2020 ALGEBRAIC GEOMETRY II

MIDTERM EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. For an effective divisor D on a curve X of genus g, show that $\dim |D| \le \deg D$ and equality holds $\iff D = 0$ or $g = 0$.

2. Let $X = V(f) \subset \mathbb{P}^2_k$ be an elliptic curve with $\operatorname{char} k = p > 0$. Show that the Hasse invariant is $0 \iff$ the term $(xyz)^{p-1}$ is not in f^{p-1}. Determine the corresponding coefficient $h_p(\lambda)$ for $f = y^2z - x(x-z)(x-\lambda z)$.

3. Show that a hyperelliptic curve can not be a complete intersection in any $\mathbb{P}^n$. Show also that any genus 2 curve is hyperelliptic.

4. Let $C/\mathbb{F}_q$ be a curve of genus g and $N = |C(\mathbb{F}_q)|$. Let $k = \overline{\mathbb{F}}_q$ and $f : C \to C$ be the k-linear Frobenius. Denote by $\Gamma, \Delta \subset C \times C$ the graph of f and the diagonal. Show that $\Gamma^2 = q(2-2g)$, $\Gamma.\Delta = N$, and $|N - (1+q)| \le 2g\sqrt{q}$.

5. Prove Grothendieck's lemma: any locally free sheaf $\mathscr{F}$ of finite rank on $\mathbb{P}^1$ is isomorphic to a direct sum of invertible sheaves $\mathscr{O}(n_i)$'s.

6. Let $\pi : X_r \to \mathbb{P}^2$ be the blowing up in $r \in [1,6]$ general points and embed X_r in $\mathbb{P}^{9-r}$ by $\mathbb{L}' = |\pi^*h - \sum_{i=1}^r E_i|$. Determine all lines in X_r and show that a general cubic surface arises in this way, hence has 27 lines.

7. Let $Y \cong \mathbb{P}^1$ be a curve in a surface X with $Y^2 < 0$. Show that there is a projective morphism $f : X \to X_0$ contracting (only) Y to a point p. Conversely, given f with $\dim X_0 = 2$, $f(Y) = p$ and $f^{-1}(p) = Y$, show that $Y^2 < 0$. .

8. (Bonus) Present an essential topic/theorem/exercise on curves and/or surfaces that you have well-prepared but not listed above.

Each problem is 15 points (total 120 pts). Be sure to show your answers/computations/proofs in details. Time: pm 6:00 – 9:30, April 30, 2020 at AMB 305.