CALCULUS 2018: MIDTERM EXAM

1. Taylor's formula in two variables. (10 + 10)

(a) Find the Taylor expansion of $f(x,y) = x^3 + 2y^3 - 2x^2y + y^2 + 5x - 3$ at $(-2,1)$.

(b) Find the Taylor expansion of $g(x,y) = \ln(x - xy)$ at $(1,0)$.

2. Gradient vector and tangent planes. (8 + 7 + 5)

(a) Given a function $w = f(x,y,z)$, define the directional derivative $\partial f/\partial \vec{u}$ of f at $P = (a,b,c)$ in the direction $\vec{u} = (u_1, u_2, u_3)$ and show that $\partial f/\partial \vec{u} = \nabla f(P) \cdot \vec{u}$.

(b) Let $P = (a,b,c)$ be a point on the surface $f(x,y,z) = 0$ in $\mathbb{R}^3$, show that $\nabla f(P)$ is the normal vector of the tangent plane at P.

(c) Find the tangent plane of $x^2 + y^2 - z^2 - 1 = 0$ at $P = (1,1,1)$.

3. Extremal value problems in two variables. (10 + 10)

(a) Find the critical points of $g(x,y) = \sin x + \sin y + \sin(x+y)$ and use the second derivative test to identify their natures.

(b) Use the method of Lagrange multiplier to find the extremal values of $f(x,y) = 2x + y^2$ subject to the constraint $(x-1)^2 + y^2 = 10$, and sketch a graph to verify your answer.

4. Double integrals and related applications. (10 + 10)

$$\text{(a)} \quad \int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy, \qquad \text{(b)} \quad \int_0^\infty e^{-x^2}\, dx.$$

5. Double integrals with general change of variables. (10 + 10)

$$\text{(a)} \quad \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-x)^2\, dy\, dx, \qquad \text{(b)} \quad \int_1^2 \int_{2u-2}^{u} e^{(v-u+1)^2}\, dv\, du.$$

6. Triple integrals. (4 + 4 + 4 + 8)

(a) Define spherical coordinates in $\mathbb{R}^3$ and show that $dA = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

(b) Calculate the volume bounded by $x^2 + y^2 + z^2 \le 1$ and $z \ge \sqrt{x^2 + y^2}$.

(c) Let $u = x + y + z$, $v = x - y$ and $w = y + 2z$. Calculate the Jacobian J such that $dx\, dy\, dz = |J|\, du\, dv\, dw$.

(d) Let $\Omega = \{(x,y,z) \mid 2x^2 + 3y^2 + 5z^2 + 6yz + 2xz \le 1\}$. Calculate

$$\iiint_\Omega (x+y+z)^2 dV.$$

Date: April 26, 2018, pm 3:30 – 6:30. Each problem is of 20 points. This is the the second semester course for Life Science by Chin-Lung Wang at NTU. **Important notices: (i) you may work on each part separately, (2) do give the details of your solutions.**