2019 ALGEBRA II - QUIZ II

1. Let $F : {}_R\mathbf{mod} \to \mathbf{Ab}$ be a (covariant) right exact functor. Define the left derived functor $(L_i F)(M)$ for $i \geq 0$ and show that it is independent of the choices of the projective resolutions $P_\bullet \to M \to 0$.

2. Let $0 \to N \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} M \to 0$, $i = 1, 2$ be two extensions of M by N. The Baer sum is defined by $E := F/K$ where $F = \{(z_1, z_2) \in E_1 \oplus E_2 \mid \beta_1 z_1 = \beta_2 z_2\}$ and $K \subset F$ consisting of $(\alpha_1 y, -\alpha_2 y)$, $y \in N$. Show that it gives an extension

$$0 \to N \xrightarrow{\alpha} E \xrightarrow{\beta} M \to 0$$

which corresponds to the sum $[E_1] + [E_2]$ in $\mathrm{Ext}^1(M, N)$.

3. Show that $\mathrm{Tor}_n^{\mathbb{Z}}(M, N)$ is a torsion group for every $n \geq 1$ and for any abelian groups M and N. (Hint: do the case $n = 1$ first and then use dimension shifting to handle the general cases.)

4. Let K be a commutative ring, M a K-module, and L a commutative K-algebra which is K-free. Show that $\mathrm{h.dim}_K M = \mathrm{h.dim}_L M_L$.

5. Let F be a field, $R = F[x_1, \ldots, x_n]$, and M be a graded R-module. Show that (i) if $M \otimes_R F = 0$ then $M = 0$, (ii) if $\mathrm{Tor}_1^R(M, F) = 0$ then M is free.

Show your answers/computations/proofs in details. Date: pm 6:00 – 7:40, May 20, 2019 at AMB 101. A course by Chin-Lung Wang at NTU..