2018 ALGEBRA QUIZ: II

1. Let R be a commutative ring and $\{e_i\}_{i=1}^n$ a base for R^n. Put $f_i = \sum a_{ij} e_j$ where $A = (a_{ij}) \in M_n(R)$. Show that $\{f_i\}_{i=1}^n$ form a base for a free submodule K of $R^n \Leftrightarrow \det A$ is not a zero divisor. Also $(\det A)\bar{x} = 0$ for any $\bar{x} \in R^n/K$.

2. Let D be a p.i.d., $a_{11}, \ldots, a_{1n} \in D$ with $\gcd(a_{11}, \ldots, a_{1n}) = 1$. Show that there exists $a_{kj} \in D$, $2 \le k \le n$, $1 \le j \le n$ such that $(a_{ij}) \in M_n(D)$ is invertible.

3. Let D be a p.i.d., N a pure* submodule of a finitely generated torsion module M over D. Show that N is a direct summand.

4. State and prove the fundamental theorem of f.g. abelian groups**.

5. Let $E = F(u)$, u transcendental, and let $K \neq F$ be a subfield of E/F. Show that u is algebraic over K.

6. Construct a splitting field E for $x^5 - 2$ over $\mathbb{Q}$ and compute $[E : \mathbb{Q}]$.

* A submodule N of M is pure if "for any $y \in N$, $a \in D$, $ax = y$ is solvable in $M \Leftrightarrow$ it is solvable in N".

** You may assume the result that every subgroup of $\mathbb{Z}^n$ is free of rank $\le n$, but not the other part of the remaining proofs.

Show your answers/computations/proofs in details. Time and place: pm 1:20 – 3:00, December 21, 2018 at AMB 101. A course by Chin-Lung Wang at NTU..