2018 ALGEBRA MIDTERM EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. Determine the conjugacy classes in S_5 and the number of elements in each class. Use it to show that the only normal subgroups of S_5 are $1, A_5, S_5$.

2. Show that there are no simple groups of order 148 or of order 56.

3. Show that (i) a finite abelian group is cyclic $\iff \exp G = |G|$ and (ii) any finite subgroup of the multiplicative group of a field is cyclic.

4. Let R be a ring. Prove that if $a \in R$ has more than one right inverse then it has infinitely many. (Hint: if $ab = 1$, show first that $1 - ba \neq 0$.) Give an example such that this fails for monoids.

5. Let I be an ideal in a ring R. Show that $M_n(I)$ is an ideal in $M_n(R)$ and every ideal in $M_n(R)$ is in this form.

6. Let $I_1, \ldots, I_n$ be pairwise coprime (two-sided) ideals in a ring R in the sense that $I_i + I_j = R$ for $i \neq j$. Show that (i) $R/(\bigcap_{i=1}^n I_i) \cong \prod_{i=1}^n R/I_i$ and (ii) if moreover that R is commutative then $\bigcap_{i=1}^n I_i = \prod_{i=1}^n I_i$.

7. Let F be a field with $|F| = q$. Show that (i) the ring of polynomial functions in r variables over F is isomorphic to $F[x_1, \ldots, x_r]/\langle x_1^q - x_1, \ldots, x_r^q - x_r\rangle$ and (ii) every function $f : F^r \to F$ is a polynomial function.

8. Let D be a Euclidean domain whose function δ satisfies: (i) $\delta(ab) = \delta(a)\delta_(b)$ and (ii) $\delta(a+b) \leq \max(\delta(a), \delta(b))$. Show that either D is a field or $D = F[x]$ where F is a field.

* You may prove the existence of Sylow p-subgroups to supplement one (and only one) problem up to 10 pts. Indicate explicitly the problem number.

** Bonus: Let $A = \mathbb{Z}[\frac{1}{2}(1+\sqrt{-19})]$. Show that (i) the norm $N : A \to \mathbb{N}$ is a Dedekind–Hasse norm and (ii) A is not a Euclidean domain.

Date: Time and place: pm 1:20 – 5:20, November 9, 2018 at AMB 101.

Note: (1) each problem is of 15 points (total 120 pts), (2) you may work on each part separately, (3) show your answers/computations/proofs in details.