2019 SPRING - ALGEBRA II: FINAL EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. Given a short exact sequence of complexes $0 \to C' \to C \to C'' \to 0$ of R-modules, show that there is a long exact sequence

$$\cdots \to H_i(C') \to H_i(C) \to H_i(C'') \xrightarrow{\Delta_i} H_{i-1}(C') \to \cdots$$

of homology modules.

2. Let $S \subset R \setminus \{0\}$ be a multiplicatively closed subset. Show that (i) $M_S \cong R_S \otimes_R M$ for any R-module M, (ii) R_S is a flat R-module, (iii) if R is a factorial domain (i.e. an UFD) then R_S is also factorial.

3. (i) Define the Zariski topology on $\operatorname{Spec} R$ and show that it is quasi-compact, (ii) $V(I) = V(J) \Leftrightarrow \sqrt{I} = \sqrt{J}$, (iii) $R \mapsto \operatorname{Spec} R$ is a contravariant functor from **Ring** to **Top**.

4. Let $E = F[u_1, \ldots, u_m]$ be a finitely generated algebra over a field F, and $G \subset \operatorname{Aut}_F E$ be a finite group. Show that $E^G \equiv \operatorname{Inv} G$ is finitely generated over F. (Hint: for each u_i, consider $f_i(x) = \prod_{g \in G}(x - gu_i)$.)

5. Let (R, J) be a regular local ring. Show that (i) R has Krull dimension $0 \iff R$ is a field, (ii) if $J \neq 0$ then $\dim R/Rx = \dim R - 1$ for any $x \in J \setminus J^2$, (iii) R is a domain.

6. Let E/F be finite Galois, $G = \operatorname{Gal} E/F$ and $k \in H^2(G, E^\times)$. Show that (i) k defines a central simple algebra $A = (E, G, k)$ over F on $A = E^{|G|}$, (ii) if $G = \langle s \rangle$ then A is a cyclic algebra (E, s, γ) for some $\gamma \in F^\times$ and $A \sim 1 \Leftrightarrow \gamma \in \operatorname{im} N_{E/F}$.

7. Show that (i) $\operatorname{Gal} \overline{\mathbf{F}}_p/\mathbf{F}_p \cong \widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/(m)$, (ii) there is a one to one correspondence between subfields of $\overline{\mathbf{F}}_p$ and the Steinitz numbers $\prod p_i^{k_i}$ which are formal product over all primes p_i where $k_i = 0, 1, \ldots$ or ∞.

8. (Bonus) Substantial topics in homological algebra, commutative algebra or field theory you have **well-prepared** but not shown above.

Date: Time and place: pm 1:20 – 5:20, June 21, 2019 at AMB 101.

Note: (1) each problem is of 15 points except problem **8** (up to 30 points), (2) you may work on each part separately, (3) show your answers in details, (4) **all rings are assumed to be commutative**.