PROBLEMS FOR CALCULUS 2009 - FINAL EXAM

CHIN-LUNG WANG

Problem 0.1. *Using Green's Theorem to derive the change of variable formula:*

$$\iint_\Omega f\,dxdy = \iint_D f \circ T \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where the map $T : (u,v) \mapsto (x(u,v), y(u,v))$ has continuous partial derivatives and maps D one to one and onto Ω.

Solution. Let $Q(x,y) = \int^x f(t,y)\,dt$. Then $Q_x = f$ and by Green's Theorem

$$\iint_\Omega f\,dxdy = \iint_\Omega Q_x\,dxdy = \int_{\partial\Omega} Q\,dy.$$

By the change of variable formula for one variable (line) integrals, this equals

$$\pm\int_{\partial D} Q(y_u\,du + y_v\,dv) = \pm\int_{\partial D} (Qy_u)\,du + (Qy_v)\,dv.$$

Here the sign depends on whether T preserves the orientation. By Green's Theorem again, now on the (u,v) plane, we get

$$\pm\iint_D \big((Qy_v)_u - (Qy_u)_v\big)\,dudv.$$

The integrand equals $(Q_x x_u + Q_y y_u)y_v + Qy_{vu} - (Q_x x_v + Q_y y_v)y_u - Qy_{uv}$, which simplifies to $Q_x(x_u y_v - x_v y_u) = f(x_u y_v - x_v y_u)$ by the definition of Q.

Finally, by the definition of orientation of T we have

$$\pm(x_u y_v - x_v y_u) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|.$$

The proof is complete. □

Problem 0.2. *The Maxwell equations without charges and currents state that*

$$\operatorname{div}\vec{E} = 0 = \operatorname{div}\vec{H}, \quad \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \quad \operatorname{curl}\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}.$$

Using them to prove the wave equation for the electronic field $\vec{E}$:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}.$$

Solution. From

$$\operatorname{curl}\operatorname{curl}\vec{E} = \nabla\operatorname{div}\vec{E} - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

(since $\operatorname{div}\vec{E} = 0$), we get

$$\nabla^2\vec{E} = -\operatorname{curl}\Big(-\frac{1}{c}\frac{\partial\vec{H}}{\partial t}\Big) = \frac{1}{c}\frac{\partial\operatorname{curl}\vec{H}}{\partial t} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2},$$

where in the middle equality we commute the derivatives in t and in x, y, z. □

Problem 0.3. *Let $\vec{F}$ be a conservative vector field on $\mathbb{R}^3$ with potential $P(x, y, z)$. State and prove the "Law of Conservation of Energy".*

Solution. For a moving particle $\vec{r}(t)$ of mass m, we need to show that

$$E(t) := \frac{1}{2}mv^2(t) + P(\vec{r}(t))$$

is a constant value in t.

Method 1: By differentiation in t and using $\vec{F} = -\nabla P$, we get

$$\begin{aligned} E'(t) &= \frac{1}{2}m(\vec{v}\cdot\vec{v})' + \nabla P \cdot \vec{r}'(t) \\ &= m\vec{a}\cdot\vec{v} - \vec{F}\cdot\vec{v} = \vec{F}\cdot\vec{v} - \vec{F}\cdot\vec{v} = 0. \end{aligned}$$

Thus $E(t)$ is independent of t.

Method 2: By computing the work $W = \int_C \vec{F}\cdot d\vec{r}$ done along C in two ways:

$$W(a, b) = \int_a^b \vec{m}\vec{a}\cdot\vec{v}\,dt = \int_a^b \frac{1}{2}m(\vec{v}\cdot\vec{v})'\,dt = \frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a).$$

On the other hand, since $\vec{F} = -\nabla P$,

$$W(a, b) = -\int_a^b \nabla P \cdot \frac{d\vec{r}}{dt}\,dt = -\int_a^b P(\vec{r}(t))'\,dt = -P(\vec{r}(b)) + P(\vec{r}(a)).$$

The conservation follows by substraction of the above two expressions. □

Problem 0.4. *Solve $y'' - 2y' + y = e^x/(1 + x^2)$.*

Solution. The characteristic equation for $y'' - 2y' + y' = 0$ is $(r - 1)^2 = 0$, hence the solutions for the homogenuous equation is given by

$$c_1e^x + c_2xe^x.$$

By variation of parameters we consider a particulat solution as

$$y_p = ue^x + vxe^x.$$

Then $y_p' = u'e^x + ue^x + v'xe^x + ve^x + vxe^x$.

We "need to" impose $u'e^x + v'xe^x = 0$, that is $u' + v'x = 0$. Then

$$y_p' = ue^x + ve^x + vxe^x = y_p + ve^x,$$

$$y_p'' - 2y_p' + y_p = (y_p + v'e^x + 2ve^x) - 2(y_p + ve^x) + y_p = v'e^x.$$

The equation to be solved becomes $v' = 1/(1 + x^2)$ and so $v = \tan^{-1} x$. Then $u' = -xv' = -x/(1 + x^2)$ and so $u = -\frac{1}{2}\ln(1 + x^2)$. So the general solutions are

$$y(x) = e^x\Big(c_1 + c_2x - \frac{1}{2}\ln(1 + x^2) + \tan^{-1} x\Big).$$

□

Problem 0.5. *Find all power series solutions to the (special case of) Bessel equation*

$$x^2y'' + xy' + x^2y = 0.$$

What can be concluded for the initial data $y(0)$ and $y'(0)$?

Solution. Let $y(x) = \sum_{n \geq 0} a_n x^n$. The equation becomes

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

There are no x^0 terms. For x^1 terms we get $a_1 x = 0$ hence $a_1 = y'(0) = 0$.

By comparing coefficients of x^k with $k \geq 2$ we get

$$a_k k(k-1) + a_k k + a_{k-2} = 0.$$

So

$$a_k = -\frac{a_{k-2}}{k^2}.$$

By iteration, if k is odd then we must have $a_k = 0$. For $k = 2m$ we get

$$a_{2m} = (-1)^m \frac{1}{(2m)^2 \times (2(m-1))^2 \cdots \times 2^2} a_0 = \frac{(-1)^m}{2^{2m}(m!)^2} a_0.$$

So all the power series solutions are

$$y(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}.$$

(The power series clearly onverges for all $x \in \mathbb{R}$.)

The initial condition $y(0) = a_0$ is arbitrary, but $y'(0) = a_1$ must be zero. □