CALCULUS - NTU 2010
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NOVEMBER 9, PM 12:30 - 3:15

1. Let $f(x) = x^3$ on $\mathbb{R}$. Give an "ϵ-δ" proof that f is a continuous function.

2. Show that a continuous function on $[a, b]$ must be uniformly continuous.

3. Use Riemann sum to evaluate

$$\int_a^b \cos x \, dx.$$

4. (a) If $y' = ay$, show that $y = ce^{ax}$ for some c. (b) If $f(x+y) = f(x)f(y)$ and f is differentiable, show that either $f(x) \equiv 0$ or $f(x) = e^{ax}$ for some a.

5. State and prove the "Chain Rule" for composite functions.

6. Integrate

$$\text{(a)} \quad \int (\log x)^3 \, dx \qquad \text{(b)} \quad \int \sinh^{-1} x \, dx.$$

7. Integrate

$$\text{(a)} \quad \int \frac{dx}{2\cos x + \sin x + 1} \qquad \text{(b)} \quad \int x^2 \sqrt{a^2 - x^2} \, dx.$$

8. Sketch the graph of the function: $f(0) = 1$, $f(x) = (x^2)^x$ for $x \neq 0$. Show that f is continuous on $\mathbb{R}$. Has the function maxima, minima, or points of inflection?

9. For what values of $s \in \mathbb{R}$ is the improper integral

$$\text{(a)} \quad \int_0^\infty \frac{x^{s-1}}{1+x} \, dx \qquad \text{(b)} \quad \int_0^\infty \frac{\sin x}{x^s} \, dx \qquad \text{(c)} \quad \int_0^\infty \frac{|\sin x|}{x^s} \, dx$$

convergent? Justify your answers in detail.

10. (a) State and prove the generalized mean value theorem. (b) Use it to prove the following form of L'Hopital rule: If $f(x)$ and $g(x)$ both tend to ∞ as $x \to a$ and $\lim_{x \to a} f'(x)/g'(x) = L$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$