CALCULUS - NTU MATH: FINAL EXAM

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1. (10) In polar coordinate, if we use θ as parameter, show that $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2$. Draw a picture of the cardioid $r = 1 + \cos\theta$ and calculate its arc length.

2. (10) Show that $\sum_{k=2}^{\infty} \frac{1}{k(\log k)^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$.

3. (10) Show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ by using the Fourier series of $f(x) = x^2$.

4. (10) If $f_n \in C(\mathbb{R})$, $n \in \mathbb{N}$ and $f_n \to f$ uniformly, show that $f \in C(\mathbb{R})$. Give an example to show that the conclusion fails if the convergence is not uniform.

5. (10) Let $f(x)$ be a polynomial and $f(a) = 0$. Show that there is a neighborhood $I = (a - \delta, a + \delta)$ of a so the Newton iteration converges for any $x_0 \in I$.

6. (10) A mass m particle slides down along a curve $\mathbf{r}$ under gravity $\mathbf{F} = (0, -mg)$. Using the fact that the equation of motion is given by $d^2s/dt^2 = -g dy/ds$:

(a) Express the period T as an integral (at $t = 0$, $\theta = -\theta_0$ and $ds/dt = 0$).
(b) For $\mathbf{r}(\theta) = (a(\theta + \pi + \sin\theta), -a(1 + \cos\theta))$, show that $T = 2\pi\sqrt{a/g}$

7. (20) Prove Taylor's theorem and estimate $\sin 35^\circ$:

(a) State and prove Taylor's theorem in Lagrange's or Cauchy's form.
(b) Compute the Taylor expansion for $f(x) = \sin x$ at $x = \pi/6$ and determine its radius of convergence.
(c) How many terms are needed in estimating $\sin 35^\circ$ with error $< 10^{-9}$?

8. (20) Assume that $\sigma_n(t) := \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos nt = \dfrac{\sin(n + \frac{1}{2})t}{2 \sin \frac{t}{2}}$.

(a) Show that $\displaystyle\int_0^\pi \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} dt = \pi$ for all $n \in \mathbb{N}$
(b) Let $f \in PC^1([a, b])$, show that $\displaystyle\lim_{\lambda \to \infty} \int_a^b f(t) \sin \lambda t \, dt = 0$.
(c) Show that $\displaystyle\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$.
(d) State and prove Fejer's theorem on trigonometric approximation of piecewise continuous periodic functions.