2019 ALGEBRA II - QUIZ I

1. Let R and S be two local rings such that $M_m(R) \cong M_n(S)$. Show that then $m = n$ and $R \cong S$. Give a counterexample with non-local R.

2. Let F be a field. Use Zorn's Lemma to show that any vector space V over F has a basis and any two basis have the same cardinality.

3. Prove a generalization of $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) \cong \mathbb{Z}/(\gcd(m,n))$ to the case when $\mathbb{Z}$ is replaced by a p.i.d. R. Then determine the structure of $M_1 \otimes_R M_2$ of finitely generated modules M_1, M_2 over R.

4. Let R, S be rings. Let $P = {}_RP$ be f.g. projective. $M = {}_RM_S$, $N = {}_SN$. Show that there is a group isomorphism

$$\eta : \hom_R(P, M) \otimes_S N \to \hom_R(P, M \otimes_S N).$$

Give a counterexample with $R = S = \mathbb{Z}$ and $P = \mathbb{Z}/(n)$, $n \in \mathbb{N}$.

5. Given a Morita context $(R, R', {}_{R'}P_R, {}_RP'_{R'}, \tau, \mu)$ with $\tau : P' \otimes_{R'} P \to R$ and $\mu : P \otimes_R P' \to R'$ being surjective. Prove part of Morita I: (1) P_R is a progenerator, (2) τ is an isomorphism, (3) $P' \cong P^*$.

Show your answers/computations/proofs in details. Date: pm 2:50 – 3:50, March 15, 2019 at AMB 101. A course by Chin-Lung Wang at NTU..