CALCULUS — Chin-Lung Wang

1. (§12.4)
   (a) Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{c})\vec{b}$.
   (b) Prove that $(\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})] = [\vec{a} \cdot (\vec{b} \times \vec{c})]^2$.

2. (§12.6) Identify and sketch the surfaces
   (a) $4x^2 - y^2 + 2z^2 + 4 = 0$.
   (b) $x = y^2 + z^2 - 2y - 4z + 5$.

3. (§13.3) Given $\vec{r}(t)$ with arc length function $s(t)$. Let $'$ be the derivative in t.
   (a) Show that $d\vec{B}/ds$ is perpendicular to $\vec{B}$ and $\vec{N}$. Thus we may define τ by $d\vec{B}/ds = -\tau\vec{N}$. Then show that $d\vec{N}/ds = -\kappa\vec{T} + \tau\vec{B}$.
   (b) Derive formulae for $\vec{r}'$, $\vec{r}''$ and $\vec{r}'''$ in terms of $\vec{T}, \vec{N}, \vec{B}$ and s, κ, τ (and their derivatives).
   (c) Show that
   $$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}.$$
   (d) Show that
   $$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{|\vec{r}' \times \vec{r}''|^2}.$$

4. (§13.4) Let $\vec{r}(t)$ be a motion governed by Newton's Law $\vec{F} = m\vec{a}$ and the gravitational force. Following the steps to prove Kepler's Laws:
   (a) Show that $\vec{h} := \vec{r} \times \vec{v}$ is a constant vector.
   (b) Let $r = |\vec{r}|$ and $\vec{u} = \vec{r}/r$. Show that $\vec{a} \times \vec{h} = GM\vec{u}'$. and then show that there is a constant vector $\vec{c}$ such that $\vec{v} \times \vec{h} = GM\vec{u} + \vec{c}$.
   (c) By arranging the basis vectors $\vec{i}, \vec{j}, \vec{k}$ so that $\vec{k}$ is in the direction of $\vec{h}$ and $\vec{i}$ is in the direction of $\vec{c}$. Let θ be the angle between $\vec{c}$ and $\vec{r}$, $c = |\vec{c}|$ and $h = |\vec{h}|$. Show that
   $$r = \frac{h^2}{GM + c\cos\theta}.$$
   (d) Show that this represents a quadratic curve on the x-y plane.

5. (§14.7)
   (a) Classify all the critical points of $f(x, y) = x^4 + y^4 - 4xy + 1$ and sketch the graph of it.
   (b) Do the same for $f(x, y) = x^2ye^{-(x^2+y^2)}$.

6. (§14.8) The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
   (a) Graph the cone, the plane and the ellipse.
   (b) Use Lagrange multiplier to find the highest and lowest points on the ellipse.