\phi_{xx} + \phi_{yy} = 0$.

Calculus A (Spring 2010) Quiz 2

March 24, 2010

Dept.____________________ ID No.____________________ Name:____________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Express $u_{xx} + u_{yy}$ in polar coordinate.

B. Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Show that $\dfrac{d}{dy}\displaystyle\int_a^b f(x, y)dx = \int_a^b \frac{\partial}{\partial y} f(x, y)dx$.

Hint. Every continuous function defined on a bounded and closed set $D \subset \mathbb{R}^n$ is uniformly continuous.

C. Consider the line integral $\displaystyle\int_\Gamma L$, where $L = Adx + Bdy + Cdz$ defined on $\mathbb{R}^3$. Prove that L is exact, that is, $L = df$ for some f on $\mathbb{R}^3$, if and only if the integral is independent to the path, which means that it only depends on the end points of Γ.

D. (a) Evaluate $\displaystyle\int zdx + xdy + ydz$ over the arc of the helix $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

(b) Evaluate $\displaystyle\int \frac{ydx + xdy}{1 + x^2y^2}$ over the arc of $y = \sin\dfrac{1}{x}$ from $(1/2\pi, 0)$ to $(1/\pi, 0)$.

Hint. Check if the differential form is exact.

Calculus A (Spring 2010) Quiz 3

April 7, 2010

Dept.________________ ID No.________________ Name:________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Let $F(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Suppose that $F(x_0,y_0) = 0$ and $F_y(x_0,y_0) > 0$, show that there exists a $\delta > 0$ and a unique C^1 function $f(x) : (x_0-\delta, x_0+\delta) \to \mathbb{R}$ s.t. $F(x, f(x)) = 0$ for all $x \in (x_0-\delta, x_0+\delta)$. That is, y can be solved uniquely as a C^1 function of x near x_0.

B. Let $F(x,y,z) = x + y + z - \sin xy - \sin yz - \sin xz$.

(a) Show that $F(x,y,z) = 0$ can be solve for $z = f(x,y)$, where f is a C^1 function, near $(0,0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$. What is the tangent plane of the surface $F(x,y,z) = 0$ at $(0,0,0)$?

C. Find the stationary points of $f(x,y) = x^3 + (y-x)(2y+x) - \dfrac{3}{2}y^2$ and determine whether they are local maximum, local minimum or saddle point.

D. Find the condition that the quadrilateral with given edges a, b, c, d includes the greatest area. And find its area.

Hint. Suppose the pairs a,b and c,d are adjacent. Let ϕ be the angle between a and b, ψ that between c and d. Express the area as a function of ϕ and ψ, and use cosine law to construct a constrain for ϕ and ψ.

Calculus A (Spring 2010) Quiz 4

April 28, 2010

Dept.________________ ID No.________________ Name:________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Consider a circle C, which lies in the xz-plane, with center $(a,0,0)$ and radius $r<|a|$. Let Γ be the *torus* obtained by rotating C about z-axis. Find the tangent plane of Γ at point $\left(\frac{a}{\sqrt{2}}+\frac{r}{2}, \frac{a}{\sqrt{2}}+\frac{r}{2}, \frac{r}{\sqrt{2}}\right)$.

B. Calculate the first fundamental form of the surface of revolution given by $r=\sqrt{x^2+y^2}=f(z)$, where f is a C^1 function, in terms of the cylindrical coordinates z and $\theta=\tan^{-1}\frac{y}{x}$.

C. Let S be the sphere $x^2+y^2+z^2=1$.

(a) Use stereographic projection from the north pole $(0,0,1)$ to the plane $z=0$ to obtain a parametric representation for $S\backslash\{(0,0,1)\}$.

(b) Show that the parametrization $\mathbf{r}(u,v):\mathbb{R}^2\to\mathbb{R}^3$ in (a) is conformal. That is, if two curves on $z=0$, which intersect at $(u,v,0)$, are orthogonal at $(u,v,0)$, then the two image curves on the sphere are also orthogonal at $\mathbf{r}(u,v)$.

D. Consider the function $\mathbf{U}=\mathbf{F}(\mathbf{X})=(x^2-y^2,xy)$.

(a) Obtain an iterative approximation $\mathbf{G}(\mathbf{X})$, which depends on given $\mathbf{U}$, for the inverse transformation $\mathbf{F}^{-1}(\mathbf{U})$ near $\mathbf{X}_0=(1,1)$ or $\mathbf{U}_0=(0,1)$. Verify that the fixed point $\mathbf{X}_{\text{fixed}}$ of $\mathbf{G}$ satisfies $\mathbf{U}=\mathbf{F}(\mathbf{X}_{\text{fixed}})$.

(b) Show that there exists a $\delta>0$ s.t. for any $\mathbf{U}\in B_\delta(\mathbf{U}_0)$ the iteration $\mathbf{X}_{n+1}=\mathbf{G}(\mathbf{X}_n)$ with initial value $\mathbf{X}_0$ converges to a limit, denoted by $\mathbf{X}(\mathbf{U})$.

