CALCULUS 2018: FINAL EXAM

1. (10 + 10) Extremal values of $g(x,y)$ under the given constraint.

(a) $xy = 2$, $g(x,y) = x + 2y$.
(b) $x^2 + y^2 \le 1$, $g(x,y) = x^2 - 3xy + y^2$.

2. (7 + 7 + 6) Solving differential equations in $y = y(t)$.

(a) $y' = y(1-y)$, $y(0) = 1/2$
(b) $y' + 2ty = 4t$, $y(0) = 5$.
(c) $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

3. (7 + 7 + 6) The Lokta–Volterra model for two species Q and P is

$$Q'(t) = aQ - bQP, \qquad P'(t) = -cP + dQP,$$

where $a, b, c, d > 0$. Assume that $Q(t_0) > 0$, $P(t_0) > 0$.

(a) Discuss the equilibrium solution and the increasing/decreasing behavior of $Q(t)$ and $P(t)$ on the phase plane (Q, P). Draw a picture.
(b) Find the solution curve $f(Q, P) = C$ on the phase plane.
(c) Show that the solution $(Q(t), P(t))$ is periodic in t.

4. (7 + 7 + 6) Gamma function and its properties.

(a) Show that $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t}$ is defined for $\alpha > 0$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.
(b) Find a formula for $\Gamma(\frac{1}{2})$ and $\Gamma(n + \frac{1}{2})$ where $n \in \mathbb{N}$.
(c) ($\Gamma(\alpha, \beta)$-distribution) Show that $f_X(t) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}$ is a probability density function and compute $\mathrm{E}(X)$, $\mathrm{Var}(X)$.

5. (7 + 7 + 6) Given two random variables X, Y with probability density $f_X(t)$ and $f_Y(t)$, $t \in \mathbb{R}$. Assume that X and Y are independent in the sense that the joint density function $f_{XY}(t,s) = f_X(t) f_Y(s)$. Let $Z = X + Y$.

(a) Show that $f_Z(t) = \int_{-\infty}^\infty f_X(t-v) f_Y(v)\, dv$.
(b) Let $f_X(t) = f_Y(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$. Find $f_Z(t)$.
(c) Let $f_X(t) = f_Y(t) = \lambda e^{-\lambda t}$ for $t \ge 0$, and $= 0$ for $t < 0$. Find $f_Z(t)$.

6. (7 + 5 + 8) Poisson distribution and normal distribution.

(a) Derive the formula $P(k, T) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$ as a limit of Bernoulli process when $N \to \infty$, where $P(1, \Delta t) = \lambda \Delta t$, $\Delta t = T/N$.
(b) The average service time at a counter is 2 minutes. Find the probability distribution $f_X(t)$ for the waiting time $X = t$. Suppose two persons are ahead of you, find the probability that you need to wait for ≥ 6 minutes.
(c) Derive the normal distribution $X \sim N(\mu, \sigma^2)$ that $f_X(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$.

Date: June 26, 2018, pm 3:30 – 6:30. Calculus for Life Science by Chin-Lung Wang at NTU. **You may work on each part separately. Give the details of your solutions.**