2019 ALGEBRAIC GEOMETRY I

MIDTERM EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. Let $C = \{ (t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k \}$ be the twisted cubic curve. Show that (i) C is a complete intersection; (ii) $\overline{C} \subset \mathbb{P}^3$ is not a complete intersection.

2. Let $C \subset \mathbb{A}^2_{\mathbb{C}}$ be the cubic curve $y^2 = x^3 - x + a$. Give examples of $a \in \mathbb{C}$ such that (i) C is rational, (ii) C is not rational.

3. Let Y be a k-variety with $k = \bar{k}$. Show that (i) $\operatorname{Sing} Y$ is a proper closed subset of Y; (ii) non-normal points of Y is also a proper closed subset.

4. Let $(X, \mathscr{O}) = (\operatorname{Spec} A, \tilde{A})$ be an affine scheme. Show that (i) for any $p \in X$, $\mathscr{O}_p \cong A_p$, (ii) for any $f \in A$, $\mathscr{O}(D(f)) \cong A_f$. In particular $\Gamma(X, \mathscr{O}) \cong A$.

5. Show that (i) a finite morphism is proper; (ii) the intersection of affine open sets in a separated scheme over $\operatorname{Spec} A$ is affine, and give examples showing that this fails if X is not separated.

6. Show that: (i) closed immersions are finite morphisms but open immersions are not; (ii) if $f : X \to Y$ is a finite morphism of Noetherian schemes then $\mathscr{F}$ coherent on $X \Rightarrow f_*\mathscr{F}$ coherent on Y; (iii) for X projective over $\operatorname{Spec} A$, $\mathscr{F}$ a coherent sheaf on X, then $\mathscr{F}(n)$ is generated by a finite number of global sections for large n.

(*) You may replace one and only one problem listed above by presenting an essential topic/theorem/exercise in algebraic geometry you have well-prepared.

Each problem is of 20 points (total 120 pts). Be sure to show your answers/computations/proofs in details. Time: pm 6:00 – 9:00, November 12, 2019 at AMB 102.