Calculus A (Spring 2010) Quiz 5

May 12, 2010

Dept.____________________ ID No.____________________ Name:____________________

Make sure to give sufficient reason in each problem or you will NOT *get any credit for your answer.*

A1. Show that S is Jordan measurable in $\mathbb{R}^n$ if and only if ∂S has Jordan measure 0 in $\mathbb{R}^n$.

A2. Let S and T be any disjoint sets whose union has area. Show that $A^+(S) + A^-(T) = A(S \cup T)$.

B. (a) Find the volume common to the two cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$.

(b) Find the volume common to the three cylinders $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$ and $x^2 + y^2 \leq 1$.

C. (a) Evaluate the integral $\int_0^1 \int_y^1 e^{x^2} dxdy$.

(b) Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} (x^2 + y^2 + z^2) xyz dxdydz$.

D1. For $a, b, c \in \mathbb{R}$, evaluate the integral $\int_{\{x^2+y^2+z^2 \leq 1\}} \cos(ax + by + cz) dxdydz$.

D2. Evaluate the integral $\int_{\{x^2+y^2+z^2 \leq 1\}} e^{x+y+z} dxdydz$.

Hint. Consider the change of variable $\begin{cases} x' = (x + y + z)/\sqrt{3} \\ y' = (x - z)/\sqrt{2} \\ z' = (x - 2y + z)/\sqrt{6} \end{cases}$.

Calculus A (Spring 2010) Quiz 6

May 26, 2011

Dept.________________ ID No.________________ Name:________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Determine whether the improper integral $\iiint_{\mathbb{R}^3} \dfrac{dxdydz}{(1+x^2+y^2+z^2)^2}$ converges or diverges. If it converges, evaluate the integral.

B. Find the area of surface $\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

C. Find the volume of the n-simplex described by $x_k \ge 0$ for $k = 1, 2, \cdots, n$ and $\dfrac{x_1}{a_1} + \dfrac{x_2}{a_2} + \cdots + \dfrac{x_n}{a_n} \le 1$.

D. (a) Evaluate the improper integral $\displaystyle\int_0^\infty e^{-tx}\cos x\,dx$ for any fixed t.

(b) Show that the integral in (a) converges uniformly in t.

(c) Evaluate the integral $\displaystyle\int_0^\infty \frac{e^{-bx} - e^{-ax}}{x}\cos x\,dx$.

E. Assume we have known that the surface area of $z = f(x,y)$ in the region $(x,y) \in R$ is

$$\int_R \sqrt{1 + f_x^2 + f_y^2}\,dxdy$$

Now, there is a surface in the parametric form

$$x = \phi(u,v), \quad y = \psi(u,v), \quad z = \chi(u,v)$$

Show that if we consider a portion R' of u,v-plane where the Jacobian $\dfrac{d(x,y)}{d(u,v)}$ is not zero, then the surface area of the image of R' is

$$\int_{R'} \sqrt{EG - F^2}\,dudv$$

Where E, F, G are the elements of the first fundamental form.

F. Let the region R be an unbounded set. Assume that we can find $\{R_n\}$ s.t. $R_n \subset R_{n+1} \subset R$ in each of where $f(x,y)$ is continuous. Assume we have

$$S \subset R_n \text{ for some } n, \qquad \int_{R_n} |f(x,y)|dA < u,$$

for every bounded, closed set $S \subset R$ and u is independent to n, show that

$$I = \lim_{n\to\infty} \int_{R_n} f(x,y)dA$$

exists and is independent to the choice of approximating sequence R_n.

Calculus A (Spring 2010) Quiz 7

June 9, 2010

Dept.____________________ ID No.____________________ Name:____________________

Make sure to give sufficient reason in each problem or you will NOT get any credit for your answer.

A. Let $L = \dfrac{-ydx + xdy}{x^2 + y^2}$.

(a) Evaluate $\displaystyle\int_{C_\varepsilon} L$, where C_ε is a circle with center $(0,0)$ and radius ε.

(b) Show that for any piecewise C^1 simple closed curve C,

$$\int_C L = \begin{cases} 2\pi & \text{if (0,0) lies in the close curve} \\ 0 & \text{if (0,0) does not lie in the close curve} \end{cases}.$$

B. Use Green's Theorem to prove the change of variable formula

$$\iint_R f(x,y)dxdy = \iint_S f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv,$$

provided that the transformation $(u,v) \mapsto (x(u,v), y(u,v))$ from S to R is one-to-one and C^2 with positive Jacobian.

C. Calculate $\displaystyle\iint_S zdx \wedge dy - xdy \wedge dz$, where S is the spherical cap $x^2 + y^2 + z^2 = 1$, $x > 1/2$, oriented positively with respect to the normal pointing to infinity.

D. Let $\mathbf{F}(x,y,z) = (0,0,z)$ and S be the helicoid $\mathbf{r}(u,v) = (r\cos\theta, r\sin\theta, \theta)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Evaluate the flux $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{n}$ is the upward normal of S.