2019 ALGEBRAIC GEOMETRY I

FINAL EXAM

A COURSE BY CHIN-LUNG WANG AT NTU

1. [II.6 Divisors] Let X be a noetherian integral separated scheme which is regular in codimension one. Show that (a) $X \times \mathbb{A}^1$ has the same property and $\operatorname{Cl} X \times \mathbb{A}^1 \cong \operatorname{Cl} X$, (b) $X \times \mathbb{P}^1$ has the same property and $\operatorname{Cl} X \times \mathbb{P}^1 \cong \operatorname{Cl} X \times \mathbb{Z}$.

2. [II.7 Projective morphisms] Let X be a noetherian scheme and Y, Z be closed subscheme, neither one containing the other. Let $\tilde{X}$ be the blowing up of X along $Y \cap Z$. Show that the strict transform $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{X}$ do not meet.

3. [II.8 Differentials] Show that (a) $0 \to \Omega_{\mathbb{P}^n_k} \to \mathscr{O}(-1)^{n+1} \to \mathscr{O} \to 0$ is exact, (b) $\omega_Y \cong \mathscr{O}_Y(d-(n+1))$ for $Y \subset \mathbb{P}^n_k$ being a non-singular hypersurface of degree d. (c) Generalized (b) to the case of for Y being a complete intersection of r equations.

4. [III.2-3 H^i of affine schemes] Let X be a noetherian scheme. Show that the following are equivalent. (i) X is affine. (ii) $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent $\mathscr{F}$ and all $i \geq 1$. (iii) $H^1(X, \mathscr{I}) = 0$ for all coherent sheaf of ideals $\mathscr{I}$.

5. [III.4-5 H^i of projective schemes] Let A be a noetherian ring and $X = \mathbb{P}^r_A$. Determine the structure of $H^i(X, \mathscr{O}(n))$ for all $i \geq 0$ and $n \in \mathbb{Z}$, including the perfect pairing $H^0(X, \mathscr{O}(n)) \times H^r(X, \mathscr{O}(-n-(r+1))) \to A$.

6. [III.6-7 Ext and duality] (a) Let X be projective over a noetherian ring A.Given coherent sheaves $\mathscr{F}, \mathscr{G}$, show that $\exists\, n_0$ such that $\operatorname{Ext}^i(\mathscr{F}, \mathscr{G}(n)) \cong \Gamma(X, \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}(n)))$ for all $n \geq n_0$. (b) Show that $H^q(\mathbb{P}^n_k, \Omega^p) = 0$ for $p \neq q$ and $\cong k$ if $0 \leq p = q \leq n$.

7. [III.8-9 $R^i f_*$ and flatness] Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F} \in \operatorname{Mod}_X$, $\mathscr{E} \in \operatorname{Mod}_Y$. (a) For all $i \geq 0$, show that $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong (R^i f_* \mathscr{F}) \otimes \mathscr{E}$ for $\mathscr{E}$ being locally free of finite rank. (b) Give a counterexample for coherent $\mathscr{E}$.

(*) You are allowed to replace ONE problem by an essential topic/theorem/exercise in the same labelled subsection(s).

8. (Bonus) Present an essential topic/theorem/exercise within II.6 to III.9 but not listed above nor in your replacement problem.

Each problem is of 15 points (total 120 pts). Be sure to show your answers/computations/proofs in details. Time: pm 6:00 – 9:50, January 9, 2020 at AMB 102. Also you are allowed to check the textbook (but not your notes, overleaf, or any website) during pm 7:30-7:40 without copying anything on the exam sheets.