CALCULUS 2017: MIDTERM EXAM I

1. (1) Draw the graph of $y = f(x) = \tan x$ and give the "standard definition" of $x = \tan^{-1} y$. Then compute $\cos(\tan^{-1} a)$.

(2) Find the largest interval of y such that the inverse function $x = g(y)$ is defined near $(x, y) = (3\pi, 0)$. Then compute $g(f(a))$ for $a = \pi/3$.

2. The Euler constant e is defined as $\lim_{n\to\infty}(1 + \frac{1}{n})^n$.

(1) Give the detailed explanation why the above limit exists.

(2) For $x > 0$, we define $\ln x = \log_e x$. Show that

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

(3) Using the fact that $y = e^x$ is the inverse function of $x = \ln y$ to show that $(e^x)' = e^x$.

3. Find dy/dx: (1) $y = \sqrt{1 + \sec(x^2)}$, (2) $y = x^{\ln x}$, (3) $\sin(xy) = x + y$ where y is regarded as an implicit function of x.

4. (1) Use linear approximations to show that

$$\sin x \sim x, \qquad \cos x \sim 1 - \tfrac{1}{2}x^2.$$

(2) By estimating the error term to show that $-\frac{1}{2}x^2 \leq \sin x - x \leq \frac{1}{2}x^2$.

5. Sketch the graph of functions by indicating the increasing property, min-max property, concave property, and asymptotic property:

$$(1) \quad y = \frac{x(x-2)}{x-1}, \qquad (2) \quad x^3 + y^3 = 1.$$

6. The Fermat principle says that light travels in its shortest time. Use this principle to derive Snell's law of refraction: when light passes through a boundary between two media M_1 and M_2 it obeys the rule that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where θ_i is the angle with the normal direction and v_i is its speed in M_i.

7. State and prove (1) Rolle's theorem, (2) generalized mean value theorem, and (3) derive the L'Hôpital rule: if $f(a) = g(a) = 0$, $f'(x), g'(x)$ exists and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$. How about if $a = \infty$?

Date: October 19, 2017, pm 3:30 – 6:00. Each problem is of 15 points. You may work on each part separately by assuming the other parts. (A course by Chin-Lung Wang at NTU.) .