Solutions to Calculus Mid Exam. 11/9.

1. $|x^3 - a^3| = |x-a| \cdot |x^2 + xa + a^2| \leq M|x-a|$

where $M = 3(|a|+1)^2$

given $\varepsilon > 0$, pick $\delta = \varepsilon/M \Rightarrow |x^3 - a^3| < \varepsilon$

whenever $|x-a| < \delta$.

2. Let $f \in C([a,b])$, If f is NOT unif. cont.

then $\exists\ \varepsilon > 0$ st for $\delta_n = 1/n$, there are x_n, y_n

$\in [a,b]$, $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Let x_{n_k}, $k = 1, 2, 3, \dots$ be a conv. subsequence $\to \alpha$ as $k \to \infty$

claim: $\lim_{k\to\infty} y_{n_k} = \alpha$ too.

pf: $|y_{n_k} - \alpha| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \alpha| < \varepsilon_1$ for $k > k_1$

(given $\varepsilon_1 > 0$, $\exists\, k_1$ st. $|x_{n_k} - \alpha| < \varepsilon_1/2$ & $1/n_k < \varepsilon_1/2$).

But then $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$

$$\overset{\parallel}{|f(\lim_{k\to\infty} x_{n_k}) - f(\lim_{k\to\infty} y_{n_k})| = 0} \quad \times$$

3. Let $n \in \mathbb{N}$, $h = \dfrac{b-a}{n}$, $x_k = a + kh$, $k = 0, \dots, n$

$$F_n = \sum_{k=1}^{n} h \cos(a + kh)$$

$$F_n \cdot \sin\frac{h}{2} = \frac{h}{2} \cdot \sum_{k=1}^{n} \left(\sin\left(a + kh + \frac{h}{2}\right) - \sin\left(a + kh - \frac{h}{2}\right)\right)$$

$$= \frac{h}{2}\left(\sin\left(b + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right)$$

$$\Rightarrow \lim_{n\to\infty} F_n = \lim_{h\to 0} \frac{h/2}{\sin(h/2)} \lim_{h\to 0}\left(\sin\left(b+\frac{h}{2}\right) - \sin\left(a+\frac{h}{2}\right)\right)$$

$$= \sin b - \sin a.$$

4. (a) $(y e^{-ax})' = y' e^{-ax} + y e^{-ax}(-a) = (ay - ay) e^{-ax} = 0$

$\Rightarrow y e^{-ax} = \text{const} = c$ i.e. $y = c e^{-ax}$.

(b) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \dfrac{f(h) - 1}{h}$

$f(0) = f(0+0) = f(0)^2 \Rightarrow f(0) = 0$ or 1

$f(0) = 0 \Rightarrow f(x) = f(x+0) = f(x) \cdot f(0) = 0 \quad \forall x$

If $f(0)=1$, then $f'(x) = f(x)\cdot f'(0)$

$\Rightarrow \quad f(x) = c\, e^{f'(0)x}$, $\quad f(0)=1 \Rightarrow c=1$.

5. Chain Rule: $z=f(y)$, $y=g(x)$

$g'(a)$ exists, $f'(g(a))$ exists $\Rightarrow (f\circ g)'(a) = f'(g(a))\cdot g'(a)$.

pf: $\quad \Delta y = g'(a)h + \varphi(h)h \quad$ with $\lim\limits_{h\to 0}\varphi(h)=0$

$\Delta z = f'(g(a))k + \psi(k)k \quad$ with $\lim\limits_{k\to 0}\psi(k)=0$

this also holds for $k=0$ by defining $\psi(0)=0$.

Then $\Delta z = \left(f'(g(a)) + \psi(\Delta y)\right)\Delta y$

$= \left(f'(g(a)) + \psi(\Delta y)\right)\cdot\left(g'(a) + \varphi(\Delta x)\right)\Delta x$

$\Rightarrow \lim\limits_{\Delta x\to 0}\dfrac{\Delta z}{\Delta x} = f'(g(a))\cdot g'(a)$.

6. (a) $\displaystyle\int (\log x)^3 dx = x(\log x)^3 - \int \cancel{x}\, 3(\log x)^2 \cancel{\tfrac{1}{x}}\, dx$

similarly, $\displaystyle\int (\log x)^2 dx = x(\log x)^2 - 2\underbrace{\int \log x\, dx}_{= x\log x - x}$

$\Rightarrow \displaystyle\int (\log x)^3 dx = x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x$.

