2018 FALL - ALGEBRA I: FINAL EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. Let R be a p.i.d. and $K \subset R^n$ be a sub R-module. Show directly from the definitions that K is free with base of $m \le n$ elements.

2. Let F be a field with $A, B \in M_n(F)$. Show that (i) A, B are similar over F if and only if $tI - A$ and $tI - B$ are equivalent in $M_n(F[t])$, (ii) A is similar to A^T over F. (iii) Give an example for $F = \mathbb{Z}$ such that the conclusion in (ii) fails. (You can apply the structure theorems in doing this problem.)

3. Let F be a field of characteristic $p \ne 0$ and let $a \in F$. Show that (i) $f(x) = x^p - x - a$ has only simple roots, (ii) $f(x)$ is irreducible in $F[x]$ if and only if $a \ne c^p - c$ for any $c \in F$, and in this case (iii) determine G_f.

4. Determine the splitting fields and the Galois groups over $\mathbb{Q}$ for (i) $x^p - a$, which is assumed to be irreducible with $a \in \mathbb{Q}$ and p a prime, (ii) $x^6 - 2$.

5. Let G be a group. Show that (i) if $|G| = p^n$ for p a prime then it is solvable, (ii) G is solvable if and only if $G^{(k)} = 1$ for some $k \ge 1$, (iii) if $K \lhd G$ then G is solvable if and only if both K and G/K are solvable.

6. Let E be a splitting field over F of $f(x) \in F[x]$. Show that E is normal over F.

7. Let $[E : F] < \infty$. Show that (i) $E = F(u)$ for some $u \in E$ (primitive generator) if and only if there are only a finite number of fields K with $F \subset K \subset E$, (ii) this is the case if E is separable over F.

8. Let $n \in \mathbb{Z}_{>0}$. Show that (i) there exists an irreducible polynomial of degree n in $\mathbb{Z}_p[x]$ for any prime p, (ii) a transitive subgroup $G \subset S_n$ containing an $(n-1)$-cycle and a 2-cycle must be S_n, (iii) there exists a monic $f(x) \in \mathbb{Z}[x]$ with $G_f = S_n$.

* In case you are not satisfied with your answers on the above 8 problems, write down what you know about the proof of the fundamental theorem of Galois theory or Galois' criterion for solvability of a polynomial equation by radicals. You can choose ONLY ONE of them and to remedy only one problem!

Date: Time and place: pm 1:20 – 5:00, January 11, 2019 at AMB 101.

Note: (1) each problem is of 15 points (total 120 pts), (2) you may work on each part separately, (3) show your answers/computations/proofs in details.