(b) $\displaystyle\int \sinh^{-1}x\, dx = x\sinh^{-1}x - \int \frac{x}{\sqrt{x^2+1}}dx \qquad$ let $u=x^2+1$, $du = 2x\,dx$

$\displaystyle = x\sinh^{-1}x - \int\frac{du}{2\sqrt{u}} \; (=\sqrt{u}) = x\sinh^{-1}x - \sqrt{x^2+1}$.

7. (a) Let $t=\tan x/2$, then $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$

$dx = \dfrac{2\,dt}{1+t^2}$

$\Rightarrow \displaystyle\int \frac{\frac{2\,dt}{1+t^2}}{\frac{2-2t^2}{1+t^2} + \frac{2t}{1+t^2} + 1} = -2\int\frac{dt}{t^2-2t-3}$

$\dfrac{1}{t^2-2t-3} = \dfrac{1}{(t-3)(t+1)} = \dfrac{a}{t-3} + \dfrac{b}{t+1} \qquad \begin{cases} a+b=0 \\ a-3b=1\end{cases} \Rightarrow \begin{array}{l} a = 1/4 \\ b=-1/4\end{array}$

integral $= -\dfrac{1}{2}\left(\displaystyle\int\frac{dt}{t-3} - \int\frac{dt}{t+1}\right) = -\dfrac{1}{2}\log\left|\dfrac{t-3}{t+1}\right| = \dfrac{1}{2}\log\left|\dfrac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}-3}\right|$.

(b) $\int x^3 \tan^{-1}x\,dx = \frac{1}{4}\int \tan^{-1}x\,dx^4$

$= \frac{1}{4}x^4\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$

$$\begin{array}{r|l} & x^2-1 \\ x^2+1 & x^4 \\ & x^4+x^2 \\ \hline & -x^2 \\ & -x^2-1 \\ \hline & 1 \end{array}$$

$= \frac{1}{4}x^4\tan^{-1}x - \frac{1}{4}\int(x^2-1)\,dx - \frac{1}{4}\tan^{-1}x$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad\quad -\frac{1}{12}x^3 + \frac{1}{4}x$.

8. $f(x) = (x^2)^x = e^{x\log x^2}$ is conti for $x \neq 0$

$\lim_{x\to 0} x\log x^2 = \lim_{x\to 0}\frac{\log x^2}{1/x} = \lim_{n\to\infty}\frac{-2\log n}{n} = 0$

hence $\lim_{x\to 0} f(x) = e^0 = 1 = f(1) \ni f$ is conti at $x=0$.

$f'(x) = e^{x\log x^2}\left(\log x^2 + x\cdot\frac{2x}{x^2}\right) = (x^2)^x\, 2(\log|x|+1)$

$f'(x) = 0 \iff \log|x| = -1$ i.e. $x = \pm 1/e$

Notice that $f'(0)$ does not exist, $\lim_{x\to 0} f'(0) = -\infty$ on both sides.

$f''(x) = (x^2)^x\cdot\left[4(\log|x|+1)^2 + \frac{2}{x}\right]$

$f''(x) > 0$ for $x > 0$

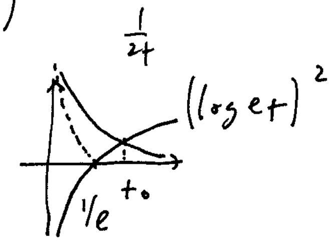
$f''(x) < 0$ for $x < 0$ and x near 0 ; > 0 when $x \to -\infty$.

let $x = -t$ $(t>0)$, claim : $\exists!\, t_0$ st $g(t) = (\log et)^2 - \frac{1}{2t} = 0$

$g'(t) = \frac{2e}{et}\log et + \frac{1}{2t^2} = \frac{2}{t}\left(\log et + \frac{1}{4t}\right)$

$g(t) \nearrow$ from $-\infty$ to $-\frac{1}{2/e}$ for $t \in (0, 1/e)$

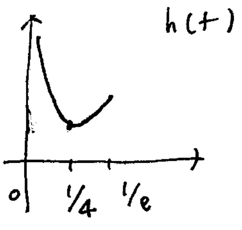
because $h(t) = \log et + \frac{1}{4t} > 0$:



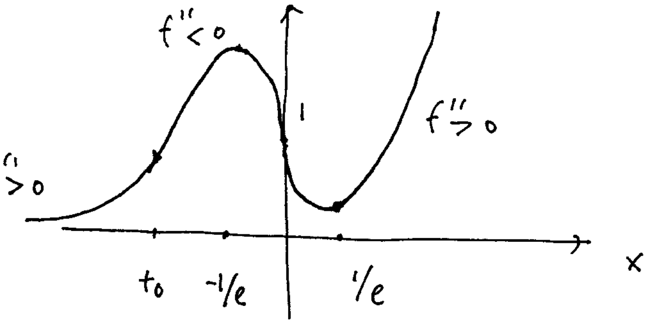
$h'(t) = \frac{1}{t} - \frac{1}{4t^2} = \frac{1}{t^2}\left(t - \frac{1}{4}\right)$

h has local min at $t = \frac{1}{4}$, $h(\frac{1}{4}) = \log\frac{e}{4} + 1$

$= \log e^2/4 > 0$.



graph of $(x^2)^x$:



maxima : $-1/e$

minima : $1/e$

inflection pts : t_0, 0.

9. (a) $\int_0^\infty \frac{x^{s-1}}{1+x}\,dx$ for $x \to 0$ need $s-1 > -1$ i.e. $s > 0$

for $x \to \infty$ get $O(x^{s-2})$ need $s-2 < -1$ ie $s < 1$.

the conv. values are $0 < s < 1$. The order is sharp for pos. fcn.

(b) $\int_0^\infty \frac{\sin x}{x^s}\,dx = \frac{1-\cos x}{x^s}\Big|_0^\infty - (-s)\int_0^\infty \frac{1-\cos x}{x^{s+1}}\,dx$

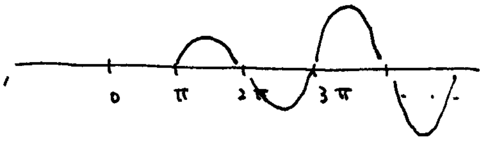
for $x \to 0$ $\frac{\sin x}{x^s} = O\left(\frac{1}{x^{s-1}}\right)$ need $s-1 < 1$ ie $s < 2$.

Since $\sin x \geq 0$ for $x \to 0$, this order estimate is sharp.

for $x \to \infty$, if $s+1 > 1$ ie. $s > 0$, get conv.

To see $s \leq 0$ is not conv., notice

$\int_0^\infty \frac{\sin x}{x^s}\,dx = \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x^s}\,dx$,



$= \sum_{n=1}^\infty a_n$, but $\lim_{n\to\infty} a_n \neq 0$.

So $a_1 + a_2 + a_3 + \cdots$ is not conv.

10. (a) GMVT : $f, g \in C([a,b])$ and f', g' exist on (a,b)

if $g' \neq 0$ on (a,b) then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

for some $\xi \in (a,b)$.

pf: Let $F(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))$

then $F(b) = 0 = F(a)$, hence $F'(\xi) = 0$ for some $\xi \in (a,b)$.

$g'(\xi) \neq 0 \Rightarrow$ GMVT *

(b) Given$_{17}$ $\varepsilon > 0$, $\exists\, \delta$ st $|x-a| < \delta \Rightarrow \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon/2$

By GMVT, for $|x-a| < \delta$, $|c-a| < \delta \Rightarrow \left|\frac{f(x)-f(c)}{g(x)-g(c)} - L\right| < \varepsilon/2$

Now $\left|\frac{f(x)}{g(x)} - L\right| \leq \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(c)}{g(x)-g(c)}\right| + \varepsilon/2$ fix one such c.

$$= \underset{\substack{\wedge \\ |L|+\varepsilon/2 \\ \wedge \\ |L|+1}}{\left|\frac{f(x)-f(c)}{g(x)-g(c)}\right|} \cdot \left| \overset{(*)}{\underset{\substack{\downarrow \\ 1}}{\frac{f(x)}{f(x)-f(c)}}} \cdot \underset{\substack{\downarrow \\ 1}}{\frac{g(x)-g(c)}{g(x)}} - 1 \right| \quad \text{as } x \to a$$

pick δ_1 st $|x-a| < \delta_1$

$\Rightarrow |(*) - 1| < \frac{\varepsilon/2}{|L|+1}$

Then $\left|\frac{f(x)}{g(x)} - